Proc. Indian Acad. Sci. (Math. Sci.) Vol. 126, No. 2, May 2016, pp. 187–197.

$\mathcal{M}^*$-supplemented subgroups of finite groups

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MS received 1 April 2013; revised 22 December 2015

Abstract. A subgroup H of a group G is said to be $\mathcal{M}^*$-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is $\mathcal{M}$-supplemented in G. In this paper, we prove as follows: Let E be a normal subgroup of a group G. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup P of $F^*(E)$ is $\mathcal{M}^*$-supplemented in G, then $E \leq Z_{\mathcal{U}\Phi}(G)$.

Keywords. Sylow subgroup; $\mathcal{M}^*$-supplemented subgroups; supersolvable group; formation.

2010 Mathematics Subject Classification. 20D10, 20D20.

1. Introduction

All the groups in this paper are supposed to be finite. Most of the notation is standard and can be found in [3] and [8]. In what follows, $\mathcal{U}$ denotes the formation of all supersolvable groups. Let $[A]B$ denote the semidirect product of groups A and B, where B is an operator group of A.

As we all know, the study of groups in which some primary subgroups satisfy a certain embedding property is one of the most important topic in finite group theory. Particularly, many authors pay more attention to supplemented subgroups satisfying additional restrictions. For instance, in 1982, Arad and Ward [1] proved that a group G is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup of G are complemented in G. In 1998, Ballester-Bolinches and Pedraza-Aguilera [2] introduced S-quasinormally embedded subgroups and showed that if all maximal subgroups of all Sylow subgroups of G are S-quasinormally embedded in G, then G is supersolvable. In 2007, Wei and Wang [15] introduced c^*-supplemented subgroups and obtained some results about p-nilpotency of a group. In 2009, Miao and Lempken [11] introduced the concept of $\mathcal{M}$-supplemented subgroups and characterized the structure of finite groups by some $\mathcal{M}$-supplemented primary subgroups. In 2013, Wang and Wei [18] introduced $c^\sharp$-normal subgroups and determined the structure of finite groups by using some $c^\sharp$-normal primary subgroups.

As a continuation of these works, we shall introduce a new embedded property called $\mathcal{M}^*$-supplementation which covers properly both complementation and $\mathcal{M}$-supplementation of subgroups.

DEFINITION 1.1 (Definition 1.1 of [11])

A subgroup H is called $\mathcal{M}$-supplemented in a finite group G, if there exists a subgroup B of G such that $G = HB$ and $H_1 B$ is a proper subgroup of G for every maximal subgroup H_1 of H.

DEFINITION 1.2

A subgroup H of a group G is said to be $\mathcal{M}^*$-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is $\mathcal{M}$-supplemented in G.

It is clear that complemented subgroups and $\mathcal{M}$-supplemented subgroups are $\mathcal{M}^*$-supplemented subgroups. But, in general, the converse is not true.

Example 1.2. Let $G = S_4$ and $H = A_4$. Then $G = HS_3$. Clearly, H is $\mathcal{M}^*$-supplemented in G, but is not $\mathcal{M}$-supplemented in G.

Example 1.3. Let $G = Q_8$ be a quaternion group of order 8 and H be a cyclic subgroup of order 4. Clearly, G is a Hamiltonian group and H is $\mathcal{M}^*$-supplemented in G. On the other hand, H is not complemented in G.

Let $\mathcal{F}$ be a class of groups and H/K be a chief factor of a group G. H/K is called Frattini provided that $H/K \leq \Phi(G/K)$. Moreover, H/K is called $\mathcal{F}$-central in G if $[H/K](G/C_G(H/K)) \in \mathcal{F}$ (otherwise, $\mathcal{F}$-eccentric). The symbol $Z_{\mathcal{F}}(G)$ denotes the $\mathcal{F}$-hypercentre of a group G which is the product of all normal subgroups of G whose G-chief factors are $\mathcal{F}$-central in G. A subgroup H of G is said to be $\mathcal{F}$-hypercentral in G if $H \leq Z_{\mathcal{F}}(G)$.

The $\mathcal{F}$-hypercentre essentially influences the structure of finite groups. Note that if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and $E \leq Z_{\mathcal{F}}(G)$, then $G \in \mathcal{F}$, for every concrete class $\mathcal{F}$.

In 2009, Shemetkov and Skiba in [13] introduced the new notion of $\mathcal{F}\Phi$-hypercentre of G and investigated the structure of $Z_{\mathcal{F}\Phi}(G)$ by using weakly s-permutable primary subgroups. Here $Z_{\mathcal{F}\Phi}(G)$ denotes the $\mathcal{F}\Phi$-hypercentre of G, which is the product of all normal subgroups of G whose non-Frattini G-chief factors are $\mathcal{F}$-central in G. The subgroup $Z_{\mathcal{F}\Phi}(G)$ is characteristic in G and every non-Frattini G-chief factor of $Z_{\mathcal{F}\Phi}(G)$ is $\mathcal{F}$-central in G.

In this paper, we shall investigate extensively the properties of $Z_{\mathcal{F}\Phi}(G)$ with $\mathcal{M}^*$-supplemented subgroups.

2. Preliminaries

Here we list some preliminary results which will be useful in the sequel.

Lemma 2.1 *Let G be a group and $H \leq G$. Then the following hold*:

(1) *If H is $\mathcal{M}^*$-supplemented in G and $H \leq M \leq G$, then H is $\mathcal{M}^*$-supplemented in M.*
(2) *Let π be a set of primes. Let N be a normal π'-subgroup and H be a π-subgroup of G. If H is $\mathcal{M}^*$-supplemented in G, then HN/N is $\mathcal{M}^*$-supplemented in G/N.*

Proof. The claims are immediate from the definition of $\mathcal{M}^*$-supplemented subgroup. □

Lemma 2.2 *Suppose that R is a solvable minimal normal subgroup of a group G and R_1 is a maximal subgroup of R. If R_1 is $\mathcal{M}^*$-supplemented in G, then R is a cyclic group of prime order.*

Proof. Since R_1 is $\mathcal{M}^*$-supplemented in G, there exists a subgroup K of G such that $G = R_1K$ and $R_1 \cap K$ is $\mathcal{M}$-supplemented in G. By hypothesis, R is a solvable minimal normal subgroup of G, $G = R_1K = RK$ and $R \cap K \in \{1, R\}$. If $R \cap K = 1$, then $R = R_1$, a contradiction. Hence we have $R \leq K = G$. Then R_1 is $\mathcal{M}$-supplemented in G and there exists a subgroup B of G such that $G = R_1B$ and $TB < R_1B$ for every maximal subgroup T of R_1. Clearly, $G = R_1B = RB$ and $|R| = p$. □

Lemma 2.3 (*Theorem* 1.8.17 *of* [4]). *Let N be a nontrivial solvable normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.4 (*Lemma* 2.11 *of* [11]). *Let p be the smallest prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if P is $\mathcal{M}$-supplemented in G.*

Lemma 2.5 (*Corollary* 2.2 *of* [12]). *If a Sylow 3-subgroup of G is $\mathcal{M}$-supplemented in G, then G is* 3*-supersolvable.*

The generalized fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G.

Lemma 2.6 *Let G be a group and N a subgroup of G. Then*

(1) *If N is normal in G, then $F^*(N) \leq F^*(G)$.*
(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G))$.*
(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*
(4) *$C_G(F^*(G)) \leq F(G)$.*
(5) *If $P \trianglelefteq G$ with $P \leq O_p(G)$, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*
(6) *If $K \leq Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Proof. Items (1)−(4) can be found in Chapter X, 13 of [9], (5) can be found in Lemma 2.17 of [14], (6) can be found in Lemma 2.3 of [10]. □

In 2009, Guo and Skiba [5] introduced the notion of quasi-$\mathcal{F}$-groups and investigated the properties of quasi-$\mathcal{F}$-groups. The following lemma is a corollary of Theorem C of [5].

Lemma 2.7 (*Theorem C of* [5]). *G is quasinilpotent if and only if $G/Z_\infty(G)$ is semisimple.*

Lemma 2.8 (*Lemma* 2.7 *of* [11]). *Let G be a finite group with normal subgroups H and L and $p \in \pi(G)$. Then the following hold*:

(1) *$\Phi(L) \leq \Phi(G)$.*
(2) *If $L \leq \Phi(G)$, then $F(G/L) = F(G)/L$.*
(3) *If $L \leq H \cap \Phi(G)$, then $F(H/L) = F(H)/L$.*
(4) *If H is a p-group and $L \leq \Phi(H)$, then $F^*(G/L) = F^*(G)/L$.*

(5) *If* $L \leq \Phi(G)$ *with* $|L| = p$, *then* $F^*(G/L) = F^*(G)/L$.
(6) *If* $L \leq H \cap \Phi(G)$ *with* $|L| = p$, *then* $F^*(H/L) = F^*(H)/L$.

Lemma 2.9 (*Theorem* 3.1 *of* [17]). *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$ *and* G *a group with a solvable normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If for any maximal subgroup* M *of* G, *either* $F(H) \leq M$ *or* $F(H) \cap M$ *is a maximal subgroup of* $F(H)$, *then* $G \in \mathcal{F}$. *The converse also holds, in the case where* $\mathcal{F} = \mathcal{U}$.

Lemma 2.10 *Let* G *be a finite group. Then* G *is solvable if and if every Sylow* 2-*subgroup and every Sylow* 3-*subgroup are* $\mathcal{M}^*$-*supplemented in* G.

Proof. If G is solvable, then every Sylow 2-subgroup and every Sylow 3-subgroup of G are complemented in G by Hall [7]. Particularly, every Sylow 2-subgroup and every Sylow 3-subgroup are $\mathcal{M}^*$-supplemented in G.

Conversely, let P be a Sylow 2-subgroup of G. By hypothesis, P is $\mathcal{M}^*$-supplemented in G. There exists a subgroup K such that $G = PK$ and $P \cap K$ is $\mathcal{M}$-supplemented in G. Obviously, $P \cap K$ is a Sylow 2-subgroup of K and $P \cap K$ is also $\mathcal{M}$-supplemented in K. By Lemma 2.4, K is 2-nilpotent and K has a Hall $2'$-subgroup K_1 and K_1 is also a Hall $2'$-subgroup of G. It follows that every Sylow 2-subgroup is complemented in G.

Let Q be a Sylow 3-subgroup of G. By hypothesis, Q is $\mathcal{M}^*$-supplemented in G and there exists a subgroup B such that $G = QB$ and $Q \cap B$ is $\mathcal{M}$-supplemented in G. Clearly, $Q \cap B$ is a Sylow 3-subgroup of B and $Q \cap B$ is also $\mathcal{M}$-supplemented in B. By Lemma 2.5, B is 3-supersolvable and so B has a Hall $3'$-subgroup B_1 in B. Here B_1 is also a Hall $3'$-subgroup of G. Now every Sylow 3-subgroup of G is also complemented in G.

Finally, we deduce that G is solvable by the Arad–Ward theorem [1]. □

Lemma 2.11 *Let* G *be a finite group. If every maximal subgroup of every Sylow* 2-*subgroup and every Sylow* 3-*group are* $\mathcal{M}^*$-*supplemented in* G, *then* G *is solvable.*

Proof. Let P_1 be a maximal subgroup of Sylow 2-subgroup of G. Then there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K$ is $\mathcal{M}$-supplemented in G. Hence there exists a subgroup B such that $G = (P_1 \cap K)B$ and $TB < G$ for every maximal subgroup T of $P_1 \cap K$. By Lemma 2.2 of [11], $|G : TB| = 2$ and $(TB)_2$ is a Sylow 2-subgroup of TB and also is a maximal subgroup of a Sylow 2-subgroup of G. By Lemma 2.1, $(TB)_2$ is $\mathcal{M}^*$-supplemented in TB. According to the proof of Lemma 2.10, TB has a Hall $2'$-subgroup $(TB)_{2'}$ and is also a Hall $2'$-subgroup of G. Similarly, G has a Hall $3'$-subgroup.

Now we have proved that every Sylow 2-subgroup and Sylow 3-subgroup of G are complemented in G and so G is solvable by the Arad–Ward theorem [1]. □

Lemma 2.12 *Let* G *be a quasinilpotent group. If every maximal subgroup of every Sylow* 2-*subgroup of* G *is* $\mathcal{M}^*$-*supplemented in* G *and every Sylow* 3-*subgroup of* G *is cyclic, then* G *is solvable. Furthermore*, G *is nilpotent by X*, *Corollary* 13.7 *of* [9].

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Furthermore, we assume that P is a Sylow 2-subgroup of G and Q is a Sylow 3-subgroup of G.

(1) *G is not a non-abelian simple group.* If every maximal subgroup of P is complemented in G, then G is 2-nilpotent by Theorem 3.4 of [6] and G is solvable, a contradiction. Let

P_1 be a maximal subgroup of Sylow 2-subgroup of G. Then there exists a subgroup K of G such that $G = P_1K$ and $1 < P_1 \cap K \leq P_1$ is $\mathcal{M}$-supplemented in G. Hence there exists a subgroup B such that $G = (P_1 \cap K)B$ and $TB < G$ for every maximal subgroup T of $P_1 \cap K$. By Lemma 2.2 of [11], $|G : TB| = 2$ and so $TB \trianglelefteq G$.
(2) *The final contradiction.* Since G is a quasinilpotent group, $G/Z_\infty(G)$ is semisimple by Lemma 2.7. We assert that $Z_\infty(G) = 1$ and G is semisimple. If $Z_\infty(G) \neq 1$, then we may pick a minimal normal subgroup L of G contained in $Z_\infty(G)$ and so $|L| = p$ where p is a prime. Then G/L satisfies the condition of lemma and G/L is solvable. Hence G is solvable, a contradiction. Set $G = A_1 \times \cdots \times A_s$, where A_i is non-abelian simple for every $i \in \{1, \ldots, s\}$. Since Q is cyclic, there exists the unique $j \in \{1, \ldots, s\}$ such that Q is a Sylow 3-subgroup of A_j. By (2), $s > 1$ and there exists an $i \in \{1, \ldots, s\}$ such that $i \neq j$ and $(3, |A_i|) = 1$. By the proof of Lemma 2.10, P is complemented in G and so $P \cap A_i$ is complemented in A_i. Then A_i is solvable by Lemma 2.10, a contradiction. □

Lemma 2.13 (*Corollary* 3.7 *of* [11]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every non-cyclic Sylow subgroups of $F^*(H)$ is $\mathcal{M}$-supplemented in G, then $G \in \mathcal{F}$.*

Lemma 2.14 (*Theorem* 1.1 *of* [16]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F^*(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

Lemma 2.15 (*Lemma* 2.3 *of* [13]). *Let $Z = Z_{\mathcal{F}\Phi}(G)$ and N and T be normal subgroups of G. Then*

(1) *Every non-Frattini G-chief factor of Z is $\mathcal{F}$-central in G.*
(2) *$ZN/N \leq Z_{\mathcal{F}\Phi}(G/N)$.*
(3) *If $TN/N \leq Z_{\mathcal{F}\Phi}(G/N)$ and $(|T|, |N|) = 1$, then $T \leq Z$.*

3. Main results

Theorem 3.1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F(H)$ is $\mathcal{M}^*$-supplemented in G, then $G \in \mathcal{F}$.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Furthermore, we have □

Claim (1). $H \cap \Phi(G) = 1$; in particular, $O_p(H)$ is elementary abelian for every $p \in \pi(F(H))$ and thus $F(H) = C_H(F(H))$ is abelian.

Suppose that $L \neq 1$ is a minimal normal subgroup of G contained in $H \cap \Phi(G)$. Clearly, $(G/L)/(H/L) \cong G/H \in \mathcal{F}$ and L is an elementary abelian p-subgroup. Moreover, $F(H/L) = F(H)/L$ by Lemma 2.8. Now let QL/L be a non-cyclic Sylow q-subgroup of H/L for some prime q. If $p \neq q$, then Q_1L/L is a maximal subgroup of QL/L where without loss $Q \in \mathrm{Syl}_q(F(H))$ and Q_1 is maximal in Q. Furthermore, since Q_1 is $\mathcal{M}^*$-supplemented in G, Q_1L/L is $\mathcal{M}^*$-supplemented in G/L by Lemma 2.1(2). If $p = q$, then $L < P_1$ where P_1 is the maximal subgroup of $O_p(H)$. Since P_1 is $\mathcal{M}^*$-supplemented in G, there exists a subgroup K such that $G = P_1K$ and $P_1 \cap K$ is $\mathcal{M}$-supplemented in G, furthermore, there exists a subgroup B such that $G = (P_1 \cap K)B$ and $TB < G$ for every maximal subgroup T of $P_1 \cap K$. Clearly, $G/L = (P_1/L)(KL/L)$ and $(P_1/L \cap KL/L) = (P_1 \cap K)L/L$ with $G/L = ((P_1 \cap K)L/L)(BL/L)$. Since $L \leq H \cap \Phi(G)$, $(TL/L)(BL/L) = TBL/L < G/L$ and so $(P_1/L \cap KL/L)$ is $\mathcal{M}$-supplemented in G/L. Hence the pair $(G/L, H/L)$ satisfies the hypothesis of the theorem. The minimal choice of G implies $G/L \in \mathcal{F}$. Since $L \leq \Phi(G)$, we get $G \in \mathcal{F}$, a contradiction which proves (1).

Since $G \notin \mathcal{F}$, Lemma 2.9 implies the existence of a maximal subgroup M of G such that $F(H) \nleq M$ and $F(H) \cap M$ is not maximal in $F(H)$. In particular, there exists a prime $p \in \pi(H)$ such that $P := O_p(H) \nleq M$ and so $G = PM$ as well as $P \cap M \trianglelefteq G$.

Claim (2). $P = R \times R_j$ with a minimal normal subgroup R_j of G such that $|R_j| > p$ and $G = R_jM$ for $R_j \cap M = 1$.

Since $P \cap \Phi(G) \leq H \cap \Phi(G) = 1$ by (1), we use Lemma 2.3 to see that $P = R_1 \times R_2 \times \cdots \times R_t$ with minimal normal subgroups $R_1, \ldots, R_t$ of G. Since $P \nleq M$ and M is maximal in G, there exists j such that $G = R_jM = PM$ with $R_j \cap M = 1$. If $|R_j| = p$, then $|G : M| = |F(H) : F(H) \cap M| = |R_j| = p$ and so $F(H) \cap M$ is a maximal subgroup of $F(H)$, contrary to the choice of M. Thus we may assume that $|R_j| > p$. Clearly, $P^* = R_1 \times R_2 \times \cdots \times {R_j}^* \times \cdots \times R_t = R \times {R_j}^*$ is a maximal subgroup of P where ${R_j}^*$ is the maximal subgroup of R_j. By hypothesis, P^* is $\mathcal{M}^*$-supplemented in G. There exists a subgroup K of G such that $G = P^*K$ and $P^* \cap K$ is $\mathcal{M}$-supplemented in G. If $P^* \cap K = 1$, then $|P \cap K| = p$. Clearly, $G = {R_j}^*RK = R_jRK$. Then $R_j \cap RK = 1$ or R_j. If $R_j \cap RK = 1$, then $R_j = {R_j}^*$, a contradiction. Hence we have $R_j \leq RK$ and $G = RK$. On the other hand, since $G = RK = (R_jR)K = PK$, it follows from $|G| = |R||K| = |R_jR||K|/|P \cap K|$ that $|R_j| = p$, also a contradiction. Furthermore, there exists a subgroup B such that $G = (P^* \cap K)B$ and $TB < G$ for every maximal subgroup T of $P^* \cap K$.

Claim (3). $P^* = L{R_j}^*X$, $L = R_1 \times \cdots \times R_{j-1}$, where R_i $(i = 1, \ldots, j-1)$ is the minimal normal subgroup of G of order p, $X = R_{j+1} \times \cdots \times R_t$, where $|R_l| > p$ and $R_l \leq B$ $(l = j+1, \ldots, t)$.

For each minimal normal subgroup R_i of G contained in P, if there exists a maximal subgroup T_i of $(P^* \cap K)$ such that $R_i \nleq T_iB$, then $|R_i| = p$ since $|G : T_iB| = p$. Otherwise, $R \leq TB$ for every maximal subgroup T of $(P^* \cap K)$. Therefore $R \leq \bigcap_{T <\cdot (P^* \cap K)}(TB) = \Phi(P^* \cap K)B = B$. Without loss of generality, $R_l \leq B$ with $|R_l| > p$ $(l = j+1, \ldots, t)$. Put $X = R_{j+1} \times \cdots \times R_t$. Then we claim that P^*/X also is $\mathcal{M}^*$-supplemented in G/X. Denote $L = R_1 \times \cdots \times R_{j-1}$. Then $P^*/X = L{R_j}^*X/X$ and $G/X = (P^*/X)(KX/X)$ and $(P^*/X) \cap (KX/X)$ is $\mathcal{M}$-supplemented in G/X. Moreover, $((P^*/X) \cap (KX/X))(B/X) = G/X$ and $TB/X < G/X$ for every maximal subgroup T of $P^* \cap K$. In brief, $\bar{G} = \bar{L}\bar{R_j}^*\bar{K}$.

Claim (4). $\bar{R_j} \leq \bar{K}$. $\bar{G} = \bar{L}\bar{R_j}^*\bar{K}$, where $\bar{L} = \prod_{i=1}^{j-1} \bar{R_i}$ and $\bar{R_i}$ is a minimal normal subgroup of $\bar{G}$ with order p. Since $\bar{G} = \bar{L}\bar{R_j}^*\bar{K}$, we have $\bar{G} = \bar{L}\bar{K}$. Otherwise, if $\bar{L}\bar{K} < \bar{G}$, then $\bar{G} = \bar{R_j}^*\bar{L}\bar{K} = \bar{R_j}\bar{L}\bar{K}$. Clearly $\bar{R_j} \cap \bar{L}\bar{K} \in \{1, \bar{R_j}\}$. If $\bar{R_j} \cap \bar{L}\bar{K} = 1$, then $\bar{R_j} = \bar{R_j}^*$, a contradiction. Thus $\bar{R_j} \leq \bar{L}\bar{K}$ and $\bar{G} = \bar{L}\bar{K}$, a contradiction. Next we consider $\bar{R_j} \cap \bar{K}$. Since $\bar{R_j} \cap \bar{K} = \bar{R_j} \cap (\bar{P} \cap \bar{K})$, we have $\bar{R_j} \cap \bar{K} = 1$ or $\bar{R_j}$. If

$\bar{R}_j \cap \bar{K} = 1$, then we consider $\bar{R}_j \cap \bar{R}_i\bar{K}$ with $\bar{R}_i \not\leq \bar{K}$. Obviously, $\bar{R}_j \cap \bar{R}_i\bar{K} = 1$ or $\bar{R}_j$. If $\bar{R}_j \leq \bar{R}_i\bar{K}$, then $\bar{R}_i\bar{K} = \bar{K}\bar{R}_j$ and hence $|\bar{R}_j| = p$, a contradiction. We repeat this step to get a similar contradiction. Therefore, $\bar{R}_j \leq \bar{K}$, $\bar{L}\bar{R_j^*} \cap \bar{K} = \bar{R_j^*}(\bar{L} \cap \bar{K})$ and $\bar{R_j^*}(\bar{L} \cap \bar{K})\bar{B} = \bar{G}$. It follows that $\bar{R_j^*}$ is $\mathcal{M}$-supplemented in $\bar{G}$ and $|\bar{R}_j| = p$ by Lemma 2.2. Therefore $|R_j| = p$, a final contradiction.

COROLLARY 3.2

Let G be a group with a solvable normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F(H)$ is $\mathcal{M}^*$-supplemented in G, then $G \in \mathcal{U}$.

COROLLARY 3.3

Let G be a group with a solvable normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F(H)$ is $\mathcal{M}$-supplemented in G, then $G \in \mathcal{U}$.

COROLLARY 3.4

Let G be a group with a solvable normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F(H)$ is complemented in G, then $G \in \mathcal{U}$.

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(H)$ is $\mathcal{M}^*$-supplemented in G, then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is false and choose G to be a counterexample of the minimal order; so in particular, $H \neq 1$. Furthermore, we have

Case I. $\mathcal{F} = U$. If every Sylow 2-subgroup of G is cyclic, then G is 2-nilpotent by Burnside theorem. Hence G is solvable and so $F^*(H)$ is solvable. If every Sylow 2-subgroup and every Sylow 3-subgroup of G are not cyclic, then G is solvable and $F^*(H)$ is solvable by Lemma 2.11. If every maximal subgroup of Sylow 2-subgroup is $\mathcal{M}^*$-supplemented in G and Sylow 3-subgroup is cyclic, then by Lemma 2.12, $F^*(H)$ is solvable. Hence $F(H) = F^*(H) \neq 1$. Since the pair (H, H) satisfies the hypothesis of the theorem in place of (G, H), the minimal choice of G implies that H is supersolvable if $H < G$; then $G \in \mathcal{U}$ by Corollary 3.2, a contradiction. Hence

(1) $H = G$ is non-solvable and $F^*(G) = F(G) \neq 1$. Particularly, $G/F^*(G)$ is non-solvable and every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(G)$ is $\mathcal{M}^*$-supplemented in G.

Let N be a proper normal subgroup of G containing $F^*(G)$. By Lemma 2.6, $F^*(G) = F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$, and so $F^*(N) = F^*(G)$. Moreover, every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(N)$ is $\mathcal{M}^*$-supplemented in N by Lemma 2.1(1). Hence N is supersolvable by the minimal choice of G. Then we have

(2) Every proper normal subgroup of G containing $F^*(G)$ is supersolvable.

Suppose now that $\Phi(O_p(G)) \neq 1$ for some $p \in \pi(F(G))$. By Lemma 2.6, we have $F^*(G/\Phi(O_p(G))) = F^*(G)/\Phi(O_p(G))$. Using Lemma 2.1, we observe that the pair $(G/\Phi(O_p(G)), F^*(G)/\Phi(O_p(G)))$ satisfies the hypothesis of the theorem. The minimal choice of G implies $G/\Phi(O_p(G)) \in \mathcal{U}$. Since $\mathcal{U}$ is a saturated formation, we get $G \in \mathcal{U}$, a contradiction. Thus we have

(3) If $p \in \pi(F(G))$, then $\Phi(O_p(G)) = 1$ and so $O_p(G)$ is elementary abelian; in particular, $F^*(G) = F(G)$ is abelian and $C_G(F(G)) = F(G)$.

Suppose that L is a minimal normal subgroup of G contained in $F(G)$ and that $|L| = p$ for some $p \in \pi(F(G))$; also set $C := C_G(L)$. Clearly, $F(G) \leq C \trianglelefteq G$. If $C < G$, then C is solvable by (2). Since G/C is cyclic, we get that G is solvable, a contradiction. Thus we have $C = G$ and so $L \leq Z(G)$. This proves

(4) Every normal subgroup of prime order of G is contained in $Z(G)$.

If $F(G) = H_1 \times \cdots \times H_r$ with cyclic Sylow subgroups $H_1, \ldots, H_r$ of $F(G)$, then $G/C_G(H_i)$ is abelian for every $i \in \{1, \ldots, r\}$ and so $G/\bigcap_{i=1}^{r} C_G(H_i) = G/C_G(F(G)) = G/F(G)$ is abelian. Therefore G is solvable, a contradiction. This proves

(5) $P := O_p(G) \in \mathrm{Syl}_p(F(G))$ is non-cyclic for some $p \in \pi(F(G))$.

If $\Phi(G) = F(G)$, then P_1 is $\mathcal{M}$-supplemented in G for every maximal subgroup of Sylow subgroup of $F(G)$. By Lemma 2.13, G is supersolvable, a contradiction. This proves

(6) $\Phi(G) < F(G)$. Since $\Phi(G) < F(G)$, there exists a maximal subgroup M of G such that $P \nleq M$ for some Sylow subgroup P of $F(G)$. Furthermore, if $P \cap \Phi(G) = 1$, then $P = R_1 \times \cdots \times R_t$ with minimal normal subgroups $R_1, \ldots, R_t$ of G by Lemma 2.3. We may assume without loss that $R_1 \nleq M$ and thus $G = R_1 M = PM$; as $R_1 \cap M \trianglelefteq G$ and R_1 is minimal normal in G, also $R_1 \cap M = 1$. Now we can employ the same arguments as in the proof of Theorem 3.1 to derive that R_1 is a normal subgroup of G of order p. By (4), $P \leq Z(G)$. Clearly, G/P satisfies the condition of the theorem and the minimal choice of G implies that $G/P \in \mathcal{U}$ and G is supersolvable, a contradiction. This eventually proves

(7) $R := P \cap \Phi(G) \neq 1$. Now suppose that $Q \in \mathrm{Syl}_q(F(G))$ for some prime $q \neq p$ and let L be a minimal normal subgroup of G contained in R. Then Q is elementary abelian by (3). By the definition of a generalized fitting subgroup, $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G/L), E(G/L)] = 1$, where $E(G/L)$ is the layer of G/L. Since $L \leq \Phi(G)$, $F(G/L) = F(G)/L$ by Lemma 2.8. Now set $E/L = E(G/L)$. Since Q is normal in G and $[F(G)/L, E/L] = 1$, $[Q, E] \leq Q \cap L = 1$, i.e. $[Q, E] = 1$. Therefore $F(G)E \leq C_G(Q) \trianglelefteq G$. If $C_G(Q) < G$, then $C_G(Q)$ is supersolvable by (2); thus $E(G/L) = E/L$ is supersolvable and consequently $F^*(G/L) = F(G)/L$. Clearly, we see that G/L satisfies the hypothesis of the theorem. By the minimal choice of G, G/L is supersolvable and so is G, a contradiction. Henceforth we have $C_G(Q) = G$, i.e. $Q \leq Z(G)$. Obviously, using the same argument as in the proof of (7), $G/Q \in \mathcal{U}$ and hence G is supersolvable, also a contradiction. Thus we have

(8) $F(G) = P$, in particular, $1 < R = P \cap \Phi(G) < P$. Now let L be a minimal normal subgroup of G contained in R and set $\bar{G} := G/L$. Clearly, $F(\bar{G}) = F(G)/L = P/L$, because $L \leq \Phi(G)$. If $F^*(\bar{G}) = F(\bar{G})$, then we easily verify that G/L satisfies the hypothesis of the theorem, and thus, by the minimal choice of G, $G/L \in \mathcal{U}$. Since $L \leq \Phi(G)$ and $\mathcal{U}$ is saturated, we get $G \in \mathcal{U}$, a contradiction. Therefore $F^*(\bar{G}) = F(\bar{G})E(\bar{G}) > F(\bar{G})$ and so there exists a perfect normal subgroup E in G such that

$E/L = E(\bar{G})$. Clearly, PE is a non-solvable normal subgroup of G; hence, by (2) and (1), $G = PE$. In particular, $\bar{G} = \bar{P}E(\bar{G}) = F^*(\bar{G})$ with $[\bar{P}, E(\bar{G})] = [F(\bar{G}), E(\bar{G})] = 1$ and so $[P, E] \leq L$. Let N be a minimal normal subgroup of G contained in P with $N \neq L$. Clearly, $[N, E] \leq N \cap L = 1$. Therefore, $PE \leq C_G(N) = G$ and $N \leq Z(G)$. Since N is a minimal normal subgroup of G contained in P, $|N| = p$. Furthermore, if $N \leq \Phi(G)$, then we easily prove that G/N satisfies the condition of the theorem and the minimal choice of G implies that $G/N \in \mathcal{U}$ and so G is supersolvable, a contradiction. Then we have the following:

(9) $G = PE$ with $L = [P, E] \leq P \cap E$, $\bar{G} = \overline{PE} = F^*(\bar{G})$ and $\bar{P} \leq Z(\bar{G})$; in particular, L is the unique minimal normal subgroup of G contained in $\Phi(G)$ and $|N| = p$ for every minimal normal subgroup of G contained in P with $N \cap \Phi(G) = 1$.

For every maximal subgroup P^* of P, by hypothesis, P^* is $\mathcal{M}^*$-supplemented in G. Hence there exists a subgroup K such that $G = P^*K$ and $P^* \cap K$ is $\mathcal{M}$-supplemented in G. If $P^* \cap K = 1$, then every maximal subgroup of P is complemented in G, then $G \in \mathcal{U}$ by Lemma 2.14, a contradiction. Thus there exists a maximal subgroup P_1 of P and there exists a subgroup E such that $G = P_1E$ with $P_1 \cap E \neq 1$ and $P_1 \cap E$ is $\mathcal{M}$-supplemented in G. If $P \cap E \cap \Phi(G) = 1$, then $P \cap E = R_1 \times R_2 \times \cdots \times R_s$, where $|R_i| = p$ $(i = 1, \ldots, s)$. Then $P \cap E \leq Z(G)$ by (4) and (9) and $P_1 \cap E$ is maximal in $P \cap E$ and also is contained in $Z(G)$. By hypothesis, there exists a subgroup B such that $G = (P_1 \cap E)B$ and $TB < G$ for every maximal subgroup T of $P_1 \cap E$. Clearly, $|G : TB| = p$ and $TB \trianglelefteq G$. Hence $P \cap TB$ is normal in G and is a maximal subgroup of P. Next we consider the factor group $G/(P \cap TB)$. Since $|O_p(G/(P \cap TB))| = p$ is cyclic, we have $G/(P \cap TB) \in \mathcal{U}$ by (5) and so G is solvable, a contradiction.

Consequently, we may assume that $P \cap E \cap \Phi(G) \neq 1$. Then $L \leq P \cap E$ by (9). Since $P_1 \cap E$ is the maximal subgroup of $P \cap E$, $P_1 \cap E \cap Ł \neq 1$, otherwise, $P \cap E = (P_1 \cap E)L$ and $|L| = p$. By (4) and Lemma 2.6(6), $L \leq Z(G)$ and G/L satisfies the condition of the theorem, the minimal choice of G implies that $G/L \in \mathcal{U}$ and so $G \in \mathcal{U}$, a contradiction. Therefore, $P_1 \cap E \cap L \neq 1$. Clearly, there exists a maximal subgroup T of $P_1 \cap E$ such that $P_1 \cap E \cap L \nleq T$, if not, $1 < P_1 \cap E \cap L \leq \bigcap_{T <\cdot P_1 \cap E} T = \Phi(P_1 \cap E) \leq \Phi(P) = 1$, a contradiction. Since $G = (P_1 \cap E)B$ and $T_iB < G$ for every maximal subgroup T_i of $P_1 \cap E$, we have $TB < G$ and $L \leq TB$ since $L \leq \Phi(G)$. Moreover, $TB = LTB = (P_1 \cap E)B = G$, a contradiction.

Case II. $\mathcal{F} \neq \mathcal{U}$. By Case I, H is supersolvable. Particularly, H is solvable and $F^*(H) = F(H)$. Therefore $G \in \mathcal{F}$ by Theorem 3.1. The final contradiction completes the proof. □

COROLLARY 3.6

Let G be a group with a normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(H)$ is $\mathcal{M}^*$-supplemented in G, then $G \in \mathcal{U}$.

COROLLARY 3.7

Let G be a group with a normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(H)$ is $\mathcal{M}$-supplemented in G, then $G \in \mathcal{U}$.

COROLLARY 3.8

Let G be a group with a normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(H)$ is complemented in G, then $G \in \mathcal{U}$.

Theorem 3.9. *Let E be a normal subgroup of a group G. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup P of $F^*(E)$ is $\mathcal{M}^*$-supplemented in G, then $E \leq Z_{\mathcal{U}\Phi}(G)$.*

Proof. Suppose that in this case the theorem is false and let (G, E) be a counterexample with $|G||E|$ minimal. Let $F = F(E)$ and $F^* = F^*(E)$. Let p be the smallest prime divisor of $|F|$.

(1) $F^* = F \neq E$.
(2) $P \leq Z_{\mathcal{U}\Phi}(G)$ and $E/P \nleq Z_{\mathcal{U}\Phi}(G/P)$.

Since P char $F = F^*$ char $E \trianglelefteq G$, $P \trianglelefteq G$. By hypothesis, $P \leq Z_{\mathcal{U}\Phi}(G)$. Therefore, $E/P \nleq Z_{\mathcal{U}\Phi}(G/P)$ by Lemma 2.15. Otherwise, $E \leq Z_{\mathcal{U}\Phi}(G)$, which is a contradiction.

(3) $E \neq G$ and E is supersolvable by Corollary 3.6.
(4) P is non-cyclic.

If P is cyclic, then the hypothesis is still true for $(G/\Phi(P), E/\Phi(P))$. Hence $E/\Phi(P) \leq Z_{\mathcal{U}\Phi}(G/\Phi(P))$ and $E \leq Z_{\mathcal{U}\Phi}(G)$ by the choice of (G, E), a contradiction. Therefore P is an elementary abelian normal subgroup of G.

(5) $\Phi(G) \cap P \neq 1$.

If $\Phi(G) \cap P = 1$, then P is the direct product of some minimal normal subgroups of G contained in P by Lemma 2.3. With a discussion similar to the one in Theorem 3.1, every minimal normal subgroup R of G contained in P is of order p. If $p = 2$, then $P \leq Z(E)$ and so $(G/P, E/P)$ satisfies the condition of the theorem, then $E/P \leq Z_{\mathcal{U}\Phi}(G/P)$. Therefore $E \leq Z_{\mathcal{U}\Phi}(G)$, a contradiction. On the other hand, for any $p \in \pi(E)$, P is the direct product of some minimal normal subgroups of G and hence every maximal subgroup of non-cyclic Sylow subgroup of $F^*(E)$ is normal in G. Therefore $E \leq Z_{\mathcal{U}\Phi}(G)$ by Theorem 1.4 of [13], a contradiction.

(6) $E = G$ is not solvable.

By (3), E is solvable if $E < G$. Let L be a minimal normal subgroup of G contained in $\Phi(G) \cap P$. By Lemma 2.8(3), $F/L = F(E/L)$. On the other hand, $F^*(E/L) = F(E/L)$, by Lemma 2.6(3). Hence by (1), $F^*(E/L) = F(E/L) = F^*/L$. Hence the hypothesis still holds for $(G/L, E/L)$. Hence $E/L \leq Z_{\mathcal{U}\Phi}(G/L)$, which implies that $E \leq Z_{\mathcal{U}\Phi}(G)$, a contradiction. □

Final contradiction. By (6), $F^* = F = F^*(G)$, G is supersolvable by Corollary 3.6. This contradiction completes our proof.

Acknowledgements

The authors would like to thank the referees for their careful reading of the manuscript and for their suggestions which helped improve the original version of the paper. This research was supported by the grant of NSFC (Grant # 11271016) and Qing Lan Project of Jiangsu Province and high-level personnel of support program of Yangzhou University and the Postgraduate Innovation Project of Jiangsu Province (No. KYLX15_1352).

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Communicating Editor: B Sury