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Characterization of $PGL(2, p)$ by its order and one conjugacy class size

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Abstract. Let p be a prime. In this paper, we do not use the classification theorem of finite simple groups and prove that the projective general linear group $PGL(2, p)$ can be uniquely determined by its order and one special conjugacy class size. Further, the validity of a conjecture of J. G. Thompson is generalized to the group $PGL(2, p)$ by a new way.

Keywords. Finite group; conjugacy class size; Thompson's conjecture.

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1. Introduction

All groups considered in this paper are finite and simple groups are nonabelian.

For a group G, we denote by $\pi(G)$ the set of prime divisors of $|G|$. A simple graph $\Gamma(G)$ called the *prime graph* of G was introduced by Gruenberg and Kegel in the middle of the 1970s: the vertex set of $\Gamma(G)$ is $\pi(G)$, two vertices p and q are joined by an edge if and only if G contains an element of order pq (see [21]). Denote the set of the connected components of $\Gamma(G)$ by $T(G) = \{\pi_i(G) | 1 \leqslant i \leqslant t(G)\}$, where $t(G)$ is the number of the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, we always assume that $2 \in \pi_1(G)$.

Denote by $N(G)$ the set of all conjugacy class sizes of a group G. The starting point for our discussion is from a conjecture of J. G. Thompson. In 1988, he posed the following conjecture in a communication letter:

Thompson's conjecture (*Problem* 12.38 *of* [18]). Let G be a group with trivial central. If L is a simple group satisfying $N(G) = N(L)$, then $G \cong L$.

In 1994, Chen proved in his Ph.D. dissertation [6] that Thompson's conjecture holds for all simple groups with disconnected prime graph (see [8–10]). For simple groups with connected prime graph, Thompson's conjecture has made some progress in recent years. Several mathematicians had established the validity of Thompson's conjecture for A_{10}, A_{16}, A_{22}, $U_4(4)$, $U_4(5)$, $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, ${}^2D_n(q)$, and $E_7(q)$

(see [1–4, 14–16, 20, 22, 23]). Lastly, Chen and his students contributed to Thompson's conjecture under a weak condition. They only used order and one or two special conjugacy class sizes of simple group, and characterized successfully sporadic simple groups, alternating group A_{10}, and projective special linear groups $L_4(4)$ and $L_2(p)$ (see [7, 12, 19]). Hence it is an interesting topic to characterize some groups with their orders and few conjugacy class sizes. Note that all the works except [14] listed above use the classification theorem of finite simple groups. In this paper, without using the classification theorem of finite simple groups we characterize the projective general linear group $PGL(2, p)$ by its order and one special class size, where p is a prime. That is the following theorem.

Main Theorem. *Let G be a group and p be a prime. Then $G \cong PGL(2, p)$ if and only if $|G| = p(p^2 - 1)$ and G has a special conjugacy class size of $(p^2 - 1)$.*

To avoid using the classification theorem of finite simple groups, we will depend on the following R. Brauer and W. F. Reynolds's result of 1958, which answered a question of E. Artin in a letter to R. Brauer: what are the simple groups of an order g which is divisible by a prime $p > g^{\frac{1}{3}}$?

Theorem 1.1 (Theorem 1 of [5]). *Let G be a simple group whose order g is divisible by a prime $p > g^{\frac{1}{3}}$. Then G is isomorphic either to $L_2(p)$ where $p > 3$ is a prime or $L_2(p - 1)$ where $p > 3$ is a Fermat prime, $p = 2^n + 1$, with integral n.*

If the main theorem is proved, then the following corollary holds, which generalizes Thompson's conjecture to $PGL(2, p)$. Note that this corollary can be obtained by other ways (see[17, 25]).

COROLLARY 1

Thompson's conjecture holds for the projective general linear group $PGL(2, p)$, where p is a prime.

Proof. Let G be a group with trivial central and $N(G) = N(PGL(2, p))$. Since $t(PGL(2, p)) > 1$, we get that $|G| = |PGL(2, p)|$ by the Lemma 1.4 of paper [10]. Hence the corollary follows from the main theorem. □

For convenience, we use $\pi(n)$ to denote the set of all primes dividing a natural number n and n_π to denote π-part of a nature number n for $\pi \subseteq \pi(n)$. In addition, we also denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of a group G and the set of all of its Sylow p-subgroups for $p \in \pi(G)$, respectively. The other notation and terminologies in this paper are standard and the reader is referred to [13] and [24], if necessary.

2. Some lemmas

A *Frobenius group* is a transitive permutation group in which the stabilizer of any two points is trivial. It is well known that there exists the Frobenius kernel and the Frobenius complement in any Frobenius group. Furthermore, a group G is a 2-*Frobenius group* if there exists a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Here we quote some known results about Frobenius group and 2-Frobenius group which are useful in the sequel.

Lemma 2.1 (*Theorem* 1 *of* [11]). *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$ and G has one of the following structures*:

(i) $2 \in \pi(H)$ *and all Sylow subgroups of K are cyclic*;
(ii) $2 \in \pi(K)$, *H is an abelian group, K is a solvable group, the Sylow subgroups of K of odd order are cyclic groups and the Sylow 2-subgroups of K are cyclic or generalized quaternion groups*;
(iii) $2 \in \pi(K)$, *H is abelian, and there exists a subgroup K_0 of K such that*

$$|K : K_0| \leqslant 2, K_0 = Z \times SL(2, 5), (|Z|, 2 \times 3 \times 5) = 1,$$

and the Sylow subgroups of Z are cyclic.

Lemma 2.2 (*Theorem* 2 *of* [11]). *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, the order of G/K divides the order of the automorphism group of K/H, and both G/K and K/H are cyclic. Especially, $|G/K| < |K/H|$ and G is solvable.*

Lemma 2.3. *Let G be a 2-Frobenius group. Then $G = ABC$, where both A and AB are normal subgroups of G, B is a normal subgroup of BC, and AB and BC are both Frobenius groups.*

Proof. By the definition of 2-Frobenius group, G has a normal series $1 \lhd A \lhd K \lhd G$ such that K and G/A are Frobenius groups with kernels A, K/A and complements B, M/A, respectively. Then $K/A \cong B$, $G/K \cong (G/A)/(K/A) \cong M/A$, and so $(|A|, |B|) = 1$, $(|B|, |G/K|) = (|K/A|, |M/A|) = 1$. Hence B is a Hall-subgroup of G. Clearly, B is also a Hall-subgroup of K. By generalized Frattini argument, $G = KN_G(B) = ABN_G(B) = AN_G(B)$. Since K is a Frobenius group with kernel A and complement B, $A \bigcap N_G(B) = N_A(B) = 1$. Thus we have $B \trianglelefteq N_G(B) \cong G/A$. In view of G/A with kernel K/A and complement M/A, and $K/A \cong B$, $N_G(B)$ is a Frobenius group with kernel B. Let C be a complement subgroup of B in $N_G(B)$. Then $N_G(B) = BC$ such that $G = ABC$, where A and AB are normal subgroups of G, B is a normal subgroup of BC, and AB and BC are two Frobenius groups.

By Lemma 2.3, a 2-Frobenius group can also be defined equivalently as follows: A group G is called a 2-Frobenius group if $G = ABC$, where both A and AB are normal subgroups of G, B is a normal subgroup of BC, and AB and BC are both Frobenius groups. Thus by Lemma 2.6 in [12], we can get the following lemma. □

Lemma 2.4 (*Lemma* 2.6 *of* [12]). *Let p be a prime and n a natural number. Suppose that G is a 2-Frobenius group with a normal series $1 \lhd H \lhd K \lhd G$, where H is an elementary abelian p-group of order p^n and K/H is a cyclic group of order $p^n - 1$. Then $|G/K| \mid n$.*

Lemma 2.5 (*Theorem A of* [21]). *Let G be a group with more than one prime graph component. Then G is one of the following*:

(i) *a Frobenius or* 2*-Frobenius group*;
(ii) *G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H.*

Lemma 2.6 (*Theorem* 4.5.3 *of* [24]). *Let G be a p-group with order p^n, $n \geq 1$, and d is the number of minimal generators of G. Then* $|\mathrm{Aut}(G)| \mid p^{d(n-d)}(p^d - 1)(p^d - p) \cdots (p^d - p^{d-1})$.

3. The proof of the main theorem

Proof of Main Theorem. Since the necessity of the theorem can be checked easily, we only need to prove the sufficiency.

If $p = 2$, then G is a nonabelian group of order 6. By the structures of the groups of order 6, one has that $G \cong S_3 \cong PGL(2,\ 2)$, where S_3 is a symmetric group of degree 3, as desired.

If $p = 3$, then G is a group of order 24 and has an element x of order 3 such that $C_G(x) = \langle x \rangle$. By the structures of the groups of order 24, one has that $G \cong S_4 \cong PGL(2,\ 3)$, where S_4 is a symmetric group of degree 4, as desired.

Let $p \geq 5$. By the hypothesis, there exists an element x of order p in G such that $C_G(x) = \langle x \rangle$. Then by Sylow theorem, we have that $C_G(y) = \langle y \rangle$ for any element $y \in G$ of order p such that $\{p\}$ is a prime graph component of G and $t(G) \geqslant 2$. Therefore, G satisfies one of the cases in Lemma 2.5 and $Z(G) = 1$. In addition, p is the maximal prime divisor of $|G|$, and is also a connected component of prime graph of G.

Suppose that G is a Frobenius group with kernel H and complement K. Then $|K| \mid (|H| - 1)$. If $p \in \pi(H)$, then, by Lemma 2.1, $|H| = p$ and $|K| = (p^2 - 1)/2$. It follows that $(p^2 - 1) \mid (p - 1)$, and thus $p = 1$, a contradiction. If $p \in \pi(K)$, then $|K| = p$ and $|H| = p^2 - 1$ by Lemma 2.1, and so $p \mid (p^2 - 2)$, which implies $p = 2$, a contradiction. Hence G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 2.2, we have that G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \{p\} = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, and $|G/K| \mid (p - 1)$. Then we have that K/H is of order p and $\pi(p + 1) \subseteq \pi(H)$.

(a) Assume p is not a Mersenne number. Then there exists an odd prime $r \in \pi(p + 1)$ such that $|H_r| < p$. By Lemma 2.6, $(p, |\mathrm{Aut}(H_r)|) = 1$, which implies that an element of order p of G can trivially act on H_r. In other words, r can be connected to p in the prime graph of G, a contradiction.
(b) Let p be a Mersenne number and $p = 2^s - 1$, $s \geq 2$. Then s is a prime and $(p - 1)/2$ is an odd number. Recall that $|G/K| \mid (p - 1)$.

If $|G/K| = p - 1$, then $|H| = p + 1 = 2^s$. If H is not an elementary abelian 2-group, then$(p, |\mathrm{Aut}(H)|) = 1$ by Lemma 2.6, a contradiction. If H is an elementary abelian 2-group, then, by Lemma 2.4, $|G/K| = (2^s - 2) \mid s$. It follows that $s = 2$, and thus $p = 3$, a contradiction.

If $|G/K| = (p-1)/2$, then $|H| = 2(p+1) = 2^{s+1}$, and so $p \mid (2^{s+1} - 1)$. Hence $s = 1$, a contradiction.

About other cases of $|G/K|$, we can always find an odd prime $r \in \pi(H)$ and $r \mid \frac{p-1}{2}$ such that $(p, |\mathrm{Aut}(H_r)|) = 1$, which implies that G has an element of order pr, a contradiction. Therefore G is not a 2-Frobenius group either.

Now, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group, G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H, and p is a connected component of prime graph of K/H, and also $K/H \le G/H \le \mathrm{Aut}(K/H)$. It follows that $p \mid |K/H| \mid |G/H| \mid |G| = p(p^2-1)$. Therefore $p > |K/H|^{\frac{1}{3}}$, and thus by Theorem 1.1, we obtain that K/H is isomorphic to $L_2(p),\ p > 3$, or $L_2(p-1),\ p > 3$ and p is a Fermat prime. Assume that K/H is isomorphic to the latter. Then the order of K/H is $p(p-1)(p-2)$ and divides $p(p^2-1)$, which implies that $p = 5$. Hence $K/H \cong L_2(4) \cong L_2(5)$. Therefore, K/H must be isomorphic to $L_2(p),\ p > 3$. Recall that $K/H \le G/H \le \mathrm{Aut}(K/H)$, we have that $L_2(p) \le G/H \le PGL(2,\ p)$, and thus $G/H \cong L_2(p)$ or $PGL(2,\ p)$. If $G/H \cong L_2(p)$, then H is a normal subgroup of order 2 of G, and then $H \subseteq Z(G)$, contradicting $Z(G) = 1$. If $G/H \cong PGL(2,\ p)$, then $H = 1$ by $|G| = |PGL(2,\ p)| = p(p^2-1)$, and so $G \cong PGL(2,\ p)$, as desired. □

Remark . In [12], the authors characterized $L_2(p)$ by its order and one special class size. But the proof of Theorem 3.2 of [12] is unpleasant. It is because the proof invokes the classification theorem of finite simple groups and needs to eliminate potential counterexamples by a tedious case by case verification. In fact, using Theorem 1.1, the proof of the theorem can be simplified greatly and we can also avoid using the classification theorem of finite simple groups.

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