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Regions of variability for a class of analytic and locally univalent functions defined by subordination

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Abstract. In this article, we consider a family $\mathcal{C}(A, B)$ of analytic and locally univalent functions on the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane that properly contains the well-known Janowski class of convex univalent functions. In this article, we determine the exact set of variability of $\log(f'(z_0))$ with fixed $z_0 \in \mathbb{D}$ and $f''(0)$ whenever f varies over the class $\mathcal{C}(A, B)$.

Keywords. Janowski class; univalent functions; variability regions.

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1. Introduction and preliminary results

Let $\mathbb{C}$ be the complex plane. We use the following notations for open and closed discs with center c and radius r in the complex plane: $\mathbb{D}(c, r) := \{z \in \mathbb{C} : |z-c| < r\}$ and $\bar{\mathbb{D}}(c, r) := \{z \in \mathbb{C} : |z - c| \leq r\}$. We also denote $\mathbb{D} := \mathbb{D}(0, 1)$ and $\bar{\mathbb{D}} := \bar{\mathbb{D}}(0, 1)$. Let $\mathcal{A}$ be the class of functions f that are analytic in $\mathbb{D}$ having the normalization $f(0) = 0 = f'(0) - 1$ and $\mathcal{S} := \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}$. A function f is said to be *locally univalent* in $\mathbb{D}$ if for any $z_0 \in \mathbb{D}$, it is univalent in some neighborhood of z_0. A necessary and sufficient condition for an analytic function to be locally univalent in $\mathbb{D}$ is $f'(z) \neq 0$ in $\mathbb{D}$. We now state the definition of *subordination* which we need for further discussion. Let h and g be analytic functions in $\mathbb{D}$. We say h is *subordinate* to g (abbreviated symbolically as $h \prec g$) if there exists an analytic function $\xi : \mathbb{D} \to \mathbb{D}$ such that $\xi(0) = 0$ and $h(z) = g(\xi(z))$. If g is one-one in $\mathbb{D}$, then $h \prec g$ if and only if $h(\mathbb{D}) \subseteq g(\mathbb{D})$ with $h(0) = g(0)$. We say the afore mentioned function $\xi : \mathbb{D} \to \mathbb{D}$, which is analytic in $\mathbb{D}$ with $\xi(0) = 0$ as the *Schwarz function* in literature.

In 1973, Witold Janowski [2] considered the following class of functions:

$$J_{A,B} : = \{f \in \mathcal{A} : \text{ there exists a Schwarz function } \psi \text{ in } \mathbb{D} \text{ such that } 1 + \frac{zf''(z)}{f'(z)} = \frac{1 + A\psi(z)}{1 + B\psi(z)}\right\},$$

where A and B are real constants with $-1 \leq A < B \leq 1$. We now observe that for $f \in \mathcal{A}$, $f \in J_{A,B}$ if and only if the quantity $\left(1 + \frac{zf''(z)}{f'(z)}\right)$ belongs to the disk which has the line segment $\left[\frac{1+A}{1+B}, \frac{1-A}{1-B}\right]$ as a diameter, i.e.,

$$1 + \frac{zf''(z)}{f'(z)} \in \mathbb{D}\left(\frac{1-AB}{1-B^2}, \frac{B-A}{1-B^2}\right) \quad \forall z \in \mathbb{D}.$$

Since the above disk is contained in the right half plane, each $f \in J_{A,B}$ is convex univalent in $\mathbb{D}$. In particular, we have $J_{A,B} \subsetneq \mathcal{S}$. We refer to the articles [2, 4] for details and many more interesting results about the Janowski class. For a function $f \in J_{A,B}$, there exists a Schwarz function ψ such that the following holds:

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= \frac{1 + A\psi(z)}{1 + B\psi(z)} \\ \Rightarrow 1 + \frac{zf''(z)}{f'(z)} &\prec \frac{1 + Az}{1 + Bz} \\ \Rightarrow \frac{zf''(z)}{f'(z)} &\prec \frac{(A-B)z}{1 + Bz}. \end{aligned}$$

Now letting $f'(z) = p(z)$, the last implication yields

$$\frac{zp'(z)}{p(z)} \prec \frac{(A-B)z}{1+Bz} =: \phi(z). \tag{1.1}$$

We now apply the result stated in Corollary 3.1d.1, p. 76 of [3] to (1.1) and get

$$f'(z) = p(z) \prec \exp \int_0^z \frac{\phi(t)}{t}\, \mathrm{d}t =: q_{A,B}(z).$$

A little computation reveals that

$$q_{A,B}(z) = \begin{cases} (1 + Bz)^{\frac{A}{B}-1}, & \text{for } B \neq 0, \\ \exp(Az), & \text{for } B = 0. \end{cases}$$

The above discussion motivates us to consider the family $\mathcal{C}(A, B)$ as defined below:

$$\begin{aligned} \mathcal{C}(A,B) = \Big\{ f : f &\text{ is analytic and locally univalent in } \mathbb{D} \text{ with } f(0) = f'(0) - 1 = 0 \\ &\text{satisfying } \log f'(z) \prec \left(\frac{A}{B} - 1\right)\log(1 + Bz),\ z \in \mathbb{D} \Big\}; \end{aligned}$$

where $-1 \leq A < B \leq 1$, $B \neq 0$. Here we clarify that, since f' is a non-vanishing function on $\mathbb{D}$, then $(f')^\lambda$ is well-defined and holomorphic once we choose the determination $(f'(0))^\lambda = 1$ for all $\lambda \in \mathbb{R}$. In other words, we chose that branch of logarithm for which $\log(f'(0)) = 0$. By definition, we have $J_{A,B} \subseteq \mathcal{C}(A, B)$. At this point, we only confirm that the class $\mathcal{C}(A, B)$ contains functions from $\mathcal{S}$. We can see this if we consider $A = 0$ and $0 < B \leq 1$. We then have from the definition of the class $\mathcal{C}(A, B)$,

$$f'(z) = \frac{1}{1 + B\omega(z)},$$

where ω is a Schwarz function. A little computation shows that

$$\operatorname{Re} f'(z) = \operatorname{Re}\left(\frac{1}{1+B\omega(z)}\right) > \frac{1}{2} \quad \text{if } 0 < B \leq 1.$$

Therefore, an application of the Noshiro–Warschawski theorem yields that f is univalent in $\mathbb{D}$. Hence, if $A = 0$ and $0 < B \leq 1$, we see that $\mathcal{C}(A, B) \subseteq \mathcal{S}$. We remark here that the class $\mathcal{C}(A, B)$ defined above may or may not be a subclass of $\mathcal{S}$ for all $-1 \leq A < B \leq 1$, $B \neq 0$. We leave this problem open.

We next claim that $J_{A,B} \subsetneq \mathcal{C}(A, B)$. In order to establish our claim, we let $f \in \mathcal{C}(A, B)$. Hence there exists a Schwarz function ψ such that

$$\log f'(z) = \left(\frac{A}{B} - 1\right)\log(1 + B\psi(z)), \quad z \in \mathbb{D}.$$

We deduce the following after differentiating the above expression,

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{(A-B)z\psi'(z)}{1 + B\psi(z)}, \quad z \in \mathbb{D}.$$

In particular, letting $\psi(z) = z^2$, the above expression simplifies into

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + (2A - B)z^2}{1 + Bz^2}, \quad z \in \mathbb{D}.$$

The function $\frac{1+(2A-B)z^2}{1+Bz^2}$ maps $\mathbb{D}$ onto the disc that have the line segment $\left[\frac{1+2A-B}{1+B}, \frac{1-2A+B}{1-B}\right]$ as a diameter. Since $\frac{1+2A-B}{1+B} < \frac{1+A}{1+B}$, it follows that $f \notin J_{A,B}$. Hence, in particular, $\mathcal{C}(A, B) \setminus J_{A,B} \neq \phi$.

Finally, for a function $f \in \mathcal{C}(A, B)$, we have

$$f'(z) \prec (1 + Bz)^{\frac{A}{B}-1}.$$

Hence there exists an analytic function ω, bounded by unity with $\omega(0) = 0$ such that

$$f'(z) = (1 + B\omega(z))^{\frac{A}{B}-1};$$

from which we have

$$\omega(z) = \frac{(f'(z))^{\frac{B}{A-B}} - 1}{B}. \tag{1.2}$$

Thereafter, we calculate

$$\omega'(0) = \frac{f''(0)}{A - B}.$$

Denoting $\omega'(0) = \lambda$, we have $|\lambda| \leq 1$ by virtue of the Schwarz lemma. Hence we fix the second Taylor coefficients for functions in $\mathcal{C}(A, B)$ as

$$f''(0) = \lambda(A - B); \quad \lambda \in \bar{\mathbb{D}}.$$

We now define the following class:

$$\begin{aligned} \mathcal{C}_\lambda(A,B) &= \{f \in \mathcal{C}(A,B) : f''(0) = \lambda(A-B)\}, \quad \text{and set} \\ V_\lambda(z_0,A,B) &= \{\log f'(z_0) : f \in \mathcal{C}_\lambda(A,B)\}; \end{aligned}$$

where $\lambda \in \bar{\mathbb{D}}$ and $z_0 \in \mathbb{D}$ is an arbitrary but fixed complex number. In this paper, we wish to determine the regions of variability $V_\lambda(z_0, A, B)$ of $\log f'(z_0)$ when f ranges over the class $\mathcal{C}_\lambda(A, B)$. Here we would like to mention that similar problems on regions of variability for some subclasses of $\mathcal{S}$ have already been studied in the articles [1, 5, 6] and some references therein.

2. Discussion on the set $V_\lambda(z_0, A, B)$ and the main result

We start this section by listing some basic properties of the set $V_\lambda(z_0, A, B)$:

(1) We first note that $\mathcal{C}_\lambda(A, B)$ is a compact subset of the class of analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. As we know that for fixed $z_0 \in \mathbb{D}$, the map $\mathcal{C}_\lambda(A, B) \ni f \mapsto \log(f'(z_0))$ is continuous, therefore we conclude that the set $V_\lambda(z_0, A, B)$ is a compact subset of $\mathbb{C}$.
(2) If $|\lambda| = 1$, then by applying Schwarz lemma, we have $\omega(z) = \lambda z$ and consequently we deduce from (1.2) that

$$V_\lambda(z_0,A,B) = \left\{\left(\frac{A-B}{B}\right)\log(1+B\lambda z_0)\right\}.$$

If $z_0 = 0$, then $V_\lambda(z_0, A, B) = \{0\}$. Now, for $\lambda \in \mathbb{D}$ and $a \in \bar{\mathbb{D}}$, we introduce the following functions

$$\begin{aligned} \delta(z,\lambda) &= \frac{z+\lambda}{1+\bar{\lambda}z}, \quad z \in \mathbb{D}, \quad \text{and} \\ F_{a,\lambda}(z) &= \int_0^z (1+B\zeta\delta(a\zeta,\lambda))^{\frac{A-B}{B}}\,\mathrm{d}\zeta. \end{aligned} \tag{2.1}$$

The above integral equation yields

$$\log F'_{a,\lambda}(z) = \left(\frac{A-B}{B}\right)\log\left(1+Bz\delta(az,\lambda)\right).$$

Thus we have

$$\log F'_{a,\lambda}(z) \prec \left(\frac{A}{B}-1\right)\log(1+Bz)$$

and $F''_{a,\lambda}(0) = \lambda(A - B)$, showing that $F_{a,\lambda} \in \mathcal{C}_\lambda(A, B)$. We observe that, for a fixed $\lambda \in \mathbb{D}$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the function $a \mapsto \log F'_{a,\lambda}(z_0)$ is a non-constant analytic function and hence is an open mapping. From this, we see that

$$\log F'_{0,\lambda}(z_0) = \left(\frac{A-B}{B}\right)\log(1+B\lambda z_0)$$

is an interior point of $\{\log F'_{a,\lambda}(z_0) : a \in \mathbb{D}\} \subset V_\lambda(z_0, A, B)$.

(3) We claim that $V_\lambda(e^{i\theta}z_0, A, B) = V_{\lambda e^{i\theta}}(z_0, A, B)$ for $\theta \in \mathbb{R}$. This will be established if we can show that $f \in \mathcal{C}_\lambda(A, B)$ if and only if $e^{-i\theta} f(e^{i\theta}z) \in \mathcal{C}_{\lambda e^{i\theta}}(A, B)$. To prove this, let $f \in \mathcal{C}_\lambda(A, B)$ and define

$$g(z) = e^{-i\theta} f(e^{i\theta}z), \quad z \in \mathbb{D}.$$

A little computation reveals that

$$\begin{aligned} &g''(0) = e^{i\theta} f''(0) = (A - B)\lambda e^{i\theta} \quad \text{and} \\ &\log g'(z) = \log f'(e^{i\theta}z) = \left(\frac{A-B}{B}\right)\log(1 + B\omega(e^{i\theta}z)). \end{aligned}$$

Hence we have

$$\log g'(z) \prec \left(\frac{A-B}{B}\right)\log(1 + Bz),$$

proving $g(z) \in \mathcal{C}_{\lambda e^{i\theta}}(A, B)$. Conversely, let

$$g(z) = e^{-i\theta} f(e^{i\theta}z) \in \mathcal{C}_{\lambda e^{i\theta}}(A, B).$$

Hence we have

$$g''(0) = \lambda e^{i\theta}(A - B) = e^{i\theta} f''(0),$$

which results in $f''(0) = \lambda(A - B)$. Next we observe the following chain of implications:

$$\begin{aligned} &\log g'(z) \prec \left(\frac{A-B}{B}\right)\log(1 + Bz) \\ &\quad \Rightarrow \log f'(e^{i\theta}z) \prec \left(\frac{A-B}{B}\right)\log(1 + Bz) \\ &\quad \Rightarrow \log f'(z) \prec \left(\frac{A-B}{B}\right)\log(1 + Bz). \end{aligned}$$

This proves $f \in \mathcal{C}_\lambda(A, B)$.

In view of the aforementioned properties of the set $V_\lambda(z_0, A, B)$, it is sufficient to determine $V_\lambda(z_0, A, B)$ for $\lambda \in [0, 1)$, as the case $|\lambda| = 1$ is completely described in the Item (2) above. We now state our main theorem of the paper.

Theorem 2.2. *For $\lambda \in [0, 1)$ and $z_0 \in \mathbb{D} \setminus \{0\}$, we have*

$$V_\lambda(z_0, A, B) = \left\{\left(\frac{A-B}{B}\right)\log\left(c(z_0, \lambda) + ar(z_0, \lambda)\right) : |a| \leq 1\right\},$$

where

$$\begin{aligned} c(z_0, \lambda) &= \frac{1 - \lambda^2|z_0|^2 + \lambda B(1 - |z_0|^2)z_0}{1 - \lambda^2|z_0|^2} \quad \textit{and} \\ r(z_0, \lambda) &= \frac{|B|(1 - \lambda^2)|z_0|^2}{1 - \lambda^2|z_0|^2}. \end{aligned}$$

The boundary $\partial V_\lambda(z_0, A, B)$ *of the above set is the Jordan curve given by*

$$\begin{aligned}(-\pi, \pi] \ni \theta \ \mapsto \ & \log F'_{e^{i\theta},\lambda}(z_0) \\ = \ & \left(\frac{A-B}{B}\right) \log(1 + Bz_0\delta(e^{i\theta} z_0, \lambda)). \end{aligned} \tag{2.2}$$

If $\log f'(z_0) = \log F'_{e^{i\theta},\lambda}(z_0)$ *for some* $f \in \mathcal{C}_\lambda(A, B)$ *and* $\theta \in (-\pi, \pi]$, *then* $f(z) = F_{e^{i\theta},\lambda}(z)$. *Here* $F_{e^{i\theta},\lambda}(z)$ *is given by* (2.1) *with* $a = e^{i\theta}$.

We prove Theorem 2.2 in the following section.

3. Description of the set $V_\lambda(z_0, A, B)$

At the beginning, we establish the following result which will help us to achieve our main goal, i.e. to prove Theorem 2.2.

PROPOSITION 3.1

For $f \in \mathcal{C}_\lambda(A, B)$ *with* $\lambda \in [0, 1)$, *we have*

$$\left|(f'(z))^{\frac{B}{A-B}} - c(z,\lambda)\right| \le r(z,\lambda), \quad z \in \mathbb{D}, \tag{3.2}$$

where

$$\begin{aligned} c(z,\lambda) &= \frac{1-\lambda^2|z|^2 + \lambda B(1-|z|^2)z}{1-\lambda^2|z|^2} \quad \text{and} \\ r(z,\lambda) &= \frac{|B|(1-\lambda^2)|z|^2}{1-\lambda^2|z|^2}. \end{aligned}$$

For each $z \in \mathbb{D}\setminus\{0\}$, *equality holds in* (3.2) *if and only if* $f = F_{e^{i\theta},\lambda}$ *for some* $\theta \in \mathbb{R}$.

Proof. Let $f \in \mathcal{C}_\lambda(A, B)$. Then, as stated in the Introduction, there exists an ω such that

$$\omega(z) = \frac{(f'(z))^{\frac{B}{A-B}} - 1}{B}, \quad z \in \mathbb{D}.$$

Since ω is a Schwarz function that satisfies the condition $\omega'(0) = \lambda$, it follows from the Schwarz lemma that

$$\left|\frac{\frac{\omega(z)}{z} - \lambda}{1 - \lambda\frac{\omega(z)}{z}}\right| \le |z|, \quad z \in \mathbb{D}. \tag{3.3}$$

Now we insert the above expression of $\omega(z)$ in the inequality (3.3) and we get

$$\left|\frac{(f'(z))^{\frac{B}{A-B}} - M(z,\lambda)}{\lambda(f'(z))^{\frac{B}{A-B}} - N(z,\lambda)}\right| \le |z|, \quad z \in \mathbb{D}, \tag{3.4}$$

where

$$\begin{cases} M(z,\lambda) = 1 + \lambda Bz, \\ N(z,\lambda) = \lambda + Bz. \end{cases} \tag{3.5}$$

A calculation shows that the inequality (3.4) is equivalent to the following inequality:

$$\left|(f'(z))^{\frac{B}{A-B}} - \frac{M-\lambda|z|^2N}{1-\lambda^2|z|^2}\right| \leq \frac{|\lambda M - N||z|}{1-\lambda^2|z|^2}, \quad z \in \mathbb{D}. \tag{3.6}$$

We now simplify the following expressions using (3.5) as

$$\frac{M-\lambda|z|^2N}{1-\lambda^2|z|^2} = \frac{1-\lambda^2|z^2| + \lambda B(1-|z|^2)z}{1-\lambda^2|z|^2}$$

and

$$\frac{|\lambda M - N||z|}{1-\lambda^2|z|^2} = \frac{|B|(1-\lambda^2)|z|^2}{1-\lambda^2|z|^2}.$$

Therefore, the inequality (3.2) follows from the last two equalities and (3.6). We now turn to the case of establishing sharpness of the inequality (3.2). Our first aim is to prove that the equality occurs for any $z \in \mathbb{D}$ in (3.2) whenever $f = F_{e^{i\theta},\lambda}$, for some $\theta \in \mathbb{R}$. To this end, we first show that if $f = F_{e^{i\theta},\lambda}$ then

$$(F'_{e^{i\theta},\lambda}(z))^{\frac{B}{A-B}} = 1 + B\left(\frac{e^{i\theta}z+\lambda}{1+\lambda z e^{i\theta}}\right)z, \quad \theta \in \mathbb{R}.$$

Astraightforward computation reveals that

$$\begin{aligned}
&(F'_{e^{i\theta},\lambda}(z))^{\frac{B}{A-B}} - c(z,\lambda) \\
&\quad = 1 + B\left(\frac{e^{i\theta}z+\lambda}{1+\lambda e^{i\theta}z}\right)z - \frac{1-\lambda^2|z^2| + \lambda B(1-|z|^2)z}{1-\lambda^2|z|^2} \\
&\quad = \left(\frac{B(1-\lambda^2)z^2}{1-\lambda^2|z|^2}\right)\left(\frac{e^{i\theta}+\lambda\bar{z}}{1+z\lambda e^{i\theta}}\right), \quad z \in \mathbb{D}.
\end{aligned} \tag{3.7}$$

The first part of the case of equality follows from (3.7) by noting that

$$\left|\frac{e^{i\theta}+\lambda\bar{z}}{1+z\lambda e^{i\theta}}\right| = 1, \quad z \in \mathbb{D};$$

for the aforesaid range of values of θ and λ. Conversely, if the equality holds for some $z \in \mathbb{D} \setminus \{0\}$ in (3.2), then the equality must hold in (3.3). Consequently, from the Schwarz lemma, there exists $\theta \in \mathbb{R}$ such that $\omega(z) = z\delta(e^{i\theta}z, \lambda)$ for all $z \in \mathbb{D}$, i.e. we have

$$\frac{(f'(z))^{\frac{B}{A-B}} - 1}{B} = z\left(\frac{e^{i\theta}z+\lambda}{1+\lambda e^{i\theta}z}\right),$$

which after solving for f results in $f(z) = F_{e^{i\theta},\lambda}(z)$, $z \in \mathbb{D}$. This completes the proof of the Proposition. □

Next, we get the following useful estimate in the case of $\lambda = 0$ that one may look for.

COROLLARY 3.8

Let $f \in \mathcal{C}_0(A,B)$. *Then we have the following sharp inequality*:

$$|(f'(z))^{\frac{B}{A-B}} - 1| \leq |B||z|^2, \quad z \in \mathbb{D}. \tag{3.9}$$

We are now ready to deliver a proof of our main theorem after all these preparation.

Proof of Theorem 2.2. We begin the proof by defining the following class of functions:

$$H^\infty(\mathbb{D}) = \{w : w \text{ is analytic in } \mathbb{D} \text{ with } |w(z)| \leq 1, z \in \mathbb{D}\}.$$

We observe that $f \in \mathcal{C}(A,B)$ if and only if there exists $\varphi \in H^\infty(\mathbb{D})$ such that

$$f'(z) = (1 + Bz\varphi(z))^{\frac{A-B}{B}}, \quad z \in \mathbb{D}.$$

Furthermore, $f \in \mathcal{C}_\lambda(A,B)$ if and only if there exists $w \in H^\infty(\mathbb{D})$ such that

$$\begin{aligned} f'(z) &= \left(1 + Bz\frac{zw(z)+\lambda}{1+\bar{\lambda}zw(z)}\right)^{\frac{A-B}{B}} \\ &= (1 + Bz\delta(zw(z),\lambda))^{\frac{A-B}{B}}, \quad z \in \mathbb{D}. \end{aligned}$$

Therefore we get

$$\begin{aligned} V_\lambda(z_0, A, B) &= \{\log f'(z_0) : f \in \mathcal{C}_\lambda(A,B)\} \\ &= \left\{\left(\frac{A-B}{B}\right)\log\left(1 + Bz_0\delta(z_0w(z_0),\lambda)\right) : w \in H_1^\infty(\mathbb{D})\right\} \\ &= \left(\frac{A-B}{B}\right)\log\left(1 + Bz_0\delta(z_0\bar{\mathbb{D}},\lambda)\right). \end{aligned}$$

Combining the above and the relation

$$1 + Bz_0\delta(z_0\bar{\mathbb{D}},\lambda) = \bar{\mathbb{D}}(c(z_0,\lambda), r(z_0,\lambda)),$$

which easily follows from the proof of the Proposition 3.1, we obtain

$$V_\lambda(z_0, A, B) = \left\{\left(\frac{A-B}{B}\right)\log\left(c(z_0,\lambda) + ar(z_0,\lambda)\right) : a \in \bar{\mathbb{D}}\right\}.$$

Since the function $\log(1+w)$ is convex univalent in $\mathbb{D}$, the set in the right-hand side of the above equation is a convex closed Jordan domain and its boundary curve is given by

$$(-\pi,\pi] \ni \theta \mapsto \left(\frac{A-B}{B}\right)\log(1 + Bz_0\delta(z_0e^{i\theta},\lambda)) = \log F'_{e^{i\theta},\lambda}(z_0).$$

This completes the proof of the theorem. □

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