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Test elements of direct sums and free products of free Lie algebras

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Abstract. We give a characterization of test elements of a direct sum of free Lie algebras in terms of test elements of the factors. In addition, we construct certain types of test elements and we prove that in a free product of free Lie algebras, product of the homogeneous test elements of the factors is also a test element.

Keywords. Test elements; free product; free Lie algebra.

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1. Introduction

Let K be a field, $X = \{x_1, \ldots, x_n\}$ and F_n be the free Lie algebra generated by X over K. An element u of F_n is a *test element* if for any endomorphism ϕ of F_n it follows from $\phi(u) = u$ that ϕ is an automorphism of F_n.

Automorphisms and test elements of free algebras and free groups have been of interest for many years. Examples of such elements were given in [1–3, 9, 10, 12, 13]. Examples of test elements of free Lie algebras were obtained in [5–7].

In [8], O'Neill and Turner characterized test elements of a direct product of certain groups in terms of test elements of the factors. In this note we prove an analog of this result for direct sum of free Lie algebras of finite rank. We also give examples of test elements in direct sum of free Lie algebras.

Given an arbitrary element v of F_n, we define the rank of v to be the least number of free generators on which the image of v under an arbitrary automorphism of F_n can depend.

In [11], Shpilrain has defined the rank of an element u of F_n. He has given a simple algorithm for finding the rank of a homogeneous element of F_n. Moreover, he proved that homogeneous elements of maximal rank are test elements for automorphisms of F_n. Mikhalev and Zolotykh [4] proved an analog of Shpilrain's result for test elements for automorphisms in the class of all monomorphisms of a free Lie algebra.

From the results of Mikhalev *et al.* [7], it follows that the element $u = [x_1, x_2, \ldots, x_n]$ is a test element of F_n. Using Shpilrain's algorithm we give certain examples of test elements which are generalizations of the element u. Moreover we prove that the product of certain test elements in a free product of free Lie algebras of finite rank is also a test element.

For any free Lie algebra G, by $\gamma_m(G)$ we denote the m-th term of its lower central series; we usually write G' for $\gamma_2(G)$. By $[x_1, x_2^{k_2}, \ldots, x_n^{k_n}]$ we mean left normed commutator

$$[[\ldots[\ldots[\ldots[[x_1, \underbrace{x_2], x_2]\ldots, x_2]}_{k_2\text{-times}}, \ldots, \underbrace{x_n]\ldots, x_n], x_n]}_{k_n\text{-times}}$$

We denote by $L_1 \oplus \cdots \oplus L_n$ and $L_1 * \cdots * L_n$ the direct sum and the free product of any Lie algebras $L_1, \ldots, L_n$ respectively.

If an element u of F involves all generators $x_1, \ldots, x_n$, we write $u = u(x_1, \ldots, x_n)$ and we say that u depends on the generators $x_1, \ldots, x_n$.

2. Preliminaries

In this section, we introduce some more notation. Let F_n be the free Lie algebra generated by the set $X = \{x_1, \ldots, x_n\}$ over a field K. Let $U(F_n)$ be the universal enveloping algebra of F_n and $\triangle$ its augmentation ideal; that is, the kernel of the natural homomorphism $\varepsilon : U(F_n) \to U(F_n)$. Then there are mappings $\frac{\partial}{\partial x_i} : U(F_n) \to U(F_n)$ satisfying the following conditions whenever $a, b \in K$ and $u, v \in U(F_n)$:

(1) $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$,
(2) $\frac{\partial}{\partial x_i}(au + bv) = a\frac{\partial u}{\partial x_i} + b\frac{\partial v}{\partial x_i}$,
(3) $\frac{\partial}{\partial x_i}(uv) = u\frac{\partial v}{\partial x_i} + \frac{\partial u}{\partial x_i}\varepsilon(v)$.

It follows immediately that $\frac{\partial a}{\partial x_i} = 0$ for any $a \in K$. We need Shpilrain's theorem and algorithm for our purpose. Let $u = u(x_1, \ldots, x_n)$ be an element of F_n which depends on the generators $x_1, \ldots, x_n$.

Theorem 2.1 (11). *Let $u = u(x_1, \ldots, x_n) \in F_n$ be a homogeneous element of degree $m \geq 2$. Then the following statements are equivalent*:

(*a*) *the element u has rank $n \geq 2$*;
(*b*) *the image of u under an arbitrary linear automorphism of F_n depends on at least n free generators*;
(*c*) *if ϕ is an endomorphism of F_n and $u \in \gamma_m(\phi(F_n))$, then $X \subset \phi(F_n)$.*

COROLLARY 2.2 (11)

Let $u = u(x_1, \ldots, x_n) \in F_n$ be a homogeneous element of degree $m \geq 2$. Then u has rank n if and only if it has the following property: *whenever $u \in \gamma_m(\phi(F_n))$ for some endomorphism ϕ of F_n, one has $\phi \in \mathrm{Aut}\, F_n$.*

In order to proceed with our results, we need the Shpilrain's algorithm [11].

Algorithm 2.3. Let $u = u(x_1, \ldots, x_n) \in F_n$ be a homogeneous element of degree m. Take all possible Fox derivatives of weight $(m-1)$ of u; this yields a set of K-linear combinations of $x_1, \ldots, x_n$. If the K-span of this set equals to K-span of $\{x_1, \ldots, x_n\}$ we deduce that the rank of u is equal to n. If not, we can find the rank r of this K-span (in the usual sense of linear algebra) and a basis. After that we define a linear automorphism α

of L such that $\alpha(u)$ depends on r free generators. This number r is now equal to the rank of u (for details, see [11]).

We use the above algorithm to obtain a series of test elements in a free Lie algebra.

3. Test elements in direct sums of free Lie algebras

Let K be a field of characteristic zero. Suppose that $L_1, \ldots, L_m$ are free Lie algebras over K and $L = L_1 \oplus \cdots \oplus L_m$.

DEFINITION 3.1

For $i = 1, \ldots, m$, let $u_i \in L_i$. Then the set of elements $\{u_1, \ldots, u_m\}$ is an independent set if for $i \neq j$ there is no homomorphism $\psi : L_i \to L_j$ so that $\psi(u_i) = u_j$.

Theorem 3.2. *Let $u = u_1 + \cdots + u_m \in L$, where $u_i \in L_i,\ i = 1, \ldots, m$. Then u is a test element of L if and only if*
(i) u_i is a test element in L_i for each i, and
(ii) the subset $\{u_1, \ldots, u_n\}$ of L is an independent set.

Proof.

(i) Let $u = u_1 + \cdots + u_m$ be a test element of L. Assume that u_1 is not a test element. Then there is an endomorphism φ_1 of L_1 such that $\varphi_1(u_1) = u_1$ and it is not an automorphism. Now consider the homomorphism $\varphi : L \to L$ defined as

$$\varphi(u) = \varphi_1(u_1) + u_2 + \cdots + u_m.$$

Clearly $\varphi(u) = u$ and kernel of φ contains kernel of φ_1, that is, φ is not an automorphism of L. This contradiction shows that u_i is a test element for each $i = 1, \ldots, m$.
(ii) Assume that the set $\{u_1, \ldots, u_m\}$ is dependent in L. Then for some $i \neq j$ there is a homomorphism $\psi : L_i \to L_j$ such that $\psi(u_i) = u_j$. Now consider the endomorphism φ of L defined by

$$\varphi(x) = x_1 + \cdots + x_{j-1} + \psi(x_i) + x_{j+1} + \cdots + x_m,$$

where $x \in L,\ x_k \in L_k,\ k = 1, \ldots, m$. Then $\varphi(u) = u$ and L_i is contained in ker φ. So u is not a test element.

Suppose now that the conditions (i) and (ii) are valid and that φ is an endomorphism of L such that $\varphi(u) = u$.

Let $\theta_j : L_j \to L$ and $\pi_i : L \to L_i$ be the natural embedding and canonical projection and define $\psi_{ij} = \pi_i \varphi \theta_j$. We decompose φ as

$$\varphi(u) = \sum_{i=1}^{m} \varphi_i(u), \qquad \text{where } \varphi_i(u) = \sum_{j=1}^{m} \psi_{ij}(u_j).$$

We claim that
(a) $\psi_{ij} = 0$, for $i \neq j$ and
(b) ψ_{ii} is an automorphism.

Now we fix i.

(a) Since $\varphi(u) = u$, there are some $k_1, \dots, k_l$ so that $\sum_{r=1}^{l} \psi_{ik_r} \neq 0$. Then

$$u_i = \varphi_i(u) = \sum_{r=1}^{l} \psi_{ik_r}(u_{k_r}). \tag{1}$$

Using (1), for any $k \in \{k_1, \dots, k_l\}$ we get

$$0 = \varphi_i([u, u_k]) = [u_i, \psi_{ik}(u_k)].$$

If $\psi_{ik}(u_k) \neq 0$ then $u_i = \alpha\psi_{ik}(u_k)$, where $0 \neq \alpha \in K$. Let $\gamma_{ik} = \alpha\psi_{ik}$. Then $\gamma_{ik}(u_k) = u_i$. Hence $k = i$ by (ii). This is claim (a).

(b) If $\psi_{ir} = 0$ for $r \neq k$, then by (1) we have

$$\psi_{ik}(u_k) = u_i.$$

Therefore $k = i$ by (ii) and ψ_{ii} is an automorphism by (i). Thus claims (a) and (b) hold for (i).

If $w = w_1 + \cdots + w_m \in \ker\varphi$, then

$$\varphi(w) = \sum_{i=1}^{m} \psi_{ii}(w_i) = 0.$$

Since all ψ_{ii} are automorphisms, we get $w = 0$. So φ is injective. Surjectivity of φ is clear. Hence φ is an automorphism and u is a test element of L. □

In the free Lie algebra F_n, we define

$$u_n = u_{nk_2\dots k_n} = [x_1, x_2^{k_2}, \dots, x_n^{k_n}], \quad k_i \geq 1, \ 2 \leq i \leq n, \ n \geq 2.$$

Lemma 3.3. *For any $n \geq 2$ and $k_i \geq 1$, the element u_n of F_n is a test element.*

Proof. We prove by induction on $n \geq 2$ that the element $u_{nk_2\dots k_n}$ has rank n. Let $n = 2$. Take all possible Fox derivatives of weight k_2 of $u_{2k_2} = [x_1, x_2^{k_2}]$. By straigtforward calculations, we see that

$$\frac{\partial^{k_2} u_{2k_2}}{\partial x_2^{k_2}} = x_1, \qquad \frac{\partial^{k_2} u_{2k_2}}{\partial x_1 \partial x_2^{k_2-1}} = (-1)^{k_2} x_2.$$

Now consider K-span $\mathrm{Sp}\{x_1, (-1)^{k_2}x_2\}$ of the set $\{x_1, (-1)^{k_2}x_2\}$. Clearly the rank of this K-span is 2. Thus u_{2k_2} has maximal rank and it is a test element of F_2, by Theorem 2.1.

Now let $n \geq 3$ and let $k = k_2 + \cdots + k_n$. Applying all possible derivatives of weight k to $u_{nk_2\dots k_n}$ and using induction hypothesis, we finally arrive at

$$\frac{\partial^k u_n}{\partial x_2^{k_2} \dots \partial x_n^{k_n}} = x_1,$$

$$\frac{\partial^k u_n}{\partial x_j^{k_j-1} \partial x_1 \partial x_2^{k_2} \dots \partial x_{j-1}^{k_{j-1}} \partial x_{j+1}^{k_{j+1}} \dots \partial x_n^{k_n}} = (-1)^{k_j} x_j, \qquad 2 \leq j \leq n.$$

So the set $\{x_1, (-1)^{k_j} x_j : 2 \le j \le n\}$ is a basis of the K-span $\mathrm{Sp}\{x_1, (-1)^{k_j} x_j : 2 \le j \le \mathrm{n}\}$. Hence rank of $u_{nk_2\ldots k_n}$ is n by Algorithm 2.3 and $u_{nk_2\ldots k_n}$ is a test element of F_n by Corollary 2.2. □

We now suppose that for $i = 1, \ldots, m$ the algebras L_i are free Lie algebras generated by the sets $X_i = \{x_{i1}, \ldots, x_{ir_i}\}$ over the field K, where $r_i \ge 1$, $X_i \cap X_j = \emptyset$ for $i \ne j$. Define

$$u_{r_i} = u_{r_i k_2 \ldots k_{r_i}} = \left[x_{i1}, x_{i2}^{k_{i2}}, \ldots, x_{ir_i}^{k_{ir_i}}\right],$$

where $i = 1, \ldots, m,\ r_i \ge 1,\ k_{is} \ge 1$.

We obtain an obvious application of Theorem 2.1.

COROLLARY 3.4

Let $L = L_1 \oplus \cdots \oplus L_m$. *Assume that there is no nontrivial relation between the elements of the free generating sets* X_i *and* X_j, *where* $1 \le i, j \le m,\ i \ne j$. *Then* $u = u_{r_1} + \cdots + u_{r_m}$ *is a test element of* L, *where* $r_i \ge 2,\ i = 1, \ldots, m$.

Proof. The elements $u_{r_1}, \ldots, u_{r_m}$ are test elements of the free Lie algebras $L_1, \ldots, L_m$ respectively, by Lemma 3.3 and these elements are independent, by asssumption. Hence the result follows. □

4. Test elements in free products of free Lie algebras

Let L_i be the free Lie algebra generated by the set $X_i = \{x_{i1}, \ldots, x_{ir_i}\}$ over a field K of characteristic zero, where $i = 1, \ldots, m$. In this section, we give a series of test elements in free products of free Lie algebras.

Lemma 4.1. *Let* $G = L_1 * \cdots * L_m,\ m \ge 2$. *If* $g_i = g_i(x_{i1}, \ldots, x_{ir_i})$ *is a homogeneous test element of* L_i *for each* i, *then* $g = [g_1, g_2, \ldots, g_m]$ *is a test element of* G.

Proof. Let α be an arbitrary linear automorphism of G. Since each g_i is a test element of L_i, its rank is r_i, by Theorem 2.1 and $\alpha(g_i)$ depends on at least r_i free generators of G. Thus the element

$$\alpha(g) = [\alpha(g_1), \alpha(g_2), \ldots, \alpha(g_m)]$$

of G depends on at least $r = r_1 + \cdots + r_m$ free generators. This shows that $\alpha(g)$ depends on just r free generators. Let φ be an endomorphism of G such that $\varphi(g) = g$. Then

$$\begin{aligned} g &= \varphi(g) \\ &= \left[\varphi(g_1), \varphi(g_2), \ldots, \varphi(g_m)\right] \\ &= \left[g_1\left(\varphi(x_{11}), \ldots, \varphi(x_{1r_1})\right),\ \ldots\ , g_m\left(\varphi(x_{m1}), \ldots, \varphi(x_{mr_m})\right)\right]. \end{aligned}$$

Thus $g \in \gamma_r(\varphi(L))$ and $\cup_{i=1}^m X_i \subset \varphi(G)$ by Theorem 2.1. This completes the proof. □

COROLLARY 4.2

Let $G = L_1 * \cdots * L_m$. *Then the element* $w = [u_{r_1}, u_{r_2}, \ldots, u_{r_m}]$ *of* G *is a test element.*

Proof. If for all $i = 1, \ldots, n$, $r_i = 1$ then L is a free Lie algebra generated by the set $\{x_{11}, x_{21}, \ldots, x_{n1}\}$ and $w = [x_{11}, x_{21}, \ldots, x_{n1}]$ is a test element of L, by Lemma 3.3. If for at least one i, $r_i > 1$ then in view of Lemma 4.1, w is a test element of L. □

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