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Optimal control problem for the extended Fisher–Kolmogorov equation

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Abstract. In this paper, the optimal control problem for the extended Fisher–Kolmogorov equation is studied. The optimal control under boundary condition is given, the existence of optimal solution to the equation is proved and the optimality system is established.

Keywords. Optimal control; extended Fisher–Kolmogorov equation; optimal solution, optimality condition.

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1. Introduction

In this paper, we consider the following initial-boundary problem of the extended Fisher–Kolmogorov (EFK) equation. We seek a real-valued function $u = u(x, t)$ defined on $\Omega \times [0, T]$.

$$u_t + k\Delta^2 u - \Delta u + f(u) = 0, \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

where $k > 0$ is given, Δ is the Laplacian operator, $f(u) = u^3 - u$ and Ω denotes an open bounded set of $\mathbb{R}^n (n = 1, 2, 3)$ with smooth boundary $\partial\Omega$.

On the basis of physical consideration, eq. (1.1) is supplemented by the following boundary conditions

$$u(x, t) = \Delta u(x, t) = 0, \quad \text{in } \partial\Omega \times (0, T), \tag{1.2}$$

and initial condition

$$u(x, 0) = u_0(x), \quad \text{in } \Omega. \tag{1.3}$$

When $k = 0$ in (1.1), we obtain the standard Fisher–Kolmogorov equation. However, by adding a stabilizing fourth order derivative term to the Fisher–Kolmogorov equation, Coullet *et al*. [1] and, Dee and van Saarloos [3] proposed (1.1) and called the model described in (1.1) as the extended Fisher–Kolmogorov equation. The extended Fisher–Kolmogorov (EFK) equation arises as the mezoscopic model of a phase transition in a binary system near the Lifshitz point (see [6, 9, 26]) and is frequently used as a model system for the study of pattern formation from an unstable spatially homogeneous state (see [1, 3]). The equation entered mathematical literature through a series of papers by Peletier and Troy [13–16]. Recently, Khiari and Omrani [8] considered the approximate solutions for the extended Fisher–Kolmogorov equation in two space dimension with Dirichlet boundary conditions by a Crank–Nicolson type finite difference scheme. Using an iteration procedure, regularity estimates for the linear semigroups, Luo [12] considered the long-time behavior of solution to the extended Fisher–Kolmogorov equation. We also noticed that some investigations of the Fisher–Kolmogorov equation were studied, such as in [2, 5, 7, 10, 17] and so on.

In past decades, the optimal control of distributed parameter system had received much attention in the academic field. A wide spectrum of problems in applications can be solved by methods of optimal control, such as chemical engineering and vehicle dynamics. Modern optimal control theories and applied models are not only represented by ODE, but also by PDE, especially nonlinear parabolic equation. Many papers have already been published to study the control problems of nonlinear parabolic equations, for example [18, 19, 21–24]. Recently, based on Lions's classical theorem on optimal control, we also studied the optimal control problem on some nonlinear PDE. (In [4], we considered the existence of optimal solutions for 1D modified Swift–Hohenberg equation; In [25], we investigated the optimal control problem for a generalized Ginzburg–Landau model equation in population problems.)

In this paper, suppose that $Q_0 \subseteq Q = (0, 1) \times (0, T)$, C is the observer and S is a real Hilbert space of observations, we are concerned with distributed optimal control problem

$$\min J(u, w) = \frac{1}{2}\|Cu - z_d\|_S^2 + \frac{\delta}{2}\|w\|_{L^2(Q_0)}^2, \tag{1.4}$$

subject to

$$\begin{cases} u_t + k\Delta^2 u - \Delta u + u^3 - u = Bw, & (x, t) \in \Omega \times (0, T), \\ u(x, t) = \Delta u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T) \\ u(0) = u_0, & x \in \Omega. \end{cases} \tag{1.5}$$

The control target is to match the given desired state z_d in L^2-sense by adjusting the body force w in a control volume Q_0 in the L^2-sense.

In the following, we introduce some notations that will be used throughout the paper. For fixed $T > 0$, $V = H^2(\Omega) \bigcap H_0^1(\Omega)$, $U = H_0^1(\Omega)$ and $H = L^2(0, 1)$, let V^*, U^* and H^* be dual spaces of V and H. Then, we obtain

$$V \hookrightarrow U \hookrightarrow H = H^* \hookrightarrow U^* \hookrightarrow V^*.$$

Clearly, each embedding is dense.

The extension operator $B \in \mathcal{L}\left(L^2(Q_0), L^2(0, T; H)\right)$ which is called the controller is introduced as

$$Bq = \begin{cases} q, & q \in Q_0, \\ 0, & q \in Q \setminus Q_0. \end{cases}$$

We supply H with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$, and define a space $W(0, T; V)$ as

$$W(0, T; V) = \left\{v : v \in L^2(0, T; V), \frac{\partial v}{\partial t} \in L^2(0, T; V^*)\right\},$$

which is a Hillbert space endowed with common inner product.

This paper is organized as follows. In § 2, we prove the existence and uniqueness of the weak solution to problem (1.5) in a special space and discuss the relation among the norms of weak solution, initial value and control item. In § 3, we consider the optimal control problem and prove the existence of the optimal solution. In § 4, the optimality conditions are shown and the optimality system is derived.

In the following, the letters c, c_i $(i = 1, 2, \ldots)$ will always denote positive constants different in various occurrences.

2. Existence and uniqueness of weak solution

In this section, we prove the existence and uniqueness of weak solution for the following equation

$$u_t + k\Delta^2 u - \Delta u + u^3 - u = Bw, \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

with the boundary value conditions

$$u(x, t) = \Delta u(x, t) = 0, \quad \text{in } \partial\Omega \times (0, T), \tag{2.2}$$

and initial condition

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \tag{2.3}$$

where $Bw \in L^2(0, T; H)$ and a control $w \in L^2(Q_0)$.

Now, we give the definition of the weak solution for problems (2.1)–(2.3) in the space $W(0, T; V)$.

DEFINITION 2.1

For all $\eta \in V, t \in (0, T)$, a function $u(x, t) \in W(0, T; V)$ is called a weak solution to problems (2.1)–(2.3), if

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} u, \eta\right) + k(\Delta u, \Delta\eta) - (\nabla u, \nabla\eta) + (u^3, \eta) - (u, \eta) = (Bw, \eta). \tag{2.4}$$

We shall give Theorem 2.1 on the existence and uniqueness of weak solution to problems (2.1)–(2.3).

Theorem 2.2. *Suppose* $u_0 \in V$, $Bw \in L^2(0, T; H)$, *then the problems* (2.1)–(2.3) *admit a unique weak solution* $u(x, t) \in W(0, T; V)$.

Proof. Galerkin method is applied to the proof.

Let $\{y_j(x)\}$ $(j = 1, 2, \ldots)$ be the orthonormal base in $L^2(\Omega)$ being composed of the eigenfunctions of the eigenvalue problem

$$\Delta y + \lambda y = 0, \, y(0) = y_0,$$

corresponding to eigenvalues $\lambda_j (j = 1, 2, \ldots)$.

Performing the Galerkin procedure for the problems (2.1)–(2.3), we obtain

$$\begin{cases} (u_{nt} + k\Delta^2 u_n + \Delta u_n + u_n^3 - u_n, y_j) = (Bw, y_j), \\ (u_n(\cdot, 0), y_j) = (u_{n0}(\cdot), y_j), \ j = 1, 2, \dots, N. \end{cases} \tag{2.5}$$

Obviously, equation (2.5) is an ordinary differential equation and according to ODE theory, there exists a unique solution $u_n(x,t) = \sum_{j=1}^{N} u_{nj}(t) y_j(x)$ to the equation (2.5) in the interval $[0, t_n)$. What we should do is to show that the solution is uniformly bounded when $t_n \to T$. We also need to show that the times t_n are not decaying to 0 as $n \to \infty$.

Then, we shall prove the existence of the solution.

Multiplying both sides of equation (2.5) by $u_{nj}(t)$, summing up the products over $j = 1, 2, \dots, N$, and using Hölder's inequality, we derive that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + k\|\Delta u_n\|^2 + \|u_n\|_4^4 &= (Bw, u_n) + \|u_n\|^2 - (\Delta u_n, u_n) \\ &\le \frac{1}{2}\|Bw\|^2 + \frac{k}{2}\|\Delta u_n\|^2 + \left(\frac{k}{2} + \frac{3}{2}\right)\|u_n\|^2. \end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + k\|\Delta u_n\|^2 \le \|Bw\|^2 + (k+3)\|u_n\|^2. \tag{2.6}$$

Since $Bw \in L^2(0, T; H)$ is the control item, we can assume that $\|Bw\| \le M$, where M is a positive constant. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + k\|\Delta u_n\|^2 \le M^2 + (k+3)\|u_n\|^2. \tag{2.7}$$

Using Gronwall's inequality, we derive that

$$\begin{aligned} \|u_n\|^2 &\le \mathrm{e}^{(k+3)t}\|u_n(0)\|^2 + \frac{M^2}{k+3} \\ &\le \mathrm{e}^{(k+3)T}\|u_n(0)\|^2 + \frac{M^2}{(k+3)} = c_1^2, \qquad \forall t \in [0, T]. \end{aligned} \tag{2.8}$$

Multiplying both sides of equation (2.5) by $\lambda_j u_{nj}(t)$, and summing up the products over $j = 1, 2, \dots, N$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n\|^2 + k\|\nabla\Delta u_n\|^2 - \|\Delta u_n\|^2 + (\nabla(u_n^3 - u_n), \nabla u_n) = -(Bw, \Delta u_n),$$

that is,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n\|^2 + k\|\nabla\Delta u_n\|^2 + (3u_n^2\nabla u_n, \nabla u_n) \\ &\quad = \|\nabla u_n\|^2 + \|\Delta u_n\|^2 - (Bw, \Delta u_n) \\ &\quad \le \|\nabla u_n\|^2 + \frac{3}{2}\|\Delta u_n\|^2 + \frac{1}{2}\|Bw\|^2 \\ &\quad \le \frac{k}{2}\|\nabla\Delta u_n\|^2 + \left(\frac{9}{8k} + 1\right)\|\nabla u_n\|^2 + \frac{1}{2}\|Bw\|^2. \end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n\|^2 + k\|\nabla\Delta u_n\|^2 \le \left(\frac{9}{4k}+2\right)\|\nabla u_n\|^2 + \|Bw\|^2, \tag{2.9}$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n\|^2 + k\|\nabla\Delta u_n\|^2 \le \left(\frac{9}{4k}+2\right)\|\nabla u_n\|^2 + M^2. \tag{2.10}$$

Using Gronwall's inequality, we derive that

$$\begin{aligned}\|\nabla u_n\|^2 &\le \mathrm{e}^{(\frac{9}{4k}+2)t}\|\nabla u_n(0)\|^2 + \frac{4kM^2}{9+8k}\\ &\le \mathrm{e}^{(\frac{9}{4k}+2)T}\|\nabla u_n(0)\|^2 + \frac{4kM^2}{9+8k} = c_2^2, \quad \forall t\in[0,T].\end{aligned} \tag{2.11}$$

Again multiplying both sides of equation (2.5) by $\lambda_j^2 u_{nj}(t)$, summing up the products over $j = 1, 2, \ldots, N$, and using Hölder's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_n\|^2 + k\|\Delta^2 u_n\|^2\\ &\quad= (\Delta u_n, \Delta^2 u_n) + (u_n^3, \Delta^2 u_n) - \|\Delta u_n\|^2 + (Bw, \Delta^2 u_n)\\ &\quad\le \frac{k}{4}\|\Delta^2 u_n\|^2 + 2\|\Delta u_n\|^2 + 3\|u_n\|_6^6 + 3\|Bw\|^2.\end{aligned}$$

Using Nirenberg's inequality, we deduce that

$$3\|u_n\|_6^6 \le 3(c'\|\Delta^2 u_n\|^{\frac{n}{12}}\|u_n\|^{1-\frac{n}{12}} + c''\|u_n\|)^6 \le \frac{k}{4}\|\Delta^2 u_n\|^2 + c_3.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_n\|^2 + k\|\Delta^2 u_n\|^2 \le 4\|\Delta u_n\|^2 + 2c_3 + 6\|Bw\|^2, \tag{2.12}$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_n\|^2 + k\|\Delta^2 u_n\|^2 \le 4\|\Delta u_n\|^2 + 2c_3 + 6M^2. \tag{2.13}$$

Using Gronwall's inequality, we derive that

$$\begin{aligned}\|\Delta u_n\|^2 &\le \mathrm{e}^{4t}\|\Delta u_n(0)\|^2 + \frac{c_3}{2} + \frac{3M^2}{2}\\ &\le \mathrm{e}^{4T}\|\Delta u_n(0)\|^2 + \frac{c_3}{2} + \frac{3M^2}{2} = c_4^2, \quad \forall t\in[0,T].\end{aligned} \tag{2.14}$$

By (2.8), (2.11), (2.14) and Sobolev's embedding theorem, we get

$$\|u_n\|_\infty \le c\|u_n\|_{H^2} \le c_5. \tag{2.15}$$

Adding (2.8), (2.11) and (2.14) gives

$$\|u_n\|^2_{L^2(0,T;V)} = \int_0^T (\|u_n\|^2 + \|\nabla u_n\|^2 + \|\Delta u_n\|^2)\mathrm{d}t \le c_6. \tag{2.16}$$

In addition, we prove a uniform $L^2(0, T; V^*)$ bound on a sequence $\{u_{n,t}\}$. Note that

$$\begin{aligned}(\Delta u_n, \eta) &= (u_n, \Delta\eta) \le \|u_n\|\|\Delta\eta\| \le \|u_n\|\|\eta\|_V,\\ (u_n^3 - u_n, \eta) &\le \|u_n^3 - u_n\|_\infty \|\eta\|_1 \le \|u_n^3 - u_n\|_\infty \|\eta\|_V,\\ (Bw, \eta) &\le \|Bw\|\|\eta\| \le \|Bw\|\|\eta\|_V.\end{aligned}$$

Therefore

$$\|u_{n,t}\|_{V^*} + k\|\Delta^2 u_n\|_{V^*} \le \|u_n\| + \|u_n^3 - u_n\|_\infty + \|Bw\| \le c_7.$$

Then, we immediately conclude

$$\|u_{n,t}\|^2_{L^2(0,T;V^*)} = \int_0^T \|u_{n,t}\|^2_{V^*} \mathrm{d}t \le c_7^2 T. \tag{2.17}$$

It then follows from (2.8), (2.11), (2.14), (2.16) and (2.17) that

(1) For every $t \in [0, T]$, the sequence $\{u_n\}_{n\in N}$ is bounded in $L^2(0, T; V)$, which is independent of the dimension of ansatz space N.
(2) For every $t \in [0, T]$, the sequence $\{u_{n,t}\}_{n\in N}$ is bounded in $L^2(0, T; V^*)$, which is independent of the dimension of ansatz space N.

Based on the above discussion, we obtain $u(x, t) \in W(0, T; V)$. It is easy to check that $W(0, T; V)$ is continuously embedded into $C(0, T; H)$ which denotes the space of continuous functions. We conclude convergence of a subsequences, again denoted by $\{u_n\}$ which is weak in $W(0, T; V)$, weak-star in $L^\infty(0, T; H)$ and strong in $L^2(0, T; H)$ to functions $u(x, t) \in W(0, T; V)$. Since the proof of uniqueness is easy, we omit it.

Then, the proof of Theorem 2.2 is completed. □

Now, we shall discuss the relation among the norm of weak solution and initial value and control item.

Theorem 2.3. *Suppose that $u_0 \in V, Bw \in L^2(0, T; H)$, then there exists positive constants C' and C'' such that*

$$\|u\|^2_{W(0,T;V)} \le C'\left(\|u_0\|^2_V + \|w\|^2_{L^2(Q_0)}\right) + C''. \tag{2.18}$$

Proof. Clearly, (2.18) means

$$\|u\|^2_{L^2(0,T;V)} + \|u_t\|^2_{L^2(0,T;V^*)} \le C'\left(\|u_0\|^2_V + \|Bw\|^2_{L^2(0,T;H)}\right) + C''. \tag{2.19}$$

Passing to the limit in (2.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + k\|\Delta u\|^2 \le \|Bw\|^2 + (k+3)\|u\|^2. \tag{2.20}$$

Using Gronwall's inequality, we obtain

$$\|u\|^2 \le \mathrm{e}^{(k+3)T}\|u_0\|^2 + \frac{1}{k+3}\|Bw\|^2, \quad \forall t \in [0, T]. \tag{2.21}$$

Thus

$$\begin{aligned}\|u\|^2_{L^2(0,T;H)} &\le T\mathrm{e}^{(k+3)T}\|u_0\|^2 + \frac{1}{k+3}\|Bw\|^2_{L^2(0,T;H)}\\ &\le c_8\|u_0\|^2 + c_9\|Bw\|^2_{L^2(0,T;H)}.\end{aligned} \tag{2.22}$$

Passing to the limit in (2.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + k\|\nabla\Delta u\|^2 \le \left(\frac{9}{4k}+2\right)\|\nabla u\|^2 + \|Bw\|^2. \tag{2.23}$$

Using Gronwall's inequality, we get

$$\|\nabla u\|^2 \le \mathrm{e}^{(\frac{9}{4k}+2)T}\|\nabla u_0\|^2 + \frac{4k}{9+8k}\|Bw\|^2. \tag{2.24}$$

Hence, we have

$$\begin{aligned}\|\nabla u\|^2_{L^2(0,T:H)} &\le T\mathrm{e}^{\left(\frac{9}{4k}+2\right)T}\|\nabla u_0\|^2 + \frac{4k}{9+8k}\|Bw\|^2_{L^2(0,T;H)}\\ &\le c_{10}\|\nabla u_0\|^2 + c_{11}\|Bw\|^2_{L^2(0,T;H)}.\end{aligned} \tag{2.25}$$

Passing to the limit in (2.12), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + k\|\Delta^2 u\|^2 \le 4\|\Delta u\|^2 + 2c_3 + 6\|Bw\|^2. \tag{2.26}$$

Using Gronwall's inequality, we derive that

$$\|\Delta u\|^2 \le \mathrm{e}^{4T}\|\Delta u_0\|^2 + \frac{c_3}{2} + \frac{3}{2}\|Bw\|^2. \tag{2.27}$$

Hence, we have

$$\begin{aligned}\|\Delta u\|^2_{L^2(0,T;H)} &\le T\mathrm{e}^{4T}\|\Delta u_0\|^2 + \frac{12}{c_3}T + \frac{3}{2}\|Bw\|^2_{L^2(0,T;H)}\\ &\le c_{12}\|\Delta u_0\|^2 + c_{13}\|Bw\|^2_{L^2(0,T;H)} + c_{14}.\end{aligned} \tag{2.28}$$

By (2.21), (2.24), (2.27) and Sobolev's embedding theorem, we conclude

$$\|u(x,t)\|_\infty \le c_{15}.$$

On the other hand, we have

$$\begin{aligned}(\Delta u,\eta) &= (u,\Delta\eta) \le \|u\|\|\Delta\eta\| \le \|u\|\|\eta\|_V,\\ (u^3-u,\eta) &\le \|u^3-u\|_\infty\|\eta\|_1 \le \|u^3-u\|_\infty\|\eta\|_V,\\ (Bw,\eta) &\le \|Bw\|\|\eta\| \le \|Bw\|\|\eta\|_V.\end{aligned}$$

Therefore

$$\|u_t\|_{V^*} + k\|\Delta^2 u\|_{V^*} \le \|u\| + \|u^3-u\|_\infty + \|Bw\| \le c_{16} + \|Bw\|.$$

Then, we immediately conclude that

$$\|u_t\|^2_{L^2(0,T;V^*)} \le c(\|Bw\|^2_{L^2(0,T;H)} + c_{16}^2 T). \tag{2.29}$$

By (2.22), (2.25), (2.28), (2.29) and the definition of extension operator B, we obtain (2.18). Hence, Theorem 2.3 is proved. □

3. Optimal control problem

In this section, based on Lions theory (see [11]), we consider the optimal control problem associated with the extended Fisher–Kolmogorov equation and prove the existence of optimal solution.

In the following, we suppose $L^2(Q_0)$ is a Hilbert space of control variables. We also suppose $B \in \mathcal{L}(L^2(Q_0), L^2(0,T;H))$ is the controller and a control $w \in L^2(Q_0)$. Consider the following control system:

$$\begin{cases} u_t + k\Delta^2 u - \Delta u + u^3 - u = Bw, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = \Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ u(0) = u_0, \quad x \in \Omega. \end{cases} \tag{3.1}$$

Here in (3.1), it is assumed that $u_0 \in V$. By virtue of Theorem 2.2, we can define the solution map $w \to u(w)$ of $L^2(Q_0)$ into $W(0,T;V)$. The solution u is called the state of the control system (3.1). The observation of the state is assumed to be given by Cu. Here $C \in \mathcal{L}(W(0,T;V), S)$ is an operator, which is called the observer, S is a real Hilbert space of observations. The cost function associated with the control system (3.1) is given by

$$J(u,w) = \frac{1}{2}\|Cu - z_d\|_S^2 + \frac{\delta}{2}\|w\|_{L^2(Q_0)}^2, \tag{3.2}$$

where $z_d \in S$ is a desired state and $\delta > 0$ is fixed. An optimal control problem about the extended Fisher–Kolmogorov equation is

$$\min J(u,w), \tag{3.3}$$

where (u,w) satisfies (3.1).

Let $X = W(0,T;V) \times L^2(Q_0)$ and $Y = L^2(0,T;V) \times H$. We define an operator $e = e(e_1, e_2) : X \to Y$, where

$$e_1 = (\Delta^2)^{-1}(u_t + k\Delta^2 u - \Delta u + u^3 - u - Bw)$$

and

$$e_2 = u(x,0) - u_0.$$

Here Δ^2 is an operator from V to V^*. Then, we write (3.3) in the following form:

$$\min J(u,w) \text{ subject to } e(u,w) = 0.$$

Theorem 3.1. *Suppose that $u_0 \in V$, $Bw \in L^2(0,T;H)$, then there exists an optimal control solution (u^*, w^*) to the problem* (3.1).

Proof. Suppose (u,w) satisfy the equation $e(u,w) = 0$. In view of (3.2), we deduce that

$$J(u,w) \geq \frac{\delta}{2}\|w\|_{L^2(Q_0)}^2.$$

By Theorem 2.3, we obtain

$$\|u\|_{W(0,T;V)} \to \infty \text{ yields } \|w\|_{L^2(Q_0)} \to \infty.$$

Therefore, we get

$$J(u, w) \to \infty, \text{ when } \|(u, w)\|_X \to \infty. \tag{3.4}$$

As the norm is weakly lower semi-continuous, we achieve that J is weakly lower semi-continuous. Since for all $(u, w) \in X$, $J(u, w) \geq 0$, there exists $\lambda \geq 0$ defined by

$$\lambda = \inf\{J(u, w) | (u, w) \in X, e(u, w) = 0\},$$

which means the existence of a minimizing sequence $\{(u^n, w^n)\}_{n\in N}$ in X such that

$$\lambda = \lim_{n\to\infty} J(u^n, w^n) \quad \text{and} \quad e(u^n, w^n) = 0, \quad \forall n \in N.$$

From (3.4), there exists an element $(u^*, w^*) \in X$ such that when $n \to \infty$,

$$u^n \to u^*, \text{ weakly}, \ u^* \in W(0, T; V), \tag{3.5}$$

$$w^n \to w^*, \text{ weakly}, \ w^* \in L^2(Q_0). \tag{3.6}$$

Since $u_n \in L^\infty(0, T; V)$, $u_{n,t} \in L^2(0, T; V^*)$ and we also have that $W(0, T; V)$ is continuously embedded into $L^2(0, T; L^\infty)$. Hence by Lemma 4 of [20], we have $u^n \to u^*$ strongly in $L^2(0, T; L^\infty)$ as $n \to \infty$, $u^n \to u^*$ strongly in $C(0, T; H)$ as $n \to \infty$. As the sequence $\{u^n\}_{n\in\mathbb{N}}$ converges weakly, then $\|u^n\|_{W(0,T;V)}$ is bounded. Based on the embedding theorem, $\|u^n\|_{L^2(0,T;L^\infty_{per})}$ is also bounded . Because $u^n \to u^*$ in $L^2(0, T; L^\infty)$ as $n \to \infty$, we know that $\|u^*\|_{L^2(0,T;L^\infty)}$ is bounded too.

It then follows from (3.5) that

$$\lim_{n\to\infty} \int_0^T (u_t^n(x, t) - u_t^*, \psi(t))_{V^*,V} \mathrm{d}t = 0, \quad \forall \psi \in L^2(0, T; V),$$

$$\lim_{n\to\infty} \int_0^T (u^n(x, t) - u^*, \psi(t))_{V^*,V} \mathrm{d}t = 0, \quad \forall \psi \in L^2(0, T; V)$$

and

$$\lim_{n\to\infty} \int_0^T (\Delta u^n(x, t) - \Delta u^*, \psi(t))_{V',V} \mathrm{d}t = 0, \quad \forall \psi \in L^2(0, T; V).$$

Using (3.6) again, we derive that

$$\left| \int_0^T \int_\Omega (Bw - Bw^*) \eta \mathrm{d}x \mathrm{d}t \right| \to 0, \quad n \to \infty, \quad \forall \eta \in L^2(0, T; H).$$

Again by (3.5), we deduce that

$$\begin{aligned}
&\left|\int_0^T\int_\Omega ((u^n)^3-(u^*)^3)\eta \mathrm{d}x\mathrm{d}t\right| \\
&\quad\le \left|\int_0^T\int_\Omega ((u^n)^2+u_nu^*+(u^*)^2)(u^n-u^*)\eta \mathrm{d}x\mathrm{d}t\right| \\
&\quad\le \left|\int_0^T \|(u^n)^2+u_nu^*+(u^*)^2\|_\infty \|u^n-u^*\|_H\|\eta\|_H \mathrm{d}t\right| \\
&\quad\le \|(u^n)^2+u_nu^*+(u^*)^2\|_{L^2(L^\infty)}\|u^n-u^*\|_{C(H)}\|\eta\|_{L^2(H)} \\
&\quad\to 0, \quad n\to\infty, \quad \forall \eta\in L^2(0,T;V).
\end{aligned}$$

Hence we have $u=u^*$ and therefore

$$J(u,w)\le \lim_{n\to\infty} J(u^n,\omega^n)=\lambda.$$

In view of the above discussions, we get

$$e_1(u^*,w^*)=0, \quad \forall n\in N.$$

Noticing that $u^*\in W(0,T;V)$, we derive that $u^*(0)\in H$. Since $u^n\to u^*$ weakly in $W(0,T;V)$, we can infer that $u^n(0)\to u^*(0)$ weakly when $n\to\infty$. Thus, we obtain

$$(u^n(0)-u^*(0),\eta)\to 0, n\to\infty, \quad \forall\eta\in H,$$

which means $e_2(u^*,w^*)=0$. Therefore, we obtain

$$e(u^*,w^*)=0, \quad \text{in } Y.$$

So, there exists an optimal solution (u^*,w^*) to the problem (3.1). Then, the proof of Theorem 3.1 is completed. □

4. Optimality conditions

It is well known that the optimality conditions for w is given by the variational inequality

$$J'(u,w)(v-w)\ge 0, \quad \text{for all } v\in L^2(Q_0), \tag{4.1}$$

where $J'(u,w)$ denotes the Gateaux derivative of $J(u,v)$ at $v=w$.

The following Lemma 4.1 is essential in deriving necessary optimality conditions.

Lemma 4.1. *The map $v\to u(v)$ of $L^2(Q_0)$ into $W(0,T;V)$ is weakly Gateaux differentiable at $v=w$ such that the Gateaux derivative of $u(v)$ at $v=w$ in the direction $v-w\in L^2(Q_0)$, say $z=\mathcal{D}u(w)(v-w)$, is a unique weak solution of the following problem*:

$$\begin{cases} z_t+k\Delta^2 z+\Delta z+3u^2z-z=B(v-w), (x,t)\in Q,\\ z(x,t)=\Delta z(x,t)=0, (x,t)\in\partial\Omega\times(0,T),\\ z(0)=0, \quad x\in\Omega. \end{cases} \tag{4.2}$$

Proof. Let $0 \le h \le 1$, u_h and u be the solutions of (3.1) corresponding to $w + h(v - w)$ and w, respectively. Then we prove the lemma in the following two steps:

Step 1. We prove $u_h \to u$ strongly in $C(0, T; H)$ as $h \to 0$. Let $q = u_h - u$, then

$$\begin{cases} \frac{dq}{dt} + k\Delta^2 q + \Delta q + (u_h^3 - u^3) - q = hB(v - w), 0 < t \le T, \\ q(x, t) = \Delta q(x, t) = 0, x \in \partial\Omega, \\ q(0) = 0, \quad x \in \Omega. \end{cases} \tag{4.3}$$

Using Theorem 2.3 and Sobolev's embedding theorem, we consider the problem in the 3D case. Then

$$\|u\|_\infty \le c, \|u_h\|_\infty \le c.$$

Taking the scalar product of (4.3) with q, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|q\|^2 + k\|\Delta q\|^2 \\ &\quad = -(\Delta q, q) - ((u_h^2 + u_h u + u^2)q, q) + \|q\|^2 + (hB(v - w), q) \\ &\quad \le \frac{k}{2}\|\Delta q\|^2 + \frac{1}{2k}\|q\|^2 + \|u_h^2 + u_h u + u^2\|_\infty \|q\|^2 + \|q\|^2 + h\|B(v - w)\|\|q\| \\ &\quad \le \frac{k}{2}\|\Delta q\|^2 + c_1\|q\|^2 + \frac{1}{2}h^2\|B(v - w)\|^2. \end{aligned}$$

Hence

$$\frac{d}{dt}\|q\|^2 + k\|\Delta q\|^2 \le 2c_1\|q\|^2 + h^2\|B(v - w)\|^2. \tag{4.4}$$

Using Gronwall's inequality, it is easy to see that $\|q\|^2 \to 0$ as $h \to 0$. Then, $u_h \to u$ strongly in $C(0, T; H)$ as $h \to 0$.

Step 2. We prove that $\frac{u_h - u}{h} \to z$ strongly in $W(0, T; V)$. Rewrite (4.3) in the following form:

$$\begin{cases} \frac{d}{dt}\left(\frac{u_h - u}{h}\right) + k\Delta^2\left(\frac{u_h - u}{h}\right) + \frac{u_h^3 - u^3}{h} - \frac{u_h - u}{h} = B(v - w), 0 < t \le T, \\ \frac{u_h - u}{h}(x, t) = \Delta\left(\frac{u_h - u}{h}\right)(x, t) = 0, (x, t) \in \partial\Omega \times (0, T), \\ \frac{u_h - u}{h}(0) = 0, \quad x \in \Omega. \end{cases}$$

We can easily verify that the above problem possesses a unique weak solution in $W(0, T; V)$. On the other hand, it is easy to check that the linear problem (4.2) possesses a unique weak solution $z \in W(0, T; V)$. Let $r = \frac{u_h - u}{h} - z$, thus r satisfies

$$\begin{cases} \frac{d}{dt}r + k\Delta^2 r + \Delta r + \left(\frac{u_h^3 - u^3}{h} - 3u^2 z\right) - r = 0, 0 < t \le T, \\ r(x, t) = \Delta r(x, t) = 0, (x, t) \in \partial\Omega \times (0, T), \\ r(0) = 0, \quad x \in \Omega. \end{cases} \tag{4.5}$$

Taking the scalar product of (4.5) with r, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|r\|^2 + k\|\Delta r\|^2 &= -(\Delta r, r) - \left(\frac{u_h^3 - u^3}{h} - 3u^2 z, r\right) + \|r\|^2 \\ &\le \frac{k}{2}\|\Delta r\|^2 - \left(\frac{u_h^3 - u^3}{h} - 3u^2 z, r\right) + \left(\frac{1}{2k} + 1\right)\|r\|^2.\end{aligned}$$

Note that

$$\begin{aligned}-\left(\frac{u_h^3 - u^3}{h} - 3u^2 z, r\right) &= -\left(3(u + \kappa(u_h - u))^2 \frac{u_h - u}{h} - 3u^2 z, r\right) \\ &\le \left\|3(u + \kappa(u_h - u))^2 \frac{u_h - u}{h} - 3u^2 z\right\| \|r\| \\ &\le \frac{1}{2}\|r\|^2 + \frac{1}{2}\left\|3(u + \kappa(u_h - u))^2 \frac{u_h - u}{h} - 3u^2 z\right\|^2,\end{aligned}$$

where $\kappa \in (0, 1)$. We have $u_h \to u$ strongly in $C(0, T; H)$ as $h \to 0$, then

$$\begin{aligned}&\left\|3(u + \kappa(u_h - u))^2 \frac{u_h - u}{h} - 3u^2 z\right\|^2 \\ &\quad \to \left\|3u^2\left(\frac{u_h - u}{h} - z\right)\right\|^2 \le c_2\|r\|^2 \text{ as } h \to 0.\end{aligned}$$

Therefore

$$-\left(\frac{u_h^3 - u^3}{h} - 3u^2 z, r\right) \le \frac{1}{2}\|r\|^2 + \frac{c_2}{2}\|r\|^2.$$

Summing up, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|r\|^2 + k\|\Delta r\|^2 \le \left(\frac{1}{k} + 3 + c_2\right)\|r\|^2.$$

Using Gronwall's inequality, it is easy to check that $\frac{u_h - u}{h}$ is strongly convergent to z in $W(0, T; V)$.

Then, Lemma 4.1 is proved. □

As in [11], we denote $\Lambda =$ canonical isomorphism of S onto S^*, where S^* is the dual spaces of S. By calculating the Gateaux derivative of (3.2) via Lemma 4.1, we see that the cost $J(v)$ is weakly Gateaux differentiable at w in the direction $v - w$. Therefore, (4.1) can be rewritten as

$$(C^*\Lambda(Cu(w) - z_d), z)_{W(V)^*, W(V)} + \frac{\delta}{2}(w, v - w)_{L^2(Q_0)} \ge 0, \quad \forall v \in L^2(Q_0), \tag{4.6}$$

where z is the solution of (4.2).

Now, we study the necessary conditions of optimality. To avoid the complexity of observation states, we consider two types of distributive and terminal value observations.

1. Case of $C \in \mathcal{L}(L^2(0,T;V);S)$. In this case, $C^* \in \mathcal{L}(S^*;L^2(0,T;V^*))$, and (4.6) may be written as

$$\int_0^T (C^*\Lambda(Cu(w)-z_d),z)_{V^*,V}\,\mathrm{d}t+\frac{\delta}{2}(w,v-w)_{L^2(Q_0)} \geq 0, \quad \forall v \in L^2(Q_0). \tag{4.7}$$

We introduce the adjoint state $p(v)$ by

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}t}p(v)+k\Delta^2 p(v)+\Delta p(v)+3[u(v)]^2 p(v)-p(v) \\ = C^*\Lambda(Cu(v)-z_d),\, (x,t) \in Q, \\ p(v) = \Delta p(v) = 0,\, x \in \partial\Omega, \\ p(x,T;v) = 0. \end{cases} \tag{4.8}$$

According to Theorem 2.2, the above problem admits a unique solution (after changing t into $T-t$).

Multiplying both sides of (4.8) (with $v = w$) by z, using Lemma 4.1, we get

$$\int_0^T \left(-\frac{\mathrm{d}}{\mathrm{d}t}p(w),z\right)_{V^*,V} \mathrm{d}t = \int_0^T \left(p(w),\frac{\mathrm{d}}{\mathrm{d}t}z\right)\mathrm{d}t,$$

$$\int_0^T \left(\Delta^2 p(w),z\right)_{V^*,V} \mathrm{d}t = \int_0^T (p(w),\Delta^2 z)\mathrm{d}t,$$

$$\int_0^T (\Delta p(w),z)_{V^*,V}\,\mathrm{d}t = \int_0^T (p(w),\Delta z)\mathrm{d}t$$

and

$$\int_0^T \left(3(u(w))^2 p(w),z\right)_{V^*,V} \mathrm{d}t = \int_0^T \left(p(w),3(u(w))^2 z)\right)\mathrm{d}t.$$

Thus, we obtain

$$\begin{aligned} &\int_0^T (C^*\Lambda(Cu(w)-z_d),z)_{V^*,V}\,\mathrm{d}t \\ &\quad = \int_0^T \left(p(w),\frac{\mathrm{d}}{\mathrm{d}t}z+k\Delta^2 z+2k\Delta z+3u^2 z-z\right)\mathrm{d}t \\ &\quad = \int_0^T (p(w),Bv-Bw)dt = (B^*p(w),v-w). \end{aligned}$$

Therefore, (4.7) may be written as

$$\int_0^T \int_\Omega B^*p(w)(v-w)\mathrm{d}x\mathrm{d}t+\frac{\delta}{2}(w,v-w)_{L^2(Q_0)} \geq 0, \quad \forall v \in L^2(Q_0). \tag{4.9}$$

Then, we have proved the following theorem:

Theorem 4.2. *Assume that $C \in \mathcal{L}(L^2(0,T;V);S)$ amd all conditions of Theorem* 3.1 *hold. Then, the optimal control w is characterized by the system of two PDEs and an inequality*: (3.1), (4.8) *and* (4.9).

2. Case of $C \in \mathcal{L}(H; S)$. In this case, we observe $Cu(v) = Du(T; v)$, $D \in \mathcal{L}(H; H)$. The associated cost function is expressed as

$$J(u, v) = \|Du(T; v) - z\|_S^2 + \frac{\delta}{2}\|v\|^2_{L^2(Q_0)}, \quad \forall v \in L^2(Q_0). \tag{4.10}$$

Then, $\forall v \in L^2(Q_0)$, the optimal control w for (4.10) is characterized by

$$(Du(T; w) - z, Du(T; v) - Du(T; w))_{V^*,V} + \frac{\delta}{2}(w, v - w)_{L^2(Q_0)} \geq 0. \tag{4.11}$$

We introduce the adjoint state $p(v)$ by

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}t} p(v) + k\Delta^2 p(v) + 2k\Delta p(v) + 3[u(v)]^2 p(v) - p(v) = 0, (x, t) \in Q, \\ p(v) = \Delta p(v) = 0, x \in \partial\Omega, \\ p(T; v) = D^*(Du(T; v) - z_d). \end{cases} \tag{4.12}$$

According to Theorem 2.2, the above problem admits a unique solution (after changing t into $T - t$).

Set $v = w$ in the above equations and scalar multiply both sides of the first equation of (4.12) by $u(v) - u(w)$ and integrate from 0 to T. A simple calculation shows that (4.11) is equivalent to

$$\int_0^T \int_\Omega B^* p(w))(v - w)\mathrm{d}x\mathrm{d}t + \frac{\delta}{2} w, v - w)_{L^2(Q_0)} \geq 0, \quad \forall v \in L^2(Q_0). \tag{4.13}$$

Thus, we obtain

Theorem 4.3. *Assume that $D \in \mathcal{L}(H; H)$ and all conditions of Theorem* 3.1 *hold. Then, the optimal control w is characterized by the system of two PDEs and an inequality*: (3.1), (4.12) *and* (4.13).

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