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Null controllability of the viscous Camassa–Holm equation with moving control

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Abstract. In this paper, we study the null controllability of the viscous Camassa–Holm equation on the one-dimensional torus. By using a moving distributed control, we obtain that the system is null controllable for a given data with certain regularity.

Keywords. Viscous Camassa–Holm equation; null controllability; moving control; moment problem.

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1. Introduction

Recently, there have been more and more studies on the one-dimensional viscous Camassa–Holm equation

$$y_t - y_{xxt} - \gamma(y_{xx} - y_{xxxx}) + 3yy_x = yy_{xxx} + 2y_x y_{xx},$$

which models the unidirectional propagation of shallow water waves over a flat bottom, $y(x, t)$ stands for the fluid velocity at time t in the spatial x direction.

This equation rose from the work on shallow water equation [1], which led to [7, 12], where the equation is derived by considering variational principles and Lagrangian averaging. In light of this derivation, the equation are sometimes called the Lagrangian averaged Navier–Stokes equation. In [4], the equation was derived as a filtered Navier–Stokes equation, which obeys a modified Kelvin circulation theorem along filtered velocities. In this setting this is sometimes referred to as the Navier–Stokes–α equation, where α is the parameter in the filter. Solutions to the viscous Camassa–Holm equation is closely related to the solution of the famous Navier–Stokes equation (NSE), but the filter allows bounds that are currently unobtainable for the NSE, making them in some ways better suited for computational turbulence study, see [8]. Various works for this equation has also been investigated in [4–6, 15, 16] and their references.

This paper is devoted to studying null controllability of viscous Camassa–Holm equation. For the sake of simplicity, we will take $\gamma = 1$ throughout this paper. All the results can be extended without difficulty to the arbitrary $\gamma > 0$.

Our goal in this paper is to prove that the equation

$$\begin{cases} y_t - y_{xxt} - y_{xx} + y_{xxxx} = b(x+ct)u(x,t), \\ y(x,0) = y_0(x) \end{cases}$$

with a moving distributed control is null controllable, where $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ is the space variable and u is a control function.

The concept of moving point control was introduced by Lions in [11] for the wave equation. One important motivation for this kind of control is that the exact controllability of the wave equation with a pointwise control and Dirichlet boundary conditions fails if the point is a zero of some eigenfunction of the Dirichlet Laplacian, while it holds when the point is moving under some (much more stable) conditions which are easy to check (see, e.g., [2]). The controllability of the wave equation (resp., of the heat equation) with a moving point control was investigated in [2, 9, 11] (resp., in [3, 10]). See also [17] for Maxwells equations.

Let $z(x,t) = y(x-ct,t)$, then z satisfies

$$\begin{cases} z_t + cz_x - z_{xxt} - cz_{xxx} - z_{xx} + z_{xxxx} = b(x)u(x-ct,t), \\ z(x,0) = y_0(x). \end{cases}$$

Throughout this paper, we assume that $c = -1$, while for any $c \neq 0$ it can be derived in the same way. Namely, we consider the following systems

$$\begin{cases} y_t - y_{xxt} - y_{xx} + y_{xxxx} = b(x-t)u(x,t), \\ y(x,0) = y_0(x) \end{cases} \tag{1.1}$$

and

$$\begin{cases} z_t - z_x - z_{xxt} + z_{xxx} - z_{xx} + z_{xxxx} = b(x)u(x+t,t), \\ z(x,0) = y_0(x). \end{cases} \tag{1.2}$$

The results are stated on the control problem (1.1) while the proofs are worked out on the control problem (1.2). The precise results are given in the following theorems.

Theorem 1.1. *Let $b \in L^2(\mathbb{T})$ be such that*

$$\rho_k = \int_{\mathbb{T}} b(x)\mathrm{e}^{-ikx}\mathrm{d}x \neq 0 \qquad \text{for } k \in \mathbb{Z}. \tag{1.3}$$

For any $T > 2\pi$ and any $y_0 \in L^2(\mathbb{T})$ decomposed as $y_0 = \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{ikx}$ with

$$\sum_{k\in\mathbb{Z}} |\rho_k|^{-1}|a_k| < +\infty, \tag{1.4}$$

there exists a control $u \in L^2(0,T)$ such that (1.1) *is null controllable.*

As a consequence of Theorem 1.1, we have the second result on the internal controllability problem.

Theorem 1.2. *Let $b = \chi_\omega$ with ω a nonempty open set of $\mathbb{T}$, where χ_ω is the characteristic function of ω. Then for any $T > 2\pi$ and any $y_0 \in H^s(\mathbb{T})$ with $s > \frac{5}{2}$, there exists a control $u \in L^2(\mathbb{T} \times (0,T))$ such that* (1.1) *is null controllable.*

The crucial point in this paper is the construction of the bi-orthogonal family, the method of proofs is inspired from [13]. First, we reduce the controllability problem to a moment problem. In order to solve this moment problem we should construct a biorthogonal family. It combines different results of complex analysis about entire functions of exponential type, sine-type functions and the Paley–Wiener theorem.

The rest of the paper is organized as follows. In §2, we introduce some useful results. A biorthogonal family is constructed in §3 and proofs of Theorems 1.1 and 1.2 are given in §4.

2. Preliminaries

In this section we give some useful results. Let us first consider the operator

$$Az := (I - \partial_x^2)^{-1}(z_x - z_{xxx} + z_{xx} - z_{xxxx}),$$

with $D(A) = H^4(\mathbb{T})$. The eigenvalues of A are obtained by solving

$$\lambda(z - z_{xx}) = z_x - z_{xxx} + z_{xx} - z_{xxxx}.$$

A simple calculation yields

$$\lambda_k = -k^2 + ik, \qquad \forall k \in \mathbb{Z}.$$

To each λ_k corresponds an eigenfunction e^{ikx}.

If $y_0 \in L^2(\mathbb{T})$ decomposed as $y_0 = \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{ikx}$, the solution z of

$$\begin{cases} z_t - z_x - z_{xxt} + z_{xxx} - z_{xx} + z_{xxxx} = 0 \\ z(x,0) = y_0(x) \end{cases} \tag{2.5}$$

can be written as

$$z(x,t) = \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\lambda_k t} \mathrm{e}^{ikx}.$$

We consider (2.5) in $H^s(\mathbb{T})(s \in \mathbb{R})$. The well-posedness of (2.5) reads as follows.

PROPOSITION 2.1

Let $s \in \mathbb{R}$. If $y_0 \in H^s(\mathbb{T})$, then the solution z of (2.5) *satisfies $z \in C([0,+\infty); H^s(\mathbb{T}))$.*

Proof. We use the same method as in Proposition 2.1 of [14]. If $y_0 \in H^s(\mathbb{T})$, we have $\sum_{k\in\mathbb{Z}} |a_k|^2 k^{2s} < +\infty$. It holds that

$$k^{2s}|a_k \mathrm{e}^{\lambda_k t}|^2 \le k^{2s}|a_k|^2 \mathrm{e}^{-2k^2 t} \le k^{2s}|a_k|^2 \mathrm{e}^{-2t}, \qquad \forall k \in \mathbb{Z},$$

then $z \in C([0,+\infty); H^s(\mathbb{T}))$. □

Inspired from [13], the strategy to prove Theorem 1.1 is connected to the method of moments. The idea is to construct a suitable family $\Phi_k(z)$ of entire functions of exponential type, satisfying

$$\Phi_k(i\lambda_l) = \delta_k^l, \qquad \forall k, l \in \mathbb{Z},$$

where δ_k^l is the Kronecker symbol.

Then using the Paley–Wiener theorem we deduce our biorthogonal family φ_k as the inverse Fourier transform of Φ_k (up to a translation in time). The family Φ_k is constructed from a single entire function having simple poles at $i\lambda_k$. This function is naturally constructed as a Weierstrass product (which turns out to be explicit here), multiplied by a function (which we will designate as a multiplier) intended to make Φ_k of relevant exponential type and with suitable behaviour on the real axis.

The following lemmas are useful when we construct Φ_k and apply the Paley–Wiener theorem.

Lemma 2.1 (*Corollary* 3.4 *of* [13]). *Assume that* $\mu_k = sgn(k)\sqrt{-\lambda_k} := k + d_k$ *where* $d_0 = 0$ *and* $d_k = d + O(k^{-1})$ *as* $|k| \to \infty$ *for some constant* $d \in \mathbb{C}$, *and that* $\mu_k \neq \mu_l$ *for* $k \neq l$. *Then* $f(z) = z\prod_{k\in\mathbb{Z}\setminus\{0\}}(1 - \frac{z}{\mu_k})$ *is an entire function of type sine.*

The definition and basic properties of the entire function of type sine can be found in §3.1 of [13].

In order to construct a multiplier m, we need the following lemma.

Lemma 2.2 (*Proposition* 3.9 *of* [13]). *There exist a function* $\tilde{U}(z)$ *and some positive constant* $C = C(a, b)$ *such that for any complex number* $z = x + iy$ *with* $y \neq 0$,

$$-C - b\pi\left(1 + \frac{1}{\sqrt{2}}\right)\sqrt{|y|} - \log^+\frac{|x|}{|y|} - \log^+\left(\frac{x^2 + y^2}{B^2}\right) - \log 2$$
$$\leq \tilde{U}(z) + b\pi\sqrt{|x|} - a\pi|y| \leq C + \log^+\frac{|x|}{|y|}.$$

Taking $a = \frac{T}{2\pi} - 1, b = 2\sqrt{2}$ and by the same method in Proposition 3.10 of [13], we have the following lemma.

Lemma 2.3. *There exists an entire function* m *on* $\mathbb{C}$ *of exponential type at most* $a\pi$, *such that the following estimates hold for some positive constant* C :

$$|m(x)| \leq C(1 + |x|)e^{-2\sqrt{2}\pi\sqrt{|x|}}, \qquad x \in \mathbb{R}, \tag{2.6}$$

$$|m(i\lambda_k)| \geq C^{-1}e^{a\pi k^2 - 4\sqrt{2}\pi|k|}, \qquad k \in \mathbb{Z}\setminus\{0\}. \tag{2.7}$$

Remark 2.1. In [13], the authors choose $b = \sqrt{2}$ in Lemma 2.2, while in this paper we choose $b = 2\sqrt{2}$. The reason is that we need $\Phi_k \in L^2(\mathbb{R})$ in the next section.

The following lemma is important to prove Theorem 1.2.

Lemma 2.4. *Let* $\sigma \in (0, 1)$ *be a quadratic irrational,* $\tilde{b}(x) = \chi_{[\varepsilon,\varepsilon+\sigma\pi]}(x)$, $\tilde{\beta}_k = \int_{\mathbb{T}}\tilde{b}(x)\mathrm{e}^{-ikx}\mathrm{d}x$. *Then there exists* $C > 0$ *such that for any* $k \in \mathbb{Z}\setminus\{0\}$,

$$|\tilde{\beta}_k| \geq \frac{C}{|k|^2}.$$

Proof. The proof is standard. However, we outline it for reader's convenience. It is easy to check that $|\tilde{\beta}_k| = \frac{2}{|k|}|\sin\frac{k\sigma}{2}\pi|$. It follows from Lemma 2.3 of [13] that

$$\left|\sin\frac{k\sigma}{2}\pi\right| \geq \frac{C}{|k|}.$$

Thus

$$|\tilde{\beta}_k| \geq \frac{C}{|k|^2}.$$

□

3. Construction of a biorthogonal family

This section is motivated by [13]. The key observation is to construct a suitable family $\{\varphi_k\}_{k\in\mathbb{Z}}$ of entire functions of exponential type, by which we can solve the null controllability problem after turning it into a moment problem.

PROPOSITION 3.1

Let $T > 2\pi$. There exists a family $\{\varphi_k\}_{k\in\mathbb{Z}}$ of functions in $L^2(-\frac{T}{2}, \frac{T}{2})$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi_k(t)\mathrm{e}^{\lambda_l t}\mathrm{d}t = \delta_k^l, \qquad k, l \in \mathbb{Z}.$$

Moreover,

$$\|\varphi_k\|_{L^2(-\frac{T}{2},\frac{T}{2})} \leq C\mathrm{e}^{4\sqrt{2}\pi|k|-a\pi k^2}, \qquad k \in \mathbb{Z}\backslash\{0\}, \tag{3.8}$$

where C denotes some positive constant.

Proof. The method of the proof is inspired from the one in [13]. Let

$$\begin{aligned}
P(z) &= z\prod_{k\in\mathbb{Z}\backslash\{0\}}\left(1-\frac{z}{i\lambda_k}\right),\\
P_1(z) &= z\prod_{k\in\mathbb{Z}\backslash\{0\}}\left(1+\frac{z}{i\lambda_k}\right),\\
P_2(z) &= z^2\prod_{k\in\mathbb{Z}\backslash\{0\}}\left(1+\frac{z^2}{i\lambda_k}\right),\\
P_3(z) &= z\prod_{k\in\mathbb{Z}\backslash\{0\}}\left(1-\frac{z}{\mu_k}\right),
\end{aligned}$$

where $\mu_k = sgn(k)\sqrt{-\lambda_k} := k + d_k$ with $d_k = -\frac{i}{2} + O(k^{-1})$. Note also that

$$\begin{aligned}
&P(z) = -P_1(-z),\\
&P_1(z) = iP_2(e^{-i\frac{\pi}{4}}\sqrt{z}),\\
&P_2(z) = -P_3(z)P_3(-z).
\end{aligned}$$

According to $\mu_k = k - \frac{i}{2} + O(k^{-1})$ with $\mu_k \neq \mu_l$ if $k \neq l$ and $\mu_0 = 0$, it follows from Lemma 2.1 that P_3 is an entire function of type sine. In particular, we have

$$|P_3(z)| \leq C\mathrm{e}^{\pi|z|}, \qquad z \in \mathbb{C}.$$

Therefore, for any $\varepsilon > 0$ and for some positive constants C_1, C_2, C_3, it holds that

$$C_1\mathrm{e}^{\pi|y|} \leq |P_3(x+iy)| \leq C_2\mathrm{e}^{\pi|y|}, \ \ \mathrm{dist}(x+iy, \{\mu_k\}) > \varepsilon, \tag{3.9}$$
$$|P_3'(\mu_k)| \geq C_3, \qquad k \in \mathbb{Z}. \tag{3.10}$$

Thus P_2 is an entire function of exponential type 2π and

$$\begin{aligned}
&|P_1(z)| \leq Ce^{2\pi\sqrt{|z|}}, \qquad z \in \mathbb{C},\\
&C_1^2 \exp(2\pi|\mathrm{Im}(\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{z})|) \leq |P_1(z)|\\
&\leq C_2^2 \exp(2\pi|\mathrm{Im}(\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{z})|), \mathrm{dist}(\pm\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{z}, \{\mu_k\}) > \varepsilon.
\end{aligned}$$

It is clear that

$$|P_1(x)| \leq C\mathrm{e}^{\sqrt{2}\pi\sqrt{|x|}}, \qquad x \in \mathbb{R}.$$

Thus,

$$|P(x)| \leq C\mathrm{e}^{\sqrt{2}\pi\sqrt{|x|}}, \qquad x \in \mathbb{R}. \tag{3.11}$$

Note that

$$P_1'(z) = \frac{\mathrm{e}^{i\frac{\pi}{4}}}{2\sqrt{z}}[P_3'(\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{z})P_3(-\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{z}) - P_3(\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{z})P_3'(-\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{z})].$$

Take $z = -i\lambda_k$, then $\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{z} = \sqrt{-\lambda_k} = sgn(k)\mu_k$, and

$$P_1'(-i\lambda_k) = \frac{1}{2\mu_k}P_3'(\mu_k)P_3(-\mu_k).$$

According to $|\mu_k + \mu_l| > \delta$ for $k \in \mathbb{Z}\backslash\{0\}$ and $l \in \mathbb{Z}$, it follows from (3.9) and (3.10) that $|P_3(-\mu_k)| \geq C$, $|P_3'(\mu_k)| \geq C > 0$. Thus we have for some constant $C > 0$,

$$|P_1'(-i\lambda_k)| \geq \frac{C}{|k|}, \qquad k \in \mathbb{Z}\backslash\{0\}.$$

Therefore,

$$|P'(i\lambda_k)| \geq \frac{C}{|k|}, \qquad k \in \mathbb{Z}\backslash\{0\}. \tag{3.12}$$

We are in a position to define the functions in a biorthogonal family. For any $k \in \mathbb{Z}$, we set

$$\Phi_k(z) = \frac{P(z)}{P'(i\lambda_k)(z - i\lambda_k)} \cdot \frac{m(z)}{m(i\lambda_k)}.$$

It is easy to see that Φ_k is an entire function of exponential type at most $\pi(1+a) = \frac{T}{2}$ and

$$\Phi_k(i\lambda_l) = \delta_k^l, \qquad \forall k, l \in \mathbb{Z}.$$

If $k \in \mathbb{Z}\backslash\{0\}$, it follows from (2.6), (2.7), (3.11) and (3.12) that

$$\begin{aligned}|\Phi_k(x)| &\leq C\frac{(1+|x|)\mathrm{e}^{-2\sqrt{2\pi}\sqrt{|x|}}.\mathrm{e}^{\sqrt{2\pi}\sqrt{|x|}}}{|x-i\lambda_k|}\cdot|k|\mathrm{e}^{4\sqrt{2\pi}|k|-a\pi k^2}\\ &\leq C|k|\mathrm{e}^{4\sqrt{2\pi}|k|-a\pi k^2}\frac{1}{|x+k|+k^2}.\end{aligned}$$

Thus $\Phi_k \in L^2(\mathbb{R})$ with

$$\|\Phi_k\|_{L^2(\mathbb{R})} \leq C\mathrm{e}^{4\sqrt{2\pi}|k|-a\pi k^2}. \tag{3.13}$$

We also know that Φ_0 belongs to $L^2(\mathbb{R})$.

Let φ_k denote the inverse Fourier transform of Φ_k. By the Paley–Wiener theorem, the functions φ_k belong to $L^2(\mathbb{R})$, and are supported in $[-\frac{T}{2}, \frac{T}{2}]$. It follows from the definition of inverse Fourier transform that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi_k(t)\mathrm{e}^{\lambda_l t}\mathrm{d}t = \Phi_k(i\lambda_l) = \delta_k^l.$$

And (3.8) follows from (3.13). □

4. Proofs of Theorems 1.1 and 1.2

4.1 *Proof of Theorem* 1.1

Let us begin with the following control problem:

$$\begin{cases} z_t - z_x - z_{xxt} + z_{xxx} - z_{xx} + z_{xxxx} = b(x)u(t) \\ z(x,0) = y_0(x). \end{cases} \tag{4.14}$$

The adjoint equation to (4.14) reads

$$-p_t + p_x - p_{xxx} + p_{xxt} - p_{xx} + p_{xxxx} = 0. \tag{4.15}$$

Multiplying (4.14) by p and integrating by parts, we obtain

$$\int_{\mathbb{T}} (zp - zp_{xx})|_0^T\mathrm{d}x = \int_0^T \int_{\mathbb{T}} bup\mathrm{d}x\mathrm{d}t. \tag{4.16}$$

If $\hat{z}(x,t) = \sum_{k\in\mathbb{Z}} a_k\mathrm{e}^{\lambda_k t}\mathrm{e}^{ikx}$ is the solution of (4.14) for $u \equiv 0$, then $p(x,t) = \hat{z}(2\pi - x, T-t)$ is the solution of (4.15). It follows from (4.16) that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} a_k(1+k^2)[(z(T),\mathrm{e}^{-ikx}) - \mathrm{e}^{\lambda_k T}(z(0),\mathrm{e}^{-ikx})]\\ &\quad= \sum_{k\in\mathbb{Z}} a_k\int_0^T \mathrm{e}^{\lambda_k(T-t)}u(t)\mathrm{d}t\int_{\mathbb{T}} b(x)\mathrm{e}^{-ikx}\mathrm{d}x.\end{aligned} \tag{4.17}$$

Since the identity (4.17) is true for each $(a_k) \in l^2$, it follows that

$$(1+k^2)[(z(T),\mathrm{e}^{-ikx}) - \mathrm{e}^{\lambda_k T}(z(0),\mathrm{e}^{-ikx})] = \rho_k\int_0^T \mathrm{e}^{\lambda_k(T-t)}u(t)\mathrm{d}t.$$

For $k \in \mathbb{Z}$, we set $\alpha_k = -\rho_k^{-1} e^{\lambda_k \frac{T}{2}} (1+k^2)(z(0), e^{-ikx})$ and introduce a function

$$\phi(t) = \sum_{k \in \mathbb{Z}} \alpha_k \varphi_k(t). \tag{4.18}$$

Now, we choose the control function

$$u(t) = \phi\left(\frac{T}{2} - t\right). \tag{4.19}$$

From (1.4) and (3.8), it is clear that $u \in L^2(0, T)$ with

$$\begin{aligned} \|u\|_{L^2(0,T)} &= \|\phi\|_{L^2\left(-\frac{T}{2}, \frac{T}{2}\right)} \\ &\leq C\left[|\alpha_0| \|\varphi_0\|_{L^2\left(-\frac{T}{2}, \frac{T}{2}\right)} + \sum_{k \in \mathbb{Z}\setminus\{0\}} |\rho_k|^{-1} |a_k| (1+k^2) e^{4\sqrt{2}\pi|k| - a\pi k^2} \right] \\ &\leq C\left[|\alpha_0| \|\varphi_0\|_{L^2\left(-\frac{T}{2}, \frac{T}{2}\right)} + \sum_{k \in \mathbb{Z}\setminus\{0\}} |\rho_k|^{-1} |a_k| \right] < +\infty. \end{aligned}$$

Then, from (1.3),(1.4),(4.18), (4.19) and Proposition 3.1, for any $k \in \mathbb{Z}$, we have

$$\begin{aligned} \rho_k \int_0^T e^{\lambda_k (T-t)} u(t) dt &= \rho_k e^{\lambda_k \frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{\lambda_k t} \phi(t) dt \\ &= \rho_k e^{\lambda_k \frac{T}{2}} \alpha_k \\ &= -(1+k^2) e^{\lambda_k T} (z(0), e^{-ikx}). \end{aligned}$$

Thus, $(z(T), e^{-ikx}) = 0 \ \forall\, k \in \mathbb{Z}$, which implies that $z(T) = 0$.

4.2 *Proof of Theorem* 1.2

Since ω is open and nonempty, it contains a small interval $[\varepsilon, \varepsilon + \sigma\pi]$ where $\sigma > 0$ is a quadratic irrational. Set

$$\tilde{b}(x) = \chi_{[\varepsilon, \varepsilon+\sigma\pi]}(x).$$

It follows from Lemma 2.4 that

$$|\tilde{\beta}_k| = \left| \int_{\mathbb{T}} \tilde{b}(x) e^{-ikx} dx \right| \geq \frac{C}{|k|^2}.$$

Since $y_0 \in H^s(\mathbb{T})$ with $s > \frac{5}{2}$, we have

$$\sum_{k \in \mathbb{Z}} |a_k|^2 |k|^{2s} < +\infty, \qquad s > \frac{5}{2}.$$

Thus

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|\tilde{\beta}_k|^{-1}|a_k| &= \sum_{k\in\mathbb{Z}\backslash\{0\}}|\tilde{\beta}_k|^{-1}|a_k| + |\tilde{\beta}_0|^{-1}|a_0|\\ &\le C\sum_{k\in\mathbb{Z}\backslash\{0\}}|a_k||k|^2 + |\tilde{\beta}_0|^{-1}|a_0|\\ &\le \left(\sum_{k\in\mathbb{Z}\backslash\{0\}}|k|^{-1-\nu}\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}\backslash\{0\}}|a_k|^2|k|^{5+\nu}\right)^{\frac{1}{2}} + |\tilde{\beta}_0|^{-1}|a_0|\\ &< +\infty,\end{aligned}$$

where $\nu = 2s - 5 > 0$. By Theorem 1.1 we can find a control $h \in L^2(0, T)$ steering y_0 to $y(x, T) = 0$. Set

$$u(x, t) = \chi_{[\varepsilon,\varepsilon+\sigma\pi]}(x - t)h(t), \quad t \in (0, T),$$

then it is easy to see that $\chi_\omega(x - t)u(x, t) = u(x, t)$. Thus $u \in L^2(\mathbb{T} \times (0, T))$ is the desired control function.

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