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# A three critical point theorem for non-smooth functionals with application in differential inclusions

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**Abstract.** A variety of three-critical-point theorems have been established for non-smooth functionals, based on a minimax inequality. In this paper, a generalized form of a recent result due to Ricceri is introduced for non-smooth functionals and by a few hypotheses, without any minimax inequality, the existence of at least three critical points with a uniform bound on the norms of solutions, is obtained. Also, as an application, our main theorem is used to obtain at least three anti-periodic solutions for a second order differential inclusion.

**Keywords.** Locally Lipschitz functions; differential inclusions; anti-periodic solution; critical point.

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## 1. Introduction

In many applications, we encounter problems with non-smooth energy functionals. These problems have attracted much attention in recent years due to interesting theoretical questions arising from them, and also to their direct applications in physics, mechanics and engineering.

In order to establish the existence of solutions for them, we need to extend the non-smooth critical point theory. In recent years, several authors have been interested in the study of this theory, see for example [4–11], [13] and [15]. Among them, Iannizzotto [9], Kristály *et al.* [13] and, Marano and Motreanu [15] extended some three critical point theorems due to Ricceri to non-smooth functionals. In all of the aforementioned works, the approach is based on a minimax inequality that is contained in hypotheses directly.

Recently, Ricceri established a three critical point theorem for $C^1$ functionals (Theorems 1, 3 of [17]), and novelty of Ricceri's theorem is that no minimax inequality appears among the hypotheses.

In the present paper, we give an extension of theorem of Ricceri to locally Lipschitz functions, providing also an application in partial differential inclusions. The notable point of our main theorem (Theorem 3.1) is that without any minimax inequalities and by a few hypotheses, the existence of at least three critical points is obtained, yielding a uniform estimate on the norms of solutions.

The key tool in the proof of our main theorem, is the multiplicity result on global minima of Ricceri [18] which obtains the existence of two local minima for a parametric functional and it is noticeable that no smoothness assumption is required on the functional. Then, by a non-smooth version of Palais–Smale condition we obtain the third critical point.

The inclusion problem given in this paper, as an application, is a second order differential inclusion with anti-periodic boundary condition of the following type:

$$\begin{cases} -(|u'|^{p-2}u')' + M|u|^{p-2}u \in J(x,u) & \text{in } [0,T] \\ u(0) = -u(T), \quad u'(0) = -u'(T). \end{cases} \tag{1.1}$$

Here $p > 1, T > 0, M \geq 0$ and $J$ is a multifunction defined in $[0,T] \times \mathbb{R}$ whose values are compact intervals in $\mathbb{R}$, measurable with respect to the first variable and upper semicontinuous with respect to the second.

We will study problem (1.1) with a general set-valued nonlinearity of the type

$$J(x,u) = \mu F(x,u) - \lambda G(x,u) + \nu H(x,u),$$

where $F, G$ and $H$ are multifunctions and $\mu, \lambda$ and $\nu$ are positive parameters. We will apply our main theorem (Theorem 3.1), in order to prove the existence for $\mu, \lambda$ and $\nu$ lying in convenient intervals of at least three anti-periodic solutions for the problem plus a uniform estimate on the norms of such solutions (see Theorem 4.2).

## 2. Preliminaries

As we already mentioned in the Introduction, our method of proof uses the non-smooth critical point theory for locally Lipschitz functionals, which is based on the subdifferential theory due to Clarke [5]. So, in this section we collect a series of results and notations from this theory which will be used throughout the paper.

Let $X$ be a Banach space whose dual is denoted by $X^*$. We recall that the generalized directional derivative $\Phi^\circ(u;v)$ of a locally Lipschitz function $\Phi : X \to \mathbb{R}$ at a point $u \in X$ and in the direction $v \in X$ is defined by

$$\Phi^\circ(u;v) = \limsup_{w \to u, \tau \to 0^+} \frac{\Phi(w+\tau v) - \Phi(w)}{\tau}.$$

The set $\partial\Phi(u) := \{u^* \in X^* : \langle u^*, v\rangle \leq \Phi^\circ(u;v) \text{ for all } v \in X\}$ denotes the generalized gradient of the function $\Phi$.

The following lemmas yield some useful properties of the above defined tools that can be found in chapter 2 of [5].

*Lemma* 2.1. *Let $\Phi \in C^1(X)$ be a functional. Then $\Phi$ is locally Lipschitz and*

(1) $\Phi^\circ(u;v) = \langle \Phi'(u), v\rangle$ *for all* $u, v \in X$;
(2) $\partial\Phi(u) = \{\Phi'(u)\}$ *for all* $u \in X$.

*Lemma* 2.2. *Let* $\Phi, H : X \to \mathbb{R}$ *be locally Lipschitz functionals. Then, for every* $u, v \in X$, *the following conditions hold*:

(1) $\partial\Phi(u)$ *is convex and* $w^*$*-compact*;
(2) *the set-valued mapping* $\partial\Phi(u) : X \to 2^{X^*}$ *is* $w^*$*-upper semicontinuous*;
(3) $\Phi^\circ(u; v) = \max_{u^* \in \partial\Phi(u)} \langle u^*, v\rangle \leq L||v||$, $v \in X$;
(4) $\partial(\lambda\Phi)(u) = \lambda\partial\Phi(u)$ *for every* $\lambda \in \mathbb{R}$;
(5) $\partial(\Phi + H)(u) \subseteq \partial\Phi(u) + \partial H(u)$;
(6) $\partial(\varphi \circ \Phi)(u) \subseteq \{\xi u^* : \xi \in \partial\varphi(\Phi(u)), u^* \in \partial\Phi(u)\}$ *for every locally Lipschitz* $\varphi : \mathbb{R} \to \mathbb{R}$.

The next lemma shows that a locally Lipschitz functional with a compact gradient, is sequentially weakly continuous and it is a technical lemma in the proof of our main result.

*Lemma* 2.3 (*Lemma* 6 *of* [9]). *Let* $\Phi : X \to \mathbb{R}$ *be a locally Lipschitz functional with compact gradient. Then* $\Phi$ *is sequentially weakly continuous.*

The next definitions generalize the notion of critical point and the definition of the Palais–Smale condition to the non-smooth functionals that fit with our main result and application.

DEFINITION 2.4

Assume $\Phi : X \to \mathbb{R}$ is a locally Lipschitz function. A vector $u \in X$ is said to be a critical point of the functional $\Phi$ if $0 \in \partial\Phi(u)$, that is,

$$\Phi^0(u; v - u) \geqslant 0, \quad \forall v \in X.$$

A number $c \in \mathbb{R}$ such that $\Phi^{-1}(c)$ contains a critical point is called a critical value of $\Phi$.

DEFINITION 2.5

The locally Lipschitz functional $\Phi$ is said to satisfy the Palais–Smale (PS) condition if every sequence $\{u_n\} \subset X$ such that $\Phi(u_n)$ is bounded and

$$\Phi^0(u_n; v - u_n) \geqslant -\varepsilon_n||v - u_n||, \quad \forall v \in X,$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$, possesses a convergent subsequence.

By the above definitions, we can easily deduce from the results obtained due to Livrea and Marano [14], the generalized Mountain Pass theorem that is formulated in the next theorem and used for our purpose.

**Theorem 2.6.** *Let* $\Phi$ *be a locally Lipschitz functional satisfying the Palais–Smale condition*, $u_0, u_1$ $(u_0 \neq u_1)$ *be local minimizers of* $\Phi$ *and*

$$\Gamma = \{\gamma \in C^0([0, 1], X) : \gamma(i) = u_i, i = 0, 1\},$$
$$e = \inf_{\gamma\in\Gamma} \sup_{\tau\in[0,1]} [\Phi(\gamma(\tau))].$$

*Then, there exists a critical point* $u_2 \in X$ *such that* $u_2 \neq u_i (i = 0, 1)$ *and* $\Phi(u_2) = e$.

The next notation, introduced by Ricceri in [17], will be used to state our main result.

If $X$ is a non-empty set and $\Gamma, \Psi, \Phi : X \to \mathbb{R}$ are three given functions, for each $\mu > 0$ and $r \in ]\inf_X \Phi, \sup_X \Phi[$, we put

$$\alpha(\mu\Gamma + \Psi, \Phi, r) = \inf_{x \in \Phi^{-1}(]-\infty, r[)} \frac{\mu\Gamma(x) + \Psi(x) - \inf_{\Phi^{-1}(]-\infty, r])} (\mu\Gamma + \Psi)}{r - \Phi(x)}$$

and

$$\beta(\mu\Gamma + \Psi, \Phi, r) = \sup_{x \in \Phi^{-1}(]r, +\infty[)} \frac{\mu\Gamma(x) + \Psi(x) - \inf_{\Phi^{-1}(]-\infty, r])} (\mu\Gamma + \Psi)}{r - \Phi(x)}.$$

When $\Psi + \Phi$ is bounded below, for each $r \in ]\inf_X \Phi, \sup_X \Phi[$ such that

$$\inf_{x \in \Phi^{-1}(]-\infty, r[)} \Gamma(x) < \inf_{x \in \Phi^{-1}(r)} \Gamma(x),$$

we put

$$\mu^*(\Gamma, \Psi, \Phi, r) = \inf\left\{ \frac{\Psi(x) - \gamma + r}{\eta_r - \Gamma(x)} : x \in X, \Phi(x) < r, \Gamma(x) < \eta_r \right\},$$

where

$$\gamma = \inf_{x \in X} (\Psi(x) + \Phi(x))$$

and

$$\eta_r = \inf_{x \in \Phi^{-1}(r)} \Gamma(x).$$

Also, as a definition, if $X$ is a topological space, a function $f : X \to \mathbb{R}$ is said to be sequentially inf-compact if, for each $r \in \mathbb{R}$, the set $f^{-1}(] - \infty, r])$ is sequentially compact.

In order to establish our main result, we recall two results due to Ricceri which are fundamental in our discussion. These two results are main tools in the proof of our main result. By Lemma 2.7, the existence of two local minima is obtained without any smoothness assumption on the functional, and causes that we can omit the minimax inequality hypothesis in our main result.

*Lemma* 2.7 (*Theorem* 2 *of* [17]). *Let $X$ be a topological space and $\Gamma, \Psi, \Phi : X \to \mathbb{R}$ three sequentially lower semicontinuous functions*, *with $\Gamma$ also sequentially inf-compact*, *satisfying the following conditions*:

(*a*) $\inf_{x \in X} (\mu\Gamma(x) + \Psi(x)) = -\infty$ *for all* $\mu > 0$;
(*b*) $\inf_{x \in X} (\Psi(x) + \Phi(x)) > -\infty$;
(*c*) *there exists* $r \in\ ]\inf_X \Phi, \sup_X \Phi[$ *such that*

$$\inf_{x \in \Phi^{-1}(]-\infty, r[)} \Gamma(x) < \inf_{x \in \Phi^{-1}(r)} \Gamma(x).$$

*Under such hypotheses, for each* $\mu > \max\{0, \mu^*(\Gamma, \Psi, \Phi, r)\}$, *one has*

$$\alpha(\mu\Gamma + \Psi, \Phi, r) = 0$$

*and*

$$\beta(\mu\Gamma + \Psi, \Phi, r) > 0.$$

*Lemma* 2.8 (*Theorem* 4 *of* [18]). *Let* $X$ *be a real, reflexive Banach space, let* $\Lambda \subseteq \mathbb{R}$ *be an interval, and let* $\varphi : X \times \Lambda \to \mathbb{R}$ *be a function satisfying the following conditions*:

(1) $\varphi(x, .)$ *is concave in* $\Lambda$ *for all* $x \in X$;
(2) $\varphi(., \lambda)$ *is continuous, coercive and sequentially weakly lower semicontinuous in* $X$ *for all* $\lambda \in \Lambda$;
(3) $\beta_1 := \sup_{\lambda\in\Lambda} \inf_{x\in X} \varphi(x, \lambda) < \inf_{x\in X} \sup_{\lambda\in\Lambda} \varphi(x, \lambda) =: \beta_2$.

*Then, for each* $\sigma > \beta_1$ *there exists a non-empty open set* $\Lambda_0 \subset \Lambda$ *with the following property*: *for every* $\lambda \in \Lambda_0$ *and every sequentially weakly lower semicontinuous function* $\Phi : X \to \mathbb{R}$ *there exists* $\mu_0 > 0$ *such that, for each* $\mu \in ]0, \mu_0[$, *the function* $\varphi(., \lambda) + \mu\Phi(.)$ *has at least two local minima lying in the set* $\{x \in X : \varphi(x, \lambda) < \sigma\}$.

At the end of this section, we present the definition of an operator of type $(S)_+$ that is one of the hypotheses in our main result.

DEFINITION 2.9

An operator $A : X \to X^*$ is of type $(S)_+$ if, for any sequence $\{u_n\}$ in $X$, $u_n \rightharpoonup u$ and $\limsup_n \langle A(u_n), u_n - u\rangle \leqslant 0$ imply $u_n \to u$.

## 3. Main result

This section is devoted to the statement and proof of our main result which extends Theorem 3 of [17] to non-smooth functionals. We obtain the existence of at least three critical points with an estimate of the critical point norms.

Before our main theorem, we introduce the following class of functions that applies in our proof. For every $N \geqslant 0$, put

$$C_N := \{g \in C^1(\mathbb{R}, \mathbb{R}) \text{ is bounded and } g(t) = t \text{ for every } t \in [-N, N]\}.$$

**Theorem 3.1.** *Let* $X$ *be a reflexive real Banach space*; $I : X \to \mathbb{R}$ *a sequentially weakly lower semicontinuous and coercive* $C^1$*-functional, bounded on each bounded subset of* $X$ *such that* $I'$ *is of type* $(S)_+$; $\Phi_i : X \to \mathbb{R}$ $(i = 1, 2)$ *be two locally Lipschitz functions with compact gradient satisfying the following conditions*:

(1) $\liminf_{||u||\to+\infty} \dfrac{\Phi_1(u)}{I(u)} = -\infty$;
(2) $\inf_{u\in X}(\Phi_1(u) + \lambda\Phi_2(u)) > -\infty$, *for all* $\lambda > 0$. *Under such hypotheses, for each* $r > \sup_M \Phi_2$, *where* $M$ *is the set of all global minima of* $I$, *for each* $\mu$ *satisfying* $\mu > \mu^*(I, \Phi_1, \Phi_2, r)$.

(3) *$\mu > \mu^*(I, \Phi_1, \Phi_2, r)$, and for each compact interval $[\lambda_1, \lambda_2] \subset ]0, \beta(\mu I + \Phi_1, \Phi_2, r)[$, there exists a number $\rho > 0$ with the following property: for every $\lambda \in [\lambda_1, \lambda_2]$ and every locally Lipschitz functional $h : X \to \mathbb{R}$ with compact gradient, there exists $\nu_0 > 0$ such that for every $\nu \in ]0, \nu_0[$, the functional $\mu I + \Phi_1 + \lambda\Phi_2 + \nu h$ has at least three critical points whose norms are less than $\rho$.*

*Proof.* We first wish to apply Lemmas 2.7 and 2.8. Using Lemma 2.3, $\Phi_1$ and $\Phi_2$ are sequentially weakly continuous and so, in particular, they are bounded on each bounded subset of $X$, due to the reflexivity of $X$. Since $I$ is coercive, by the reflexivity of $X$, this implies that the set $M$ is non-empty and bounded. As a consequence, $\Phi_2$ is bounded in $M$. Let $r > \sup_M \Phi_2$. Since $\Phi_2^{-1}(r)$ is non-empty and sequentially weakly closed, there exists $\bar{u} \in \Phi_2^{-1}(r)$ such that

$$I(\bar{u}) = \inf_{u \in \Phi_2^{-1}(r)} I(u).$$

The choice of $r$ implies that $\bar{u} \notin M$. So, we infer that

$$\inf_{u \in \Phi_2^{-1}(]-\infty, r[)} I(u) < \inf_{u \in \Phi_2^{-1}(r)} I(u).$$

It is clear that, by (1), there exists a sequence $\{u_n\}$ in $X$ such that

$$\lim_{n\to\infty} ||u_n|| = +\infty, \quad \lim_{n\to\infty} \frac{\Phi_1(u_n)}{I(u_n)} = -\infty. \tag{3.1}$$

For any $c \in \mathbb{R}$ and for $n$ large enough, we have

$$cI(u_n) + \Phi_1(u_n) = I(u_n)(c + \frac{\Phi_1(u_n)}{I(u_n)}). \tag{3.2}$$

Now, fix $\mu$ as in (3), and clearly from coercivity of $I$, (3.1) and (3.2), it follows that

$$\lim_{n\to\infty} (\mu I(u_n) + \Phi_1(u_n)) = -\infty.$$

So, if we consider $X$ endowed with the weak topology, then the function $I$ satisfies the assumptions of Lemma 2.7 as functional $\Gamma$, and by coercivity of $I$, it is bounded and sequentially weakly closed on $\{u \in X : I(u) \leqslant q\}$, for all $q \in \mathbb{R}$. Hence it is sequentially weakly compact by the Eberlin–Smulyan theorem that is inf-compact. So, all the assumptions of Lemma 2.7 are satisfied, and we have

$$\alpha(\mu I + \Phi_1, \Phi_2, r) < \beta(\mu I + \Phi_1, \Phi_2, r).$$

But, by Theorem 1 of [6], this inequality is equivalent to

$$\sup_{\lambda \geqslant 0} \inf_{u \in X} (\mu I + \Phi_1 + \lambda(\Phi_2(u) - r)) < \inf_{u \in X} \sup_{\lambda \geqslant 0} (\mu I + \Phi_1 + \lambda(\Phi_2(u) - r)).$$

Now, let $\Lambda = ]0, \beta(\mu I + \Phi_1, \Phi_2, r)[$ and $\varphi$ be defined by putting for all $(u, \lambda) \in X \times \Lambda$,

$$\varphi(u, \lambda) = \mu I(u) + \Phi_1(u) + \lambda(\Phi_2(u) - r).$$

From the fact that $\Phi_1$, $\Phi_2$ are sequentially weakly continuous, it follows that the function $\varphi$ is sequentially weakly lower semicontinuous on $X$ and in view of (2) and coercivity of $I$, one has

$$\lim_{||u||\to+\infty} (\mu I(u) + \Phi_1(u) + \lambda\Phi_2(u)) = +\infty, \tag{3.3}$$

that shows $\varphi(u, \lambda)$ is coercive. So, clearly the function $\varphi$ satisfies the hypotheses of Lemma 2.8.

Now, fix $\sigma > \sup_\Lambda \inf_X \varphi$ and consider a nonempty open set $\Lambda_0$ with the property expressed in Lemma 2.8. Fix also a compact interval $[a, b] \subset \Lambda_0$ and choose $\lambda \in [a, b]$ and a locally Lipschitz functional $h : X \to \mathbb{R}$ with compact gradient. The functional $h$ is sequentially weakly continuous by Lemma 2.3 and hence is bounded on a bounded set.

Note that

$$\bigcup_{\lambda\in[a,b]} \{u \in X : \varphi(u, \lambda) < \sigma\} \subseteq \{u \in X : \mu I(u) + \Phi_1(u) + a\Phi_2(u) < \sigma - ar\}$$
$$\cup\{u \in X : \mu I(u) + \Phi_1(u) + b\Phi_2(u) < \sigma - br\}.$$

By (3.3), the set on the right is bounded. Consequently, there is some $\eta > 0$, such that

$$\bigcup_{\lambda\in[a,b]} \{u \in X : \varphi(u, \lambda) < \sigma\} \subseteq B_\eta,$$

where $B_\eta = \{u \in X : ||u|| < \eta\}$.

Since $\bar{B}_\eta$ is bounded and sequentially weakly compact, invoking Lemma 2.3, we may define $\delta \in \mathbb{R}$ by putting

$$\delta = \mu \sup_{u\in\bar{B}_\eta} |I(u)| + \sup_{u\in\bar{B}_\eta} |\Phi_1(u)| + b \sup_{u\in\bar{B}_\eta} |\Phi_2(u)|$$

and set $\rho > \eta$ large enough such that for any $\lambda \in [a, b]$, we have

$$\{\mu I(u) + \Phi_1(u) + \lambda\Phi_2(u) \leqslant \delta + 2\} \subseteq B_\rho. \tag{3.4}$$

Furthermore, set $s = \sup_{u\in\bar{B}_\rho} |h(u)|$, and fix a function $g \in C_s$.

Now, we apply an argument analogous to that of [16]. Put $\Psi(u) := g(h(u))$, for every $u \in X$. Clearly $\Psi : X \to \mathbb{R}$ is a locally Lipschitz functional and by the chain rule (6) of Lemma 2.2, we get for every $u \in X$,

$$\partial\Psi(u) \subseteq g'(h(u))\partial h(u).$$

We prove that $\partial\Psi : X \to 2^{X^*}$ is a compact set-valued mapping. Let $\{u_n\}$ be a bounded sequence in $X$ and $u_n^* \in \partial\Psi(u_n)$ for every $n \in \mathbb{N}$, then there is a sequence $\{v_n^*\}$ in $X^*$ such that for every $n \in \mathbb{N}$, we have $v_n^* \in \partial h(u_n)$ and $u_n^* = g'(h(u_n))v_n^*$; since $\partial h$ is compact, up to a subsequence $v_n^* \to v^* \in X^*$, while, up to a subsequence, $g'(h(u_n)) \to m \in [0, 1]$ (by the Bolzano–Weierstrass theorem), so $u_n^* \to mv^*$.

Moreover, $\Psi$ is a bounded mapping on $X$, namely $\sup_{u\in X} |\Psi(u)| \leq N$. Hence, $\Psi$ is sequentially weakly continuous, then there exists $\nu_1 > 0$ such that, for each $\nu \in ]0, \nu_1[$, the functional $\varphi(u, \lambda) + \nu\Psi(u)$ restricted to $X$ has two local minimizers $u_1$ and $u_2$, lying in the set $\{u \in X : \varphi(u, \lambda) < \sigma\}$.

Set

$$\nu_0 = \min\left\{\nu_1, \frac{1}{\sup_{\mathbb{R}} |g|}\right\},$$

and fix $\nu \in ]0, \nu_0[$. Now, by applying Theorem 2.6, we are going to find a third critical point. With this aim in mind, we prove that $E_{(\lambda,\mu,\nu)}(u) = \mu I(u) + \Phi_1(u) + \lambda\Phi_2(u) + \nu\Psi(u)$ that $E_{(\lambda,\mu,\nu)}$ is a locally Lipschitz functional, satisfies the Palais–Smale condition.

Let $\{u_n\}$ be a sequence in $X$ complying conditions in Definition 2.5, by Lemma 2.2, implies for every $n \in \mathbb{N}$, $v \in X$,

$$\mu\langle I'(u_n), v - u_n\rangle + \Phi_1^0(u_n; v - u_n) + \lambda\Phi_2^0(u_n; v - u_n) + \nu\Psi^0(u_n; v - u_n) + \varepsilon_n||v - u_n|| \geqslant 0. \tag{3.5}$$

It follows from (3.3) and boundedness of $\Psi$ that $E_{(\lambda,\mu,\nu)}$ is coercive, so $\{u_n\}$ is bounded and, up to a subsequence $u_n \rightharpoonup u \in X$, choose $R > 0$ such that for every $n \in \mathbb{N}$,

$$||u_n - u|| < R,$$

and choose sequences $\{v_n^*\}$, $\{w_n^*\}$ and $\{z_n^*\}$ in $X^*$, such that, for every $n \in \mathbb{N}$, $v_n^* \in \partial\Phi_1(u_n)$, $w_n^* \in \partial\Phi_2(u_n)$ and $z_n^* \in \partial\Psi(u_n)$ and

$$\Phi_1^0(u_n; u_n - u) = \langle v_n^*, u_n - u\rangle,$$

$$\Phi_2^0(u_n; u_n - u) = \langle w_n^*, u_n - u\rangle,$$

$$\Psi^0(u_n; u_n - u) = \langle z_n^*, u_n - u\rangle.$$

By compactness of $\partial\Phi_1, \partial\Phi_2, \partial\Psi$ up to a subsequence $v_n^* \to v^* \in X^*$, $w_n^* \to w^* \in X^*$ and $z_n^* \to z^* \in X^*$. Fix $\varepsilon > 0$: from what was stated above, for $n \in \mathbb{N}$ big enough, we get

$$||v_n^* - v^*||_* < \frac{\varepsilon}{5R},\ ||w_n^* - w^*||_* < \frac{\varepsilon}{5\lambda R},\ ||z_n^* - z^*||_* < \frac{\varepsilon}{5\nu R},\ \varepsilon_n < \frac{\varepsilon}{5R},$$

and since $u_n \rightharpoonup u \in X$, for $v^* + \lambda w^* + \nu z^* \in X^*$, we have

$$< v^* + \lambda w^* + \nu z^*, u - u_n >< \frac{\varepsilon}{5}.$$

Then, from (3.5) we easily get for $n \in \mathbb{N}$ big enough $\langle I'(u_n), u_n - u\rangle < \varepsilon$, that is, $\limsup_n \langle I'(u_n), u_n - u\rangle \leqslant 0$. Since $I'$ is of type $(S)_+$, this implies that $u_n \to u$, thus $E_{(\lambda,\mu,\nu)}$ satisfies the Palais–Smale condition.

Define, as in Theorem 2.6,

$$\Gamma = \{\gamma \in C^0([0, 1], X) : \gamma(0) = u_1, \gamma(1) = u_2\},$$
$$e = \inf_{\gamma\in\Gamma} \sup_{\tau\in[0,1]} E_{(\lambda,\mu,\nu)}(\gamma(\tau)).$$

Then, there exists a critical point $u_3 \in X$ for $E_{(\lambda,\mu,\nu)}$ such that $u_3 \neq u_i$ $(i = 1, 2)$ and $E_{(\lambda,\mu,\nu)}(u_3) = e$. Now, we prove that $u_3 \in B_\rho$, define $\bar{\gamma} \in \Gamma$ by putting for every $l \in [0, 1]$, $\bar{\gamma}(l) = u_1 + l(u_2 - u_1)$, so $\bar{\gamma}(l) \in B_\eta$. We have

$$\begin{aligned}
\mu I(u_3) + \Phi_1(u_3) + \lambda\Phi_2(u_3) &= E_{(\mu,\lambda,\nu)}(u_3) - \nu\Psi(u_3) \\
&= c - \nu\Psi(u_3) \\
&\leqslant \sup_{l\in[0,1]} E_{(\mu,\lambda,\nu)}(\bar{\gamma}(l)) - \nu\Psi(u_3) \\
&= \sup_{l\in[0,1]} (\mu I(\bar{\gamma}(l)) + \Phi_1(\bar{\gamma}(l)) + \lambda\Phi_2(\bar{\gamma}(l)) \\
&\quad +\nu\Psi(\bar{\gamma}(l))) - \nu\Psi(u_3) \\
&\leqslant \sup_{u\in\bar{B}_\eta} (\mu I(u) + \Phi_1(u) + b|\Phi_2(u)|) \\
&\quad +2\nu_0 \sup_{\mathbb{R}} |g| \\
&\leqslant \delta + 2,
\end{aligned}$$

and from (3.4) it follows that $u_3 \in B_\rho$. Therefore, $u_i$ $(i = 1, 2, 3)$ are critical points for $E_{(\mu,\lambda,\nu)}$, all belonging to the ball $B_\rho$. It remains to prove that these elements are critical points not only for $E_{(\mu,\lambda,\nu)}$ but also for $\tilde{E}_{(\mu,\lambda,\nu)} = \mu I(u) + \Phi_1 + \lambda\Phi_2 + \nu h$. Let $u = u_i$, $i \in \{1, 2, 3\}$. Since $u \in B_\rho$, we have $|h(u)| \leqslant s$, and notice that for a function $g \in C_s$, $g(h(u)) = h(u)$. Consequently, on the open set $B_\rho$ the functionals $E_{(\mu,\lambda,\nu)}$ and $\tilde{E}_{(\mu,\lambda,\nu)}$ coincide, which completes the proof. □

## 4. An application

In this section, we apply Theorem 3.1 to ensure, under convenient assumptions, the existence of at least three solutions for a differential inclusion subject to anti-periodic boundary conditions. Our result also yields an estimate on the norms of the solutions.

Differential inclusions arise in models for control systems, mechanical systems, economical systems, game theory and biological systems, we refer the reader to [1–3] and [12].

Let us introduce the data of our problem: let $1 < p < +\infty$, $M \geqslant 0$ and $X$ is Banach space $\{u \in W^{1,p}([0, T]) : u(0) = -u(T)\}$ endowed with the norm

$$||u||_X = \left(\int_0^T |u'(x)|^p + M|u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \quad \text{for all } u \in X.$$

Since $X$ is a closed subset of $W^{1,p}([0, T])$, it is a reflexive Banach space.

We denote by $\mathcal{A}$ the class of all multifunctions $F : [0, T] \times \mathbb{R} \to 2^{\mathbb{R}}$ such that

(i) $F$ is upper semicontinuous (u.s.c.) with compact convex values;
(ii) $\min F, \max F : [0, T] \times \mathbb{R} \to \mathbb{R}$ are Borel measurable;
(iii) $|\xi| \leqslant a(1 + |s|^{p-1})$ for a.e. $x \in [0, T]$, all $s \in \mathbb{R}$ and $\xi \in F(x, s)$, $(p > 1, a > 0)$.

Now, consider the following problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' + M|u(x)|^{p-2}u(x) \in \mu F(x, u(x)) - \lambda G(x, u(x)) + \nu H(x, u(x)) \\ u(0) = -u(T), \quad u'(0) = -u'(T) \end{cases} \tag{4.1}$$

where $F, G, H \in \mathcal{A}$ and in first inclusion $x \in [0, T]$ and $\lambda, \mu, \nu$ are positive parameters. Let us define some functionals and fix some further notations that apply in the following theorem. We introduce for a.e. $x \in [0, T]$ and all $s \in \mathbb{R}$, the Aumann-type set-valued integrals

$$\int_0^s F(x,t)\mathrm{d}t = \left\{ \int_0^s f(x,t)\mathrm{d}t :\ f : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } F \right\},$$

$$\int_0^s G(x,t)\mathrm{d}t = \left\{ \int_0^s g(x,t)\mathrm{d}t :\ g : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } G \right\}$$

and

$$\int_0^s H(x,t)\mathrm{d}t = \left\{ \int_0^s h(x,t)\mathrm{d}t :\ h : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } H \right\}.$$

Also, set

$$\begin{aligned} \Phi_1(u) &= \int_0^T \min \int_0^u F(x,s)\,\mathrm{d}s\,\mathrm{d}x \quad \text{for all } u \in L^p[0,T], \\ \Phi_2(u) &= \int_0^T \min \int_0^u G(x,s)\,\mathrm{d}s\,\mathrm{d}x \quad \text{for all } u \in L^p[0,T], \\ \Psi(u) &= \int_0^T \min \int_0^u H(x,s)\,\mathrm{d}s\,\mathrm{d}x \quad \text{for all } u \in L^p[0,T]. \end{aligned}$$

For each $r > 0$, set

$$\tilde{\mu}(p, \Phi_1, \Phi_2, r) = p \inf \left\{ \frac{-\Phi_1(u) - \tilde{\gamma} + r}{\tilde{\eta}_r - ||u||_X^p} : u \in X,\ \Phi_2(u) < r, ||u||_X^p < \tilde{\eta}_r \right\},$$

where

$$\tilde{\gamma} = \inf_{x \in X} (\Phi_2(u) - \Phi_1(u))$$

and

$$\tilde{\eta}_r = \inf_{u \in \Phi_2^{-1}(r)} ||u||_X^p.$$

Finally, for each $\mu \in \left]0, \dfrac{1}{\max\{0, \tilde{\mu}(p, \Phi_1, \Phi_2, r)\}}\right[$, put

$$\tilde{\beta}(\mu, p, \Phi_1, \Phi_2, r) = \frac{1}{p} \sup_{x \in \Phi_2^{-1}(]r, +\infty[)} \frac{||u||_X^p - \mu p \Phi_1(u) - \inf_{u \in \Phi_2^{-1}(]-\infty, r])} (\|u\|_X^p - \mu p \Phi_1(u))}{r - \Phi_2(u)}.$$

DEFINITION 4.1

A function $u \in X$ is a (weak) solution of problem (4.1), if there exists $u^* \in L^{\gamma}[0, T]$ (for some $\gamma > 1$) such that

$$\int_0^T (|u'(x)|^{p-2}u'(x)v'(x)+M|u(x)|^{p-2}u(x)v(x)-u^*(x)v(x))\mathrm{d}x = 0 \quad (4.2)$$

for all $v \in X$ and $u^*(x) \in \mu F(x, u(x)) - \lambda G(x, u(x)) + \nu H(x, u(x))$ for a.e $x \in [0, T]$.

Here, our aim is to establish the existence of at least three solutions to problem (4.1) by the following theorem.

**Theorem 4.2.** *Let $F,\ G : [0, T] \times \mathbb{R} \to 2^{\mathbb{R}}$ be set-valued functions belonging to $\mathcal{A}$. We set*

$$F_x(s) = \min \int_0^s F(x, t)\mathrm{d}t \quad \textit{for all } s \in \mathbb{R}, \textit{a.e. } x \in [0, T],$$

*and*

$$G_x(s) = \min \int_0^s G(x, t)\mathrm{d}t \quad \textit{for all } s \in \mathbb{R}, \textit{a.e. } x \in [0, T].$$

*Moreover, assume that*

$(F_1)$ $\displaystyle\lim_{s\to+\infty} \frac{\inf_{x\in[0,T]} F_x(s)}{s^p} = +\infty$ *for all $s \in \mathbb{R}$;*

$(F_2)$ *there is $c > 0$ such that $F_x(s) \leqslant c(1 + |s|^l)$, $(1 < p < l < +\infty)$ for all $s \in \mathbb{R}$, a.e. $x \in [0, T]$;*

$(G)$ *for each $b > 0$, there exists a constant $d_b > 0$ such that $G_x(s) \geqslant b|s|^l - d_b$, $(1 < p < l < +\infty)$ for all $s \in \mathbb{R}$, a.e. $x \in [0, T]$.*

*Then, under such hypotheses, for each $r > 0$, for each $\mu \in \left]0, \frac{1}{\max\{0, \tilde{\mu}(p, \Phi_1, \Phi_2, r)\}}\right[$ and for each compact interval $[\lambda_1, \lambda_2] \subset ]0, \tilde{\beta}(\mu, p, \Phi_1, \Phi_2, r)[$, there exists a number $\rho > 0$ with the following property: for every $\lambda \in [\lambda_1, \lambda_2]$ and every function $H \in \mathcal{A}$, there exists $\nu_0 > 0$ such that for each $\nu \in [0, \nu_0]$, the problem* (4.1) *admits at least three distinct solutions in $X$ whose norms are less than $\rho$.*

Before the proof of Theorem 4.2 we have some lemmas. First, we set

$$I(u) = \frac{||u||^p}{p}, \quad \text{for all } u \in X.$$

So, $I \in C^1(X)$ is weakly lower semi-continuous.

*Lemma* 4.3. *$I : X \to X^*$ is of type $(S)_+$.*

*Proof.* By definition of $I(u) = \frac{1}{p}||u||_X^p$ and its gradient, one has

$$\langle I'(u), v\rangle = \int_0^T |u'(x)|^{p-2}u'(x)v'(x) + M|u(x)|^{p-2}u(x)v(x)\mathrm{d}x$$

for every $v \in X$, it is clear that if $u_n \rightharpoonup u$ and $\limsup_n \langle I'(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$. □

*Lemma* 4.4 (*Lemma* 3.1 *of* [10]). *The functionals* $\Phi_1, \Phi_2, \Psi : L^p[0, T] \to \mathbb{R}$ *are well defined and Lipschitz on any bounded subset of* $L^p[0, T]$. *Moreover for all* $u \in L^p[0, T]$, *and all* $u^* \in \partial\Phi_1(u) - \partial\Phi_2(u) + \partial\Psi(u)$, $u^*(x) \in F(x, u(x)) - G(x, u(x)) + H(x, u(x))$ *a.e* $x \in [0, T]$.

In order to obtain existence of solutions to problem (4.1), we consider the following functional:

$$\mathcal{N}(u) := \frac{1}{p}||u||_X^p - \mu\Phi_1(u) + \lambda\Phi_2(u) - \nu\Psi(u) \quad \text{for all, } u \in X.$$

*Lemma* 4.5 *The functional* $\mathcal{N} : X \to \mathbb{R}$ *is locally Lipschitz. Moreover, if* $u \in X$ *is a critical point of functional* $\mathcal{N}(u)$, *then* $u$ *is a solution of* (4.1).

*Proof.* Let $\mathcal{N}(u) = I(u) + \mathcal{N}_1(u)$, where $\mathcal{N}_1(u) = -\mu\Phi_1(u) + \lambda\Phi_2(u) - \nu\Psi(u)$. By Lemma 2.1, $I$ is locally Lipschitz on $X$, and by Lemma 4.4, $\mathcal{N}_1$ is locally Lipschitz on $L^p[0, T]$. Moreover, $X$ is compactly embedded into $L^p[0, T]$, so $\mathcal{N}_1$ is locally Lipschitz on $X$. Therefore $\mathcal{N}(u)$ is locally Lipschitz on $X$.

Now, let $u \in X$ be a critical point of $\mathcal{N}$: this means that

$$0 \in \partial\mathcal{N} = \{u^* \in X^* : \langle u^*, v\rangle \leqslant \mathcal{N}^\circ(u; v) \quad \text{for all } v \in X\}. \tag{4.3}$$

By (2) in Lemma 2.1 and (4), (5) in Lemma 2.2, condition (4.3) implies that $0 \in I'(u) - \partial(\mu\Phi_1(u) - \lambda\Phi_2(u) + \nu\Psi(u))$ i.e. there exists $u^* \in \partial(\mu\Phi_1(u) - \lambda\Phi_2(u) + \nu\Psi(u))$ satisfying

$$I'(u) = u^* \quad \text{in } X^*. \tag{4.4}$$

We extend $u^*$ to an element of $L^q[0, T]$ ($\frac{1}{p} + \frac{1}{q} = 1$). Here, we regard $X$ as a closed subspace of $L^p[0, T]$. First, we observe that $u^*$, as a linear functional on $X$, is continuous also with respect to the topology induced by the norm $||.||_{L^p}$. Indeed, by Lemma 4.4, $\mathcal{N}_1(u)$ admits a Lipschitz constant $L$ around $u$ with respect to $||.||_{L^p}$. Then, by (3) of Lemma 2.2, we get

$$\langle u^*, v\rangle \leqslant L||v||_p \quad \text{for all } v \in X. \tag{4.5}$$

Moreover $(\mathcal{N}_1)^\circ(u; .)$ is subadditive and positively homogeneous on $L^p[0, T]$ and

$$\langle u^*, v\rangle \leqslant (\mathcal{N}_1)^\circ(u; .) \quad \text{for all } v \in X. \tag{4.6}$$

By the Hahn–Banach theorem, $u^*$ extends to a bounded linear functional defined on $L^p[0, T]$ satisfying (4.6) for all $v \in L^p[0, T]$. This implies that we may assume $u^* \in L^q[0, T]$ and rephrase (4.4) as

$$\int_0^T (|u'(x)|^{p-2}u'(x)v'(x) + M|u(x)|^{p-2}u(x)v(x) - u^*(x)v(x))\mathrm{d}x = 0,$$

for all $v \in X$, and also by Lemma 4.4, we have $u^*(x) \in \mu F(x, u(x)) - \lambda G(x, u(x)) + \nu H(x, u(x))$ for a.e $x \in [0, T]$. Thus by Definition 4.1, $u$ is a solution of (4.1). □

*Lemma* 4.6 *$\partial\Phi_1, \partial\Phi_2 : [0, T] \times X \to 2^{X^*}$ are compact.*

*Proof.* We prove compactness of $\partial\Phi_1$ and proving of compactness of $\partial\Phi_2$ is similar. Let us fix a bounded sequence $\{u_n\}$ in $X$ and $u_n^* \in \partial\Phi_1(u_n)$ for all $n \in \mathbb{N}$. Let $L > 0$ be a Lipschitz constant for $\Phi_1$, then $||u_n^*||_{X^*} \leqslant L$ for all $n \in \mathbb{N}$. Hence, up to a subsequence $u_n^* \rightharpoonup u^* \in X^*$. We shall prove that (up to a subsequence) the convergence is strong. Arguing by contradiction, let us assume that there is some $\varepsilon > 0$ such that $||u_n^* - u^*||_* > \varepsilon$ for every $n \in \mathbb{N}$, and hence for all $n \in \mathbb{N}$ there is a $v_n \in X$ with $||v_n||_X < 1$ such that

$$\langle u_n^* - u^*, v_n\rangle > \varepsilon. \tag{4.7}$$

Passing if necessary to a subsequence, we can assume that $v_n \rightharpoonup v \in X$, while $||v_n - v||_p \to 0$ and $||v_n - v||_1 \to 0$. From (iii) we easily get

$$\begin{aligned}\langle u_n^* - u^*, v_n\rangle &= \langle u_n^*, v_n - v\rangle + \langle u_n^* - u^*, v\rangle + \langle u^*, v - v_n\rangle \\ &\leqslant a_1(||v_n - v||_1 + ||v_n - v||_p) + \langle u_n^* - u^*, v\rangle \\ &\quad + \langle u^*, v - v_n\rangle\end{aligned}$$

($a_1 > 0$), and the latter tends to 0 as $n \to \infty$, which contradicts (4.7). □

Now we are ready to prove Theorem 4.2.

*Proof of Theorem* 4.2

*Proof.* By definition of functional $I$ and Lemma 4.3, clearly the assumptions of Theorem 3.1 satisfy for $I$. Also, by Lemmas 4.4 and 4.6, $\Phi_1, \Phi_2 : X \to \mathbb{R}$ are locally Lipschitz functional with compact gradients.

Now, we shall verify the conditions (1) and (2) in Theorem 3.1. From the assumption $(F_1)$ it follows that

$$\limsup_{||u||\to+\infty} \frac{\Phi_1(u)}{||u||^p} = +\infty,$$

which is proved in the proof of Theorem 4 of [17]. Moreover, from the assumptions $(F_2)$ and $(G)$, it clearly follows that for each $\lambda > 0$, the function $\lambda G_x - F_x$ is bounded below, and so the functional $\lambda\Phi_2 - \Phi_1$ is bounded below in $X$. Therefore, if we consider $-\Phi_1(u)$ as $\Phi_1$ in Theorem 3.1, conditions (1) and (2) are fulfilled.

Let $r > 0$, $\mu$ and $[\lambda_1, \lambda_2]$ be as in Theorem 4.2. Let us choose $\lambda \in [\lambda_1, \lambda_2]$ and a multifunction $H \in \mathcal{A}$. Set

$$\Psi(u) = \int_0^T \min \int_0^u H(x, s)\mathrm{d}s\mathrm{d}x \quad \text{for all } u \in X.$$

By Lemma 4.4 and compactly embedding $X$ into $L^p[0, T]$, it follows that the functional $\Psi(u) : X \to \mathbb{R}$ is locally Lipschitz. Similar to the argument analogous of Lemma 4.6, $\partial\Psi$ is compact. Then assumptions of Theorem 3.1 satisfy and there is $\nu_0 > 0$ such that, for all, $\nu \in [0, \nu_0]$, the functional $\mathcal{N}(u) = \frac{1}{p}||u||_X^p - \mu\Phi_1(u) + \lambda\Phi_2(u) - \nu\Psi(u)$ admits at least three critical points $u_1, u_2, u_3 \in X$ with $||u_i||_X < \rho$ $(i = 1, 2, 3)$, such that by Lemma 4.5, $u_1, u_2, u_3$ are three solutions of problem (4.1). □

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## References

[1] Blagodatskikh V I and Filippov A F, Differential inclusions and optimal contal, in: Topology, differential equations, dynamical systems, *Tr. Mat. Inst. Steklova* **169** (1985) 194–252

[2] Bulgakov A I, Integral inclusions with nonconvex images and their applications to boundary value problems for differential inclusions, *Mat. Sb. 183* **10** (1992) 63–86

[3] Carlson D A, Carathéodory's method for a class of dynamic games, *J. Math. Anal. Appl.* **276(2)** (2002) 561–588

[4] Chang K C, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80** (1981) 102–129

[5] Clarke F H, Optimization and nonsmooth analysis (1983) (Hohn Wiley Sons)

[6] Cordaro G, On a minimax problem of Ricceri, *J. Inequal. Appl.* **6** (2001) 261–285

[7] Faraci F and Iannizzotto A, Three nonzero periodic solutions for a differential inclusion, *Discrete Contin. Dyn. Syst. Ser. S* **5** (2012) 779–788

[8] Faraci F and Iannizzotto A, Three solutions for a Dirichlet problem with one-sided growth conditions on the nonlinearities, *Nonlinear Anal.* **78** (2013) 121–129

[9] Iannizzotto A, Three critical points for perturbed nonsmooth functionals, *Nonlinear Anal.* **72** (2010) 1319–1338

[10] Iannizzotto A, Three periodic solutions for an ordinary differential inclusion with two parameters, *Ann. Polon. Math.* **103(1)** (2011) 89–100

[11] Iannizzotto A, Three solutions for a partial differential inclusion via nonsmooth critical point theory, *Set-Valued Var. Anal.* **19** (2011) 311–327

[12] Krasovskiĭ N N and Subbotin A I, Game-theoretical control problems, in: Springer Series in Soviet Mathematics (1988) (Springer, New York)

[13] Kristály A, Marzantowicz W and Varga C, A non-smooth three critical points theorem with applications in differential inclusions, *J. Glob. Optim.* **46** (2010) 49–62

[14] Livrea R and Marano S A, Existence and classification of critical points for nondifferentiable functions, *Adv. Diff. Equa.* **9** (2004) 961–978

[15] Marano S A and Motreanu D, On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems, *Nonlinear Anal.* **48** (2002) 37–52

[16] Ricceri B, A three critical points theorem revisited, *Nonlinear Anal.* **70** (2009) 3084–3089

[17] Ricceri B, A further refinement of a three critical points theorem, *Nonlinear Anal.* **74** (2011) 7446–7454

[18] Ricceri B, Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set, *Topol. Appl.* **153** (2006) 3308–3312

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