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Nehari manifold for non-local elliptic operator with concave–convex nonlinearities and sign-changing weight functions

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Abstract. In this article, we study the existence and multiplicity of non-negative solutions of the following p-fractional equation:

$$\begin{cases} -2\displaystyle\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{n+p\alpha}} \mathrm{d}y = \lambda h(x)|u|^{q-1}u + b(x)|u|^{r-1}u \text{ in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \quad u \in W^{\alpha,p}(\mathbb{R}^n) \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with continuous boundary, $p \geq 2$, $n > p\alpha$, $\alpha \in (0,1)$, $0 < q < p-1 < r < p^*-1$ with $p^* = np(n-p\alpha)^{-1}$, $\lambda > 0$ and h, b are sign-changing continuous functions. We show the existence and multiplicity of solutions by minimization on the suitable subset of Nehari manifold using the fibering maps. We find that there exists λ_0 such that for $\lambda \in (0, \lambda_0)$, it has at least two non-negative solutions.

Keywords. Non-local operator; p-fractional Laplacian; sign-changing weight functions; Nehari manifold; fibering maps.

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1. Introduction

We consider the following p-fractional Laplace equation

$$\begin{cases} -2\displaystyle\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{n+p\alpha}} \mathrm{d}y = \lambda h(x)|u|^{q-1}u + b(x)|u|^{r-1}u \quad \text{in } \Omega, \\ u \geq 0 \quad \text{in } \Omega, \quad u \in W^{\alpha,p}(\mathbb{R}^n), \\ u = 0 \qquad \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with continuous boundary, $p \geq 2$, $n > p\alpha$, $0 < q < p-1 < r < p^*-1$ with $p^* = np(n-p\alpha)^{-1}$, $\lambda > 0$, h and b are sign-changing continuous functions.

Recently a lot of attention has been given to the study of fractional and non-local operators of elliptic type due to concrete real world applications in finance, thin obstacle problem, optimization, quasi-geostrophic flow etc. The Dirichlet boundary value problem in the case of fractional Laplacian with polynomial type nonlinearity using variational methods has been studied in [7, 13–17, 24] recently. Also, existence and multiplicity results for non-local operators with convex–concave type nonlinearity are shown in [18]. In case of square root of Laplacian, existence and multiplicity results for sublinear and superlinear type of nonlinearity with sign-changing weight functions are studied in [24]. In [24], the author used the idea of Caffarelli and Silvestre [8], which gives a formulation of the fractional Laplacian through Dirichlet–Neumann maps. Recently, eigenvalue problem related to p-Laplacian has been studied in [11, 12].

In particular, for $\alpha = 1$, a lot of work has been done for multiplicity of positive solutions of semilinear elliptic problems with positive nonlinearities [1–3, 19]. Moreover multiplicity results with polynomial type nonlinearity with sign-changing weight functions using Nehari manifold and fibering map analysis is also studied in many papers (see [4–6, 10, 19–23]. In this work, we use fibering map analysis and Nehari manifold approach to solve the problem (1.1). Our work is motivated by the works of Servadei and Valdinoci [13] and by Brown and Wu [6].

The aim of this paper is to study the existence and multiplicity of non-negative solutions for the following equation driven by non-local operator $\mathcal{L}_K$ with convex–concave type nonlinearities

$$\left.\begin{aligned} -\mathcal{L}_K u &= \lambda h(x)|u|^{q-1}u + b(x)|u|^{r-1}u \text{ in } \Omega,\\ u &= 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{aligned}\right\} \tag{1.1}$$

The non-local operator $\mathcal{L}_K$ is defined as

$$\mathcal{L}_K u(x) = 2\int_{\mathbb{R}^n} |u(y)-u(x)|^{p-2}(u(y)-u(x))K(x-y)\mathrm{d}y \quad \text{for all } x \in \mathbb{R}^n,$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ satisfying:

(i) $mK \in L^1(\mathbb{R}^n)$, where $m(x) = \min\{1, |x|^p\}$,
(ii) there exist $\theta > 0$ and $\alpha \in (0, 1)$ such that $K(x) \geq \theta |x|^{-(n+p\alpha)}$,

More precisely, we study the problem to find $u \in W^{\alpha,p}(\mathbb{R}^n)$ such that for every $v \in W^{\alpha,p}(\mathbb{R}^n)$,

$$\begin{aligned} &\int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))K(x - y)\mathrm{d}x\mathrm{d}y \\ &\quad = \lambda \int_\Omega h(x)|u|^{q-1}uv\mathrm{d}x + \int_\Omega b(x)|u|^{r-1}uv\mathrm{d}x \end{aligned}$$

holds. Now we first introduce a space and some notations, then we have the existence result. Define the space

$$\begin{aligned} X_0 = \{u|\ &u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^p(\Omega), \\ &(u(x) - u(y))\sqrt[p]{K(x - y)} \in L^p(Q), u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}, \end{aligned}$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. In the next section, we study the properties of X_0 in detail.

$$B^{\pm} := \left\{ u \in X_0 : \int_\Omega b(x)|u|^{r+1} \mathrm{d}x \gtrless 0 \right\}, \; B_0 := \left\{ u \in X_0 : \int_\Omega b(x)|u|^{r+1} \mathrm{d}x = 0 \right\},$$

$$H^{\pm} := \left\{ u \in X_0 : \int_\Omega h(x)|u|^{q+1} \mathrm{d}x \gtrless 0 \right\}, \; H_0 := \left\{ u \in X_0 : \int_\Omega h(x)|u|^{q+1} \mathrm{d}x = 0 \right\},$$

and $H_0^{\pm} := H^{\pm} \cup H_0$, $B_0^{\pm} := B^{\pm} \cup B_0$.

Theorem 1.1. *There exists* $\lambda_0 > 0$ *such that for* $\lambda \in (0, \lambda_0)$, (1.1) *admits at least two non-negative solutions.*

Here λ_0 is the maximum of λ such that for $0 < \lambda < \lambda_0$, the fibering map $t \mapsto J_\lambda(tu)$ has exactly two critical points for each $u \in B^+ \cap H^+$.

The paper is organized as follows: In §1, we study the properties of the space X_0. In §2, we introduce Nehari manifold and study the behavior of Nehari manifold by carefully analysing the associated fibering maps. Section 3 contains the existence of non-trivial solutions.

2. Functional analytic settings

In this section, we first define the function space and prove some properties which are useful to find the solution of problem (1.1). For this, we define $W^{\alpha,p}(\Omega)$, the usual fractional Sobolev space $W^{\alpha,p}(\Omega) := \left\{ u \in L^p(\Omega); \frac{(u(x)-u(y))}{|x-y|^{\frac{n}{p}+\alpha}} \in L^p(\Omega \times \Omega) \right\}$ endowed with the norm

$$\|u\|_{W^{\alpha,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p\alpha}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}. \tag{2.1}$$

To study fractional Sobolev space in detail, we refer to [9].

Due to the non-localness of the operator $\mathcal{L}_K$, we define the linear space as follows:

$$X = \{u \mid u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^p(\Omega) \text{ and } (u(x) - u(y)) \sqrt[p]{K(x-y)} \in L^p(Q)\},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. In case of $p = 2$, the space X was firstly introduced by Servadei and Valdinoci [13]. The space X is a normed linear space endowed with the norm $\|\cdot\|_X$ defined as

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_Q |u(x) - u(y)|^p K(x-y) \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}. \tag{2.2}$$

It is easy to check that $\|.\|_X$ is a norm on X. For this, we first show that if $\|u\|_X = 0$, then $u = 0$ a.e. in $\mathbb{R}^n$. Indeed, if $\|u\|_X = 0$, then $\|u\|_{L^p(\Omega)} = 0$ which implies that

$$u = 0 \qquad \text{a.e. in } \Omega \tag{2.3}$$

and $\int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y = 0$. Thus $u(x) = u(y)$ a.e in Q means u is a constant in Q. Hence by (2.3), we have $u = 0$ a.e. in $\mathbb{R}^n$. Also triangle inequality follows from the inequality $|a+b|^p \leq |a+b|^{p-1}|a| + |a+b|^{p-1}|b|\ \forall\ a, b \in \mathbb{R},\ p \geq 1$ and Hölders inequality. Moreover, other properties of norms are obvious. □

Now we define

$$X_0 = \{u \in X : u = 0 \qquad \text{a.e. in } \mathbb{R}^n \setminus \Omega\}$$

with the norm

$$\|u\|_{X_0} = \left(\int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} \tag{2.4}$$

which is a reflexive Banach space. We note that, if $K(x) = |x|^{n+p\alpha}$, then norms in (2.1) and (2.2) are not the same, because $\Omega \times \Omega$ is strictly contained in Q.

Observation. The spaces X and X_0 are non-empty as $C_c^2(\Omega) \subseteq X_0$. Such type of spaces were introduced for $p = 2$ by Servadei and Valdinoci [13]. For a proof of this, we consider

$$\begin{aligned}\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y &= \left(\int_{\Omega\times\Omega} + 2\int_{\Omega\times\Omega^c}\right) |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \\ &\leq 2\int_{\Omega\times\mathbb{R}^n} |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

As $u \in C_c^2$, we have

$$|u(x) - u(y)| \leq 2\|u\|_{L^\infty(\mathbb{R}^n)}, \quad |u(x) - u(y)| \leq \|\nabla u\|_{L^\infty(\mathbb{R}^n)}|x-y|.$$

Thus

$$|u(x) - u(y)| \leq 2\|u\|_{C^1(\mathbb{R}^n)} \min\{1, |x-y|\}.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y &\leq 2^{p+1}\|u\|^p_{C^1(\mathbb{R}^n)} \int_{\Omega\times\mathbb{R}^n} m(x-y)K(x-y)\mathrm{d}x\mathrm{d}y \\ &\leq 2^{p+1}\|u\|^p_{C^1(\mathbb{R}^n)}|\Omega| \int_{\mathbb{R}^n} m(z)K(z)\mathrm{d}z < \infty,\end{aligned}$$

as required. □

Now we prove some properties of the spaces X and X_0. Proofs of these are easy to extend as in [13] but for completeness, we give the proof.

Lemma 2.1. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ be a function satisfying (ii). Then*

(1) *if $u \in X$, then $u \in W^{\alpha,p}(\Omega)$ and moreover $\|u\|_{W^{\alpha,p}(\Omega)} \leq c(\theta)\|u\|_X$;*
(2) *if $u \in X_0$, then $u \in W^{\alpha,p}(\mathbb{R}^n)$ and moreover $\|u\|_{W^{\alpha,p}(\Omega)} \leq \|u\|_{W^{\alpha,p}(\mathbb{R}^n)} \leq c(\theta)\|u\|_X$.*

In both the cases, $c(\theta) = \max\{1, \theta^{-1/p}\}$, where θ is given as in (ii).

Proof.

(1) Let $u \in X$, then by (ii) we have

$$\begin{aligned}\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\mathrm{d}x\mathrm{d}y &\leq \frac{1}{\theta}\int_{\Omega\times\Omega} |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \\ &\leq \frac{1}{\theta}\int_{Q} |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y < \infty.\end{aligned}$$

Thus

$$\|u\|_{W^{\alpha,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} \leq c(\theta)\|u\|_X.$$

(2) Let $u \in X_0$ then $u = 0$ on $\mathbb{R}^n \setminus \Omega$. So $\|u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\Omega)}$. Hence

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\mathrm{d}x\mathrm{d}y &= \int_{Q} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\mathrm{d}x\mathrm{d}y \\ &\leq \frac{1}{\theta}\int_{Q} |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y < +\infty,\end{aligned}$$

as required. □

Lemma 2.2. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ be a function satisfying (ii). Then there exists a positive constant c depending on n and α such that for every $u \in X_0$, we have*

$$\|u\|^p_{L^{p^*}(\Omega)} = \|u\|^p_{L^{p^*}(\mathbb{R}^n)} \leq c\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}dxdy,$$

where $p^* = np(n-p\alpha)^{-1}$ is the fractional critical Sobolev exponent.

Proof. Let $u \in X_0$, then by Lemma 2.1, $u \in W^{\alpha,p}(\mathbb{R}^n)$. Also we know that $W^{\alpha,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ (see [9]). Then we have

$$\|u\|^p_{L^{p^*}(\Omega)} = \|u\|^p_{L^{p^*}(\mathbb{R}^n)} \leq c\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}}\mathrm{d}x\mathrm{d}y$$

and hence the result. □

Lemma 2.3. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ be a function satisfying (ii). Then there exists some $C > 1$, depending only on n, α, p, θ and Ω such that for any $u \in X_0$,*

$$\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \leq \|u\|^p_X \leq C\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y$$

i.e.

$$\|u\|^p_{X_0} = \int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \tag{2.5}$$

is a norm on X_0 and is equivalent to the norm on X.

Proof. Clearly $\|u\|_X^p \geq \int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y$. Now by Lemma 2.2 and (ii), we get

$$\begin{aligned}
\|u\|_X^p &= \left(\|u\|_{L^p(\Omega)} + \left(\int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \right)^{1/p} \right)^p \\
&\leq 2^{p-1}\|u\|^p_{L^p(\Omega)} + 2^{p-1} \int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \\
&\leq 2^{p-1}|\Omega|^{1-\frac{p}{p^*}} \|u\|^p_{L^{p^*}(\Omega)} + 2^{p-1} \int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \\
&\leq 2^{p-1}\, c|\Omega|^{1-\frac{p}{p^*}} \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\alpha}} \mathrm{d}x\mathrm{d}y + 2^{p-1} \int_Q |u(x) \\
&\quad -u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \\
&\leq 2^{p-1} \left(\frac{c|\Omega|^{1-\frac{p}{p^*}}}{\theta} + 1 \right) \int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \\
&= C \int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where $C > 1$ as required. Now we show that (2.5) is a norm on X_0. For this we need only to show that if $\|u\|_{X_0} = 0$, then $u = 0$ a.e. in $\mathbb{R}^n$ as other properties of norm are obvious. Indeed, if $\|u\|_{X_0} = 0$, then $\int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y = 0$ which implies that $u(x) = u(y)$ a.e in Q. Therefore, u is a constant in Q and hence $u = c \in \mathbb{R}$ a.e in $\mathbb{R}^n$. Also by definition of X_0, we have $u = 0$ on $\mathbb{R}^n \setminus \Omega$. Thus $u = 0$ a.e. in $\mathbb{R}^n$. □

Lemma 2.4. *The space* $(X_0, \|.\|_{X_0})$ *is a reflexive Banach space.*

Proof. Let $\{u_k\}$ be a Cauchy sequence in X_0. Then by Lemma 2.3, (ii) and Lemma 2.2, $\{u_k\}$ is a Cauchy sequence in $L^p(\Omega)$ and so $\{u_k\}$ has a convergent subsequence. Thus we assume that $u_k \to u$ strongly in $L^p(\Omega)$. Since $u_k = 0$ in $\mathbb{R}^n \setminus \Omega$, we define $u = 0$ in $\mathbb{R}^n \setminus \Omega$. Then $u_k \to u$ strongly in $L^p(\mathbb{R}^n)$ as $k \to \infty$. So, there exists a subsequence denoted by u_k such that $u_k \to u$ a.e. in $\mathbb{R}^n$. Therefore, one can easily show by Fatou's lemma and using the fact that u_k is a Cauchy sequence that $u \in X_0$. Moreover, using the same fact one can verify that $\|u_k - u\|_{X_0} \to 0$ as $k \to \infty$. Hence X_0 is a Banach space. Reflexivity of X_0 follows from the fact that X_0 is a closed subspace of reflexive Banach space $W^{\alpha,p}(\mathbb{R}^n)$. □

Thus we have

$$X_0 = \{u \in X : u = 0 \qquad \text{a.e. in } \mathbb{R}^n \setminus \Omega\}$$

with the norm

$$\|u\|_{X_0} = \left(\int_Q |u(x) - u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}} \tag{2.6}$$

is a reflexive Banach space. Note that the norm $\|.\|_{X_0}$ involves the interaction between Ω and $\mathbb{R}^n \setminus \Omega$.

Lemma 2.5. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ be a function satisfying (ii) and let $\{u_k\}$ be a bounded sequence in X_0. Then, there exists $u \in L^{\beta}(\mathbb{R}^n)$ such that up to a subsequence, $u_k \to u$ in $L^{\beta}(\mathbb{R}^n)$ as $k \to \infty$ for any $\beta \in [1, p^*)$.*

Proof. Let $\{u_k\}$ be bounded in X_0. Then by Lemmas 2.1 and 2.3, $\{u_k\}$ is bounded in $W^{\alpha,p}(\Omega)$ and in $L^p(\Omega)$. Also by assumption on Ω and Corollary 7.2 of [4], there exists $u \in L^{\beta}(\Omega)$ such that up to a subsequence $u_k \to u$ strongly in $L^{\beta}(\Omega)$ as $k \to \infty$ for any $\beta \in [1, p^*)$. Since $u_k = 0$ on $\mathbb{R}^n \setminus \Omega$, we can define $u := 0$ in $\mathbb{R}^n \setminus \Omega$. Then we get $u_k \to u$ in $L^{\beta}(\mathbb{R}^n)$. □

3. Nehari manifold and Fibering map analysis for (1.1)

The Euler functional $J_\lambda : X_0 \to \mathbb{R}$ associated to the problem (1.1) is defined as

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y - \frac{\lambda}{q+1}\int_\Omega h(x)|u|^{q+1}\mathrm{d}x \\ &\quad -\frac{1}{r+1}\int_\Omega b(x)|u|^{r+1}\mathrm{d}x. \end{aligned}$$

Then J_λ is Fréchet differentiable and

$$\begin{aligned} \langle J'_\lambda(u), v\rangle &= \int_Q |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y \\ &\quad -\lambda\int_\Omega h(x)|u|^{q-1}uv\mathrm{d}x - \int_\Omega b(x)|u|^{r-1}uv\mathrm{d}x, \end{aligned}$$

which shows that the weak solutions of (1.1) are critical points of the functional J_λ.

It is easy to see that the energy functional J_λ is not bounded below on the space X_0, but we show that it is bounded below on an appropriate subset of X_0 and a minimizer on subsets of this set gives rise to solutions of (1.1). In order to obtain the existence results, we introduce the Nehari manifold

$$\mathcal{N}_\lambda := \left\{u \in X_0 : \langle J'_\lambda(u), u\rangle = 0\right\},$$

where $\langle\ ,\ \rangle$ denotes the duality between X_0 and its dual space. Therefore $u \in \mathcal{N}_\lambda$ if and only if

$$\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y - \lambda\int_\Omega h(x)|u|^{q+1}\mathrm{d}x - \int_\Omega b(x)|u|^{r+1}\mathrm{d}x = 0. \tag{3.1}$$

We note that $\mathcal{N}_\lambda$ contains every solution of (1.1). Now we know that the Nehari manifold is closely related to the behavior of the functions $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ defined as $\phi_u(t) = J_\lambda(tu)$. Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [10]. For $u \in X_0$, we have

$$\begin{aligned} \phi_u(t) &= \frac{t^p}{p}\|u\|^p_{X_0} - \frac{\lambda t^{q+1}}{q+1}\int_\Omega h(x)|u|^{q+1}\mathrm{d}x - \frac{t^{r+1}}{r+1}\int_\Omega b(x)|u|^{r+1}\mathrm{d}x, \\ \phi'_u(t) &= t^{p-1}\|u\|^p_{X_0} - \lambda t^q\int_\Omega h(x)|u|^{q+1}\mathrm{d}x - t^r\int_\Omega b(x)|u|^{r+1}\mathrm{d}x, \end{aligned}$$

$$\phi_u''(t) = (p-1)t^{p-2}\|u\|_{X_0}^p - q\lambda t^{q-1}\int_\Omega h(x)|u|^{q+1}\mathrm{d}x - rt^{r-1}\int_\Omega b(x)|u|^{r+1}\mathrm{d}x.$$

Then it is easy to see that $tu \in \mathcal{N}_\lambda$ if and only if $\phi_u'(t) = 0$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\phi_u'(1) = 0$. Thus it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection. For this we set

$$\mathcal{N}_\lambda^{\pm} := \left\{u \in \mathcal{N}_\lambda : \phi_u''(1) \gtrless 0\right\} = \{tu \in X_0 : \phi_u'(t) = 0,\ \phi_u''(t) \gtrless 0\},$$

$$\mathcal{N}_\lambda^{0} := \left\{u \in \mathcal{N}_\lambda : \phi_u''(1) = 0\right\} = \{tu \in X_0 : \phi_u'(t) = 0,\ \phi_u''(t) = 0\}.$$

Now we study the fiber map ϕ_u according to the sign of $\int_\Omega h(x)|u|^{q+1}\mathrm{d}x$ and $\int_\Omega b(x)|u|^{r+1}\mathrm{d}x$.

Case 1. $u \in H_0^- \cap B_0^-$. In this case $\phi_u(0) = 0$, $\phi_u'(t) > 0$ for all $t > 0$ which implies that ϕ_u is strictly increasing and hence no critical point.

Case 2. $u \in H_0^- \cap B^+$. In this case, firstly we define $m_u : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$m_u(t) = t^{p-1-q}\|u\|_{X_0}^p - t^{r-q}\int_\Omega b(x)|u|^{r+1}\mathrm{d}x.$$

Clearly, for $t > 0$, $tu \in \mathcal{N}_\lambda$ if and only if t is a solution of

$$m_u(t) = \lambda\int_\Omega h(x)|u|^{q+1}\mathrm{d}x. \tag{3.2}$$

Then we have $m_u(t) \to -\infty$ as $t \to \infty$ and

$$m_u'(t) = (p-1-q)t^{p-2-q}\|u\|_{X_0}^p - (r-q)t^{r-1-q}\int_\Omega b(x)|u|^{r+1}\mathrm{d}x.$$

Therefore $m_u'(t) > 0$ near zero. Since $u \in H^-$, there exists $t_*(u)$ such that $m_u(t_*) = \lambda\int_\Omega h(x)|u|^{q+1}\mathrm{d}x$. Thus for $0 < t < t_*$, $\phi_u'(t) = t^q(m_u(t) - \lambda\int_\Omega h(x)|u|^{q+1}\mathrm{d}x) > 0$ and for $t > t_*$, $\phi_u'(t) < 0$. Hence ϕ_u is increasing on $(0, t_*)$ and decreasing on (t_*, ∞). Since $\phi_u(t) > 0$ for t close to 0 and $\phi_u(t) \to -\infty$ as $t \to \infty$, ϕ_u has exactly one critical point $t_1(u)$, which is a global maximum point. Hence $t_1(u)u \in \mathcal{N}_\lambda^-$.

Case 3. $u \in H^+ \cap B_0^-$. In this case $m_u(0) = 0$, $m_u'(t) > 0$ for all $t > 0$ which implies that m_u is strictly increasing. Since $u \in H^+$, there exists a unique $t_1 = t_1(u) > 0$ such that $m_u(t_1) = \lambda\int_\Omega h(x)|u|^{q+1}\mathrm{d}x$. This implies that $\phi_u(t)$ is decreasing on $(0, t_1)$, and increasing on (t_1, ∞) and $\phi_u'(t_1) = 0$. Thus ϕ_u has exactly one critical point $t_1(u)$, corresponding to global minimum point. Hence $t_1(u)u \in \mathcal{N}_\lambda^+$.

Case 4. $u \in H^+ \cap B^+$. If λ is sufficiently large, then (3.2) has no solution and so, ϕ_u has no critical point. Moreover in this case ϕ_u is a decreasing function. If λ is sufficiently small then there are exactly two solutions $t_1(u) < t_2(u)$ of (3.2) with $m_u'(t_1(u)) > 0$ and $m_u'(t_2(u)) < 0$. It follows that ϕ_u has exactly two critical points $t_1(u)$ and $t_2(u)$ corresponding to a local minimum and a local maximum point respectively such that $t_1(u)u \in \mathcal{N}_\lambda^+$ and $t_2(u)u \in \mathcal{N}_\lambda^-$. Moreover, ϕ_u is decreasing in $(0, t_1)$, increasing in (t_1, t_2) and decreasing in (t_2, ∞).

In the following lemma, we show that for small λ, ϕ_u takes a positive value for all non zero u.

Lemma 3.1. *There exists $\lambda_0 > 0$ such that $\lambda < \lambda_0$, ϕ_u takes a positive value for all non-zero $u \in X_0$.*

Proof. For $u \in B^+$, we define $F_u : \mathbb{R}^+ \to \mathbb{R}$,

$$F_u(t) := \frac{t^p}{p}\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y - \frac{t^{r+1}}{r+1}\int_\Omega b(x)|u|^{r+1}\mathrm{d}x.$$

Then

$$F_u'(t) = t^{p-1}\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y - t^r\int_\Omega b(x)|u|^{r+1}\mathrm{d}x$$

and F_u attains its maximum value at $t_* = \left(\frac{\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y}{\int_\Omega b(x)|u|^{r+1}\mathrm{d}x}\right)^{\frac{1}{r-p+1}}$. Moreover,

$$F_u(t_*) = \left(\frac{1}{p} - \frac{1}{r+1}\right)\left(\frac{(\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y)^{r+1}}{(\int_\Omega b(x)|u|^{r+1}\mathrm{d}x)^p}\right)^{\frac{1}{r-p+1}}$$

and

$$F_u''(t_*) = (p-r-1)\frac{(\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y)^{\frac{r-1}{r-p+1}}}{\left(\int_\Omega b(x)|u|^{r+1}\mathrm{d}x\right)^{\frac{p-2}{r-p+1}}} < 0.$$

Let S_{r+1} be the best constant of Sobolev embedding $W^{\alpha,p}(\mathbb{R}^n) \hookrightarrow L^{r+1}(\mathbb{R}^n)$, then

$$\|u\|_{L^{r+1}(\Omega)} = \|u\|_{L^{r+1}(\mathbb{R}^n)} \le S_{r+1}\|u\|_{W^{\alpha,p}(\mathbb{R}^n)}.$$

By Lemmas 2.1 and 2.3, we have $\|u\|_{W^{\alpha,p}(\mathbb{R}^n)} \le C(\theta)\|u\|_{X_0} = M\|u\|_{X_0}$. Combining the above two inequalities, we get

$$\frac{1}{(MS_{r+1})^{p(r+1)}} \le \frac{(\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y)^{r+1}}{(\int_\Omega |u|^{r+1}\mathrm{d}x)^p}.$$

Hence

$$F_u(t_*) \ge \frac{r-p+1}{p(r+1)}\left(\frac{1}{\|b^+\|_\infty^p (MS_{r+1})^{p(r+1)}}\right)^{\frac{1}{r-p+1}}. \tag{3.3}$$

We now show that there exists $\lambda_0 > 0$ such that $\phi_u(t_*) > 0$. Using Sobolev embedding of fractional order spaces, we get

$$\begin{aligned}&\frac{t_*^{q+1}}{q+1}\int_\Omega h(x)|u|^{q+1}\mathrm{d}x\\ &\quad\le \frac{1}{q+1}\|h\|_\infty (MS_{q+1})^{q+1}\left(\frac{\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y}{\int_\Omega b(x)|u|^{r+1}\mathrm{d}x}\right)^{\frac{q+1}{r-p+1}}\|u\|^{q+1}\end{aligned}$$

$$= \frac{1}{q+1}\|h\|_\infty (MS_{q+1})^{q+1}\left[\frac{(\int_Q |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y)^{r+1}}{(\int_\Omega b(x)|u|^{r+1}\mathrm{d}x)^p}\right]^{\frac{q+1}{p(r-p+1)}}$$

$$= \frac{1}{q+1}\|h\|_\infty (MS_{q+1})^{q+1}\left(\frac{p(r+1)}{r-p+1}\right)^{\frac{q+1}{p}} (F_u(t_*))^{\frac{q+1}{p}} = cF_u(t_*)^{\frac{q+1}{p}},$$

where c is a constant independent of u. Thus by (3.3) and $u \in H^+$, we get

$$\phi_u(t_*) \geq F_u(t_*) - \lambda c F_u(t_*)^{\frac{q+1}{p}} = F_u(t_*)^{\frac{q+1}{p}}(F_u(t_*)^{\frac{p-1-q}{p}} - \lambda c)$$
$$\geq \delta^{\frac{q+1}{p}}(\delta^{\frac{p-1-q}{p}} - \lambda c),$$

where $\delta = \frac{r-p+1}{p(r+1)}\left(\frac{1}{\|b^+\|_\infty^p (MS_{r+1})^{p(r+1)}}\right)^{\frac{1}{r-p+1}}$. Let $\lambda_0 = \frac{\delta^{\frac{p-1-q}{p}}}{c}$. Then for every $\lambda < \lambda_0$, we get the required result. □

Lemma 3.2. *If* $\lambda < \lambda_0$, *then* $\inf_{\mathcal{N}_\lambda^-} J_\lambda(u) > 0$.

Proof. Let $u \in \mathcal{N}_\lambda^-$. Then ϕ_u has a positive global maximum at $t = 1$. If $u \in H^+ \cap B^+$, then

$$J_\lambda(u) = \phi_u(1) = \phi_u(t_*) \geq F_u(t_*)^{\frac{q+1}{p}}(F_u(t_*)^{\frac{p-1-q}{p}} - \lambda c) \geq \delta^{\frac{q+1}{p}}(\delta^{\frac{p-1-q}{p}} - \lambda c)$$

where δ is same as in Lemma 3.1. So the infimum of J_λ over $H^+ \cap B^+$ is positive. If $u \in B^+ \cap H_0^-$, then by Case 2, ϕ_u has a unique global maximum at $t = 1$ and

$$J_\lambda(u) = \frac{r+1-p}{p(r+1)}\|u\|_{X_0}^p - \lambda\frac{r-q}{(q+1)(r+1)}\int_\Omega h|u|^{q+1} > 0. \tag{3.4}$$

Moreover, $\inf_{u\in\mathcal{N}_\lambda^- \cap B^+ \cap H^+} J_\lambda(u) > 0$. Indeed, if this infimum is zero, then from (3.4), minimizing sequence $\{u_k\}$ converges strongly in X_0 to 0 and $0 \notin \mathcal{N}_\lambda^-$, a contradiction.

Since ϕ_u has a unique global minimum if $u \in B_0^- \cap H^+$ (by Case 3), we infer $u \notin B_0^- \cap H^+ \cap \mathcal{N}_\lambda^-$. Similarly, by Case 1, there is no critical point for ϕ_u if $u \in B_0^- \cap H_0^-$. So $u \notin B_0^- \cap H_0^- \cap \mathcal{N}_\lambda^-$. Therefore, $\inf_{\mathcal{N}_\lambda^-} J_\lambda(u) > 0$. □

Lemma 3.3. *If* $0 < \lambda < \lambda_0$, *then* $\mathcal{N}_\lambda^0 = \{0\}$.

Proof. Let $0 \not\equiv u \in \mathcal{N}_\lambda^0$. Then 1 is the critical point of ϕ_u. If $u \in H_0^- \cap B_0^-$, then ϕ_u has no critical point. If $u \in H_0^- \cap B^+$ or $u \in H^+ \cap B_0^-$ or $u \in H^+ \cap B^+$, then ϕ_u has critical point(s) corresponding to local maxima or local minima. So 1 is the critical point corresponding to local minima or local maxima of ϕ_u. That is, either $u \in \mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$, a contradiction. Hence the proof. □

In the following lemma we show that the minimizers on subsets of $\mathcal{N}_\lambda$ are solutions of (1.1).

Lemma 3.4. *Let u be a local minimizer for J_λ on subsets $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$, then u is a critical point for J_λ.*

Proof. Since u is a minimizer for J_λ under the constraint $I_\lambda(u) := \langle J_\lambda'(u), u\rangle = 0$, by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $J_\lambda'(u) = \mu I_\lambda'(u)$. Thus $\langle J_\lambda'(u), u\rangle = \mu\ \langle I_\lambda'(u), u\rangle = \mu\phi_u''(1)$=0, but $u \notin \mathcal{N}_\lambda^0$ and so $\phi_u''(1) \neq 0$. Hence $\mu = 0$ completes the proof. □

Lemma 3.5. *J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. On $\mathcal{N}_\lambda$,

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{r+1}\right)\|u\|_{X_0}^p - \lambda\left(\frac{1}{q+1} - \frac{1}{r+1}\right)\int_\Omega h(x)|u|^{q+1}\mathrm{d}x \\ &\geq c_1\|u\|_{X_0}^p - c_2\|u\|_{X_0}^{q+1}. \end{aligned}$$

Hence J_λ is bounded below and coercive on $\mathcal{N}_\lambda$. □

4. Existence of solutions

In this section, we show the existence of minimizers in $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ for $\lambda \in (0, \lambda_0)$.

Lemma 4.1. *If $\lambda < \lambda_0$, then J_λ achieves its minimum on $\mathcal{N}_\lambda^+$.*

Proof. Since J_λ is bounded below on $\mathcal{N}_\lambda$, and so also on $\mathcal{N}_\lambda^+$. Then there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^+$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u).$$

As J_λ is coercive on $\mathcal{N}_\lambda$, $\{u_k\}$ is a bounded sequence in X_0. Therefore $u_k \rightharpoonup u_\lambda$ weakly in X_0 and $u_k \to u_\lambda$ strongly in $L^\alpha(\mathbb{R}^n)$ for $1 \leq \alpha < np(n-p\alpha)^{-1}$. If we choose $u \in X_0$ such that $\displaystyle\int_\Omega h(x)|u|^{q+1}\mathrm{d}x > 0$, then there exist $t_1(u) > 0$ such that $t_1(u)u \in \mathcal{N}_\lambda^+$ and $J_\lambda(t_1(u)u) < 0$ and hence $\displaystyle\inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) \leq J_\lambda(t_1(u)u) < 0$. Now on $\mathcal{N}_\lambda$,

$$J_\lambda(u_k) = \left(\frac{1}{p} - \frac{1}{r+1}\right)\|u_k\|_{X_0}^p - \lambda\left(\frac{1}{q+1} - \frac{1}{r+1}\right)\int_\Omega h(x)|u_k|^{q+1}\mathrm{d}x$$

and so

$$\lambda\left(\frac{1}{q+1} - \frac{1}{r+1}\right)\int_\Omega h(x)|u_k|^{q+1}\mathrm{d}x = \left(\frac{1}{p} - \frac{1}{r+1}\right)\|u_k\|_{X_0}^p - J_\lambda(u_k).$$

Letting $k \to \infty$, we get $\int_\Omega h(x)|u_\lambda|^{q+1}\mathrm{d}x > 0$. Next we claim that $u_k \to u_\lambda$. Suppose this is not true, then

$$\int_Q |u_\lambda(x)-u_\lambda(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y < \liminf_{k\to\infty}\int_Q |u_k(x)-u_k(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y.$$

Thus

$$\phi'_{u_k}(t) = t^{p-1}\|u_k\|_{X_0}^p - \lambda t^q \int_\Omega h(x)|u_k|^{q+1}\mathrm{d}x - t^r \int_\Omega b(x)|u_k|^{r+1}\mathrm{d}x$$

and

$$\phi'_{u_\lambda}(t) = t^{p-1}\|u_\lambda\|_{X_0}^p - \lambda t^q \int_\Omega h(x)|u_\lambda|^{q+1}\mathrm{d}x - t^r \int_\Omega b(x)|u_\lambda|^{r+1}\mathrm{d}x.$$

It follows that $\phi'_{u_k}(t_\lambda(u_\lambda)) > 0$ for sufficiently large k. So, we must have $t_\lambda > 1$. But we have $t_\lambda(u_\lambda)u_\lambda \in \mathcal{N}_\lambda^+$ and so

$$J_\lambda(t_\lambda(u_\lambda)u_\lambda) < J_\lambda(u_\lambda) < \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u),$$

which is a contradiction. Hence we must have $u_k \to u_\lambda$ in X_0. Moreover, $u_\lambda \in \mathcal{N}_\lambda^+$, since $\mathcal{N}_\lambda^0 = \emptyset$. Hence u_λ is a minimizer for J_λ on $\mathcal{N}_\lambda^+$. □

Lemma 4.2. *If $\lambda < \lambda_0$, then J_λ achieves its minimum on $\mathcal{N}_\lambda^-$.*

Proof. Let $u \in \mathcal{N}_\lambda^-$, then from Lemma 3.2 we have $J_\lambda(u) \geq \delta_1$. So there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^-$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) > 0.$$

Since $J_\lambda(u_k)$ is coercive, $\{u_k\}$ is a bounded sequence in X_0. Therefore $u_k \rightharpoonup u_\lambda$ weakly in X_0 and $u_k \to u_\lambda$ strongly in L^β for $1 \leq \beta < \frac{np}{n-p\alpha}$. Then

$$J_\lambda(u_k) = \left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_k\|_{X_0}^p + \left(\frac{1}{q+1} - \frac{1}{r+1}\right)\int_\Omega b(x)|u_k|^{r+1}\mathrm{d}x.$$

Since $\lim_{k\to\infty} J_\lambda(u_k) > 0$ and $\lim_{k\to\infty}\int_\Omega b(x)|u_k|^{r+1}\mathrm{d}x = \int_\Omega b(x)|u_\lambda|^{r+1}\mathrm{d}x$, we must have $\int_\Omega b(x)|u_\lambda|^{r+1}\mathrm{d}x > 0$. Hence ϕ_{u_λ} has a global maximum at some point $\tilde{t}$ so that $\tilde{t}(u_\lambda)u_\lambda \in \mathcal{N}_\lambda^-$. On the other hand, $u_k \in \mathcal{N}_\lambda^-$ implies that 1 is a global maximum point for ϕ_{u_k}. That is, $\phi_{u_k}(t) \leq \phi_{u_k}(1)$ for every $t > 0$. Thus we have

$$\begin{aligned}
&J_\lambda(\tilde{t}(u_\lambda)u_\lambda)\\
&\quad= \frac{1}{p}(\tilde{t}(u_\lambda))^p\|u_\lambda\|_{X_0}^p - \frac{\lambda(\tilde{t}(u_\lambda))^{q+1}}{q+1}\int_\Omega h(x)|u_\lambda|^{q+1}\mathrm{d}x\\
&\qquad -\frac{(\tilde{t}(u_\lambda))^{r+1}}{r+1}\int_\Omega b(x)|u_\lambda|^{r+1}\mathrm{d}x,\\
&\quad< \liminf_{k\to\infty}\left(\frac{1}{p}(\tilde{t}(u_\lambda))^p\|u_k\|_{X_0}^p - \frac{\lambda(\tilde{t}(u_\lambda))^{q+1}}{q+1}\int_\Omega h|u_k|^{q+1}\mathrm{d}x\right.\\
&\qquad \left.-\frac{(\tilde{t}(u_\lambda))^{r+1}}{r+1}\int_\Omega b|u_k|^{r+1}\mathrm{d}x\right),\\
&\quad\leq \lim_{k\to\infty} J_\lambda(\tilde{t}(u_\lambda)u_k) \leq \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u),
\end{aligned}$$

which is a contradiction as $\tilde{t}(u_\lambda)u_\lambda \in \mathcal{N}_\lambda^-$. Hence $u_k \to u_\lambda$ strongly in X_0 and moreover $u_\lambda \in \mathcal{N}_\lambda^-$, since $\mathcal{N}_\lambda^0 = \{0\}$. □

Next, we prove the existence of non-negative solutions. For this, we first define some notations.

$$F^+ = \int_0^t f^+(x, s)\mathrm{d}s,$$

where

$$f^+(x, t) = \begin{cases} f(x, t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Let $J_\lambda^+(u) = \|u\|_{X_0}^p - \int_\Omega F^+(x, u)\mathrm{d}x$. Then the functional $J_\lambda^+(u)$ is well defined and it is Fréchet differentiable in $u \in X_0$ and for any $v \in X_0$,

$$\langle J_\lambda'^+(u), v\rangle = \int_Q |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y - \int_\Omega f^+(x, u)v\mathrm{d}x. \tag{4.1}$$

If $f(x, t) := \lambda h(x)|t|^{q-1}t + b(x)|t|^{r-1}t$, then $J_\lambda^+(u)$ satisfies all the above lemmas. So for $\lambda \in (0, \lambda_0)$, there exists two non-trivial critical points $u_\lambda \in \mathcal{N}_\lambda^+$ and $v_\lambda \in \mathcal{N}_\lambda^-$.

Now we claim that both u_λ and v_λ are non-negative in $\mathbb{R}^n$. Take $v = u^-$ in (4.1), then

$$\begin{aligned} 0 &= \langle J_\lambda'^+(u), u^-\rangle \\ &= \int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y))(u^-(x) - u^-(y))K(x - y)\mathrm{d}x\mathrm{d}y \\ &= \int_Q |u(x)-u(y)|^{p-2}((u^-(x)- u^-(y))^2+2u^-(x)u^+(y))K(x-y)\mathrm{d}x\mathrm{d}y \\ &\geq \int_Q |u^-(x) - u^-(y)|^p K(x - y)\mathrm{d}x\mathrm{d}y \\ &= \|u^-\|_{X_0}^p. \end{aligned}$$

Thus $\|u^-\|_{X_0} = 0$ and hence $u = u^+$. So by taking $u = u_\lambda$ and $u = v_\lambda$ respectively, we get the non-negative solutions of (1.1).

Proof of Theorem 1.1. Lemmas 4.1, 4.2, 3.4 and the above discussion complete the proof. □

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