Proc. Indian Acad. Sci. (Math. Sci.) Vol. 125, No. 4, November 2015, pp. 477–485.

Yoneda algebras of almost Koszul algebras

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MS received 4 September 2013; revised 14 April 2014

Abstract. Let k be an algebraically closed field, A a finite dimensional connected (p, q)-Koszul self-injective algebra with $p, q \geq 2$. In this paper, we prove that the Yoneda algebra of A is isomorphic to a twisted polynomial algebra $A^![t; \beta]$ in one indeterminate t of degree $q+1$ in which $A^!$ is the quadratic dual of A, β is an automorphism of $A^!$, and $tb = \beta(b)t$ for each $t \in A^!$. As a corollary, we recover Theorem 5.3 of [2].

Keywords. Almost Koszul algebras; Yoneda algebras; self-injective algebras.

2000 Mathematics Subject Classification. Primary: 16E05; Secondary: 16G20, 16E40.

1. Introduction

Let k be an algebraically closed field, Q a Dynkin quiver in bipartite orientation with more than two vertices. Denote the coxeter number of the underlying Dynkin graph by h_Q, the trivial extension algebra of kQ by $T(kQ)$. By Theorem 5.3 of [2], $A = T(kQ)$ is a left $(2, h_Q - 2)$-Koszul algebra (see Definition 2.1 below), and the Yoneda algebra of A is isomorphic to a twisted polynomial algebra $A^![t; \beta]$ in one indeterminate t of degree $h_Q - 1$, in which $A^!$ is the quadratic dual of A, β is a Nakayama automorphism of $A^!$, and $tb = \beta(b)t$ for each $b \in A^!$.

The result mentioned above is very interesting, so we wonder whether it holds in more general settings. In other words, we consider this question: for a finite dimensional connected left (p, q)-Koszul self-injective algebra A, is the Yoneda algebra of A isomorphic to a twisted polynomial algebra of $A^!$? Note that if A is a left (p, q)-Koszul algebra with $p, q \geq 2$, then A is a quadratic algebra (Proposition 3.7 of [2]); however, if $q = 1$, then A is not a quadratic algebra in general (Proposition 3.5 of [2]). Thus we should restrict p, q to be integers larger than 1. Also notice that for a graded self-injective algebra A, Guo introduced the concept of *stable bound quiver* to characterize the bound quiver of A [3]. In this note, we will use this tool to give an affirmative answer to the question mentioned above.

The paper is organized as follows. In §2, we give some notions and lemmas needed in later section. In §3, we give our main results in this paper. Let A be a finite dimensional connected left (p, q)-Koszul self-injective algebra with $p, q \geq 2$. Then the Yoneda algebra of A is isomorphic to a twisted polynomial algebra $A^![t; \overline{\tau^{-1}}\nu^{-1}]$ in one indeterminate t of degree $q+1$ in which $A^!$ is the quadratic duality of A, ν is a Nakayama automorphism of A, and $\overline{\tau^{-1}}$ is an automorphism of A induced by τ^{-1} the inverse of a Nakayama automorphism τ of $A^!$. As a corollary, we recover Theorem 5.3 of [2].

2. Preliminaries

Let k be an algebraically closed field. A k-algebra Γ is said to be a *graded algebra* if $\Gamma = \bigoplus_{t=0}^{\infty} \Gamma_t$ are vector spaces and satisfies the following conditions:

(1) Γ is generated by Γ_0 and Γ_1;
(2) $\Gamma_0 = \bigoplus_{i=1}^{n} ke_i$ such that $1 = e_1 + \cdots + e_n$ is a decomposition of 1 as a sum of orthogonal primitive idempotents.

So $\Gamma_0 \cong k^n$, and $\Gamma_s\Gamma_t = \Gamma_{s+t}$ for all $s, t \geq 0$. And for a graded algebra Γ, its Yoneda algebra is the vector space $\mathrm{Ext}_\Gamma^\bullet(\Gamma_0, \Gamma_0) = \coprod_{t\geq 0} \mathrm{Ext}_\Gamma^t(\Gamma_0, \Gamma_0)$ with multiplication defined by the Yoneda product.

In the rest of the paper, we fix a finite dimensional k-algebra $S \cong k \times k \times \cdots \times k$. Without otherwise stated and unadorned $\otimes$ means $\otimes_S$. Let V be a finite dimensional S–S-bimodule. There are three different ways to get an S–S-bimodule dual to V:

(1) $V^* = \mathrm{Hom}_k(V, k)$, with bimodule action $(s \cdot f \cdot t)(v) = f(tvs)$;
(2) $V^{*r} = \mathrm{Hom}_S(V_S, S_S)$ with bimodule action $(s \cdot f \cdot t)(v) = sf(tv)$;
(3) $V^{*l} = \mathrm{Hom}_S({}_SV, {}_SS)$ with bimodule action $(s \cdot f \cdot t)(v) = f(vs)t$.

However, there are natural isomorphisms L, R from V^* to V^{*l}, V^{*r}, respectively. We give the descriptions of L and R (see also section 2.1 of [1]). Let $e_1, \ldots, e_n$ be a complete set of orthogonal primitive idempotents in S. Then for $\theta \in V^*$ and $v \in V$, we define functions $L : V^* \to V^{*l}$ and $R : V^* \to V^{*r}$ by $L(\theta)(v) := \Sigma_{i=1}^n \theta(e_i v)e_i$, and $R(\theta)(v) := \Sigma_{i=1}^n \theta(ve_i)e_i$. It is not hard to see that L and R are S–S-bimodule isomorphisms. In the following, we just identify the above three dualities. The algebra $A = T(V)/(R)$ is quadratic if R is a subspace of $V \otimes V$. The quadratic dual $A^!$ of A is defined to be $T(V^*)/(R^\perp)$, where V^* is the dual bimodule of V and $R^\perp$ is the orthogonal complement of R in $V^* \otimes V^*$.

Let τ be a graded algebra automorphism of a quadratic algebra $A = T(V)/(R)$, that is, an automorphism that preserves the degree of homogeneous element. Let $\tau_1 : V_1 \to V_1$ be the component of degree 1 of τ. The transpose $\bar{\tau}_1$ of τ_1 from V^* to V^* can be extended to a graded algebra automorphism of $T(V^*)$. So $\bar{\tau}_1$ induces a graded algebra automorphism. Let $\{v_1, v_2, \ldots, v_n\}$ be a k-basis of V, $\{v_1^*, \ldots, v_n^*\}$ the corresponding dual basis in V^*. Let $\tau(v_i) = \Sigma_{j=1}^n c_{ij}v_j$, $1 \leq i \leq n$, where $c_{ij} \in k$. Then $\bar{\tau}(v_i^*) = \Sigma_{j=1}^n c_{ji}v_j^*$. We simply call the automorphism $\bar{\tau}$ constructed in the above way as the automorphism induced by τ.

DEFINITION 2.1 (Definition 3.1 of [2])

A graded algebra $A = \oplus_{i\geq 0} A_i$ with $A_0 = S$ is called a left almost Koszul if there exist integers $p, q \geq 1$ such that

(1) $A_i = 0$,for all $i > p$, and

(2) there is a graded exact complex of left A-modules

$$0 \to W \to P_q \xrightarrow{d^q} \cdots \to P_1 \xrightarrow{d^1} P_0 \to A_0 \to 0,$$

where P_i is generated by its component of degree i for all $i = 0, \ldots, q$, and $W = A_p \otimes (P_q)_q$. In this case, A is also called a left (p, q)-Koszul algebra.

Let us recall the definition of a Koszul complex. Let $A \cong T(V)/(R)$ be a quadratic algebra. For $i \geq 0$, define a S–S-submodule K^i of $V^{\otimes i}$ as follows,

$$K^0 = S, K^1 = V, K^2 = R, K^{i+1} = VK^i \cap K^i V, \quad i \geq 2.$$

The left Koszul complex of A is

$$\cdots \longrightarrow A \otimes K^n \longrightarrow \cdots \longrightarrow A \otimes K^1 \longrightarrow A \otimes K^0 \longrightarrow 0,$$

where the differentials are the compositions of the following maps:

$$A \otimes K^n \hookrightarrow A \otimes VK^{n-1} \longrightarrow AA_1 \otimes K^{n-1} \hookrightarrow A \otimes K^{n-1}.$$

The quadratic algebra A is a (p, q)-Koszul algebra if and only if $A_n = 0$ for all $n > p$, $K^m = 0$ for all $m > q$ and the only nonzero homology of the left Koszul complex of A are A_0 in degree 0, and $A_p \otimes K^q$ in degree $p + q$ (Proposition 3.9 of [2]).

We need the following lemma, (see Lemma. 2.2 of [6]).

Lemma 2.2. *Let A be a left (p, q)-Koszul algebra with $p, q \geq 2$. We have the following exact sequence*:

$$0 \to A_p \otimes A_q^{!*} \to A \otimes A_q^{!*} \xrightarrow{d^q} \cdots \to A \otimes A_1^{!*} \xrightarrow{d^1} A \otimes A_0^{!*} \to A_0 \to 0.$$

PROPOSITION 2.3 (Proposition 3.2 of [2])

Let A be a left (p, q)-Koszul algebra with $p > 1$. Then $B = \oplus_{i=0}^q \mathrm{Ext}_A^i(S, S)$ is a subalgebra of $\mathrm{Ext}_A^\bullet(S, S)$. Furthermore, B is generated by B_0 and B_1, and $\mathrm{Ext}_A^\bullet(S, S)$ is generated by B and $\mathrm{Ext}_A^{q+1}(S, S)$.

Let A be a left (p, q)-Koszul algebra with $p, q \geq 2$. Then A must be nonsemisimple and quadratic (Propostion 3.7 of [2]). In this case, $A^! \cong \oplus_{i=0}^q \mathrm{Ext}_A^i(S, S)$ (Remark 3.12 of [2]). Thus we have the following lemma.

Lemma 2.4. *Let A be a left (p, q)-Koszul self-injective algebra with $p, q \geq 2$. Then A-modules $A_p \otimes A_q^{!*}$ is isomorphic to A_0. In particular, $A^!$ is a left (q, p)-Koszul self-injective algebra.*

Proof. Let $A_0 \cong \oplus_{i=1}^n S_i$, S_i an indecomposable direct summand of A_0. Since A is a self-injective algebra, by Lemma 2.2, $\Omega^{q+1}(S_i)$ is an indecomposable direct summand of $\Omega^{q+1}(A_0) \cong A_p \otimes A_q^{!*}$. Since $A_p \otimes A_q^{!*}$ is semisimple, $\Omega^{q+1}(S_i)$ is simple, and hence there exists $t \in \{1, \ldots, n\}$ such that $\Omega^{q+1}(S_i) \cong S_t$. Since A is connected, the semisimple module A_0 has no projective summands. It follows that for $i \neq j$, $\Omega^{q+1}(S_i) \ncong$

$\Omega^{q+1}(S_j)$. Then there exists a permutation σ of $\{1,\dots,n\}$ such that $\Omega^{q+1}(S_i)\cong S_{\sigma(i)}$. Then $\Omega^{q+1}(A_0)\cong\Omega^{q+1}(\oplus_{i=1}^{n}S_i)\cong\oplus_{i=1}^{n}S_{\sigma(i)}\cong A_0$. Thus $A_p\otimes A_q^{!*}\cong A_0$. By Proposition 3.11 of [2], $A^!$ is a left (q,p)-Koszul algebra. Combining Proposition 3.3 of [6] with Proposition 2.3, $A^!$ is a graded self-injective algebra. We get the desired assertion. □

3. Yoneda algebra

Let Q be a finite quiver, denote by Q_0 the vertex set and Q_1 the arrow set. By Proposition 1.1.1 of [4], for a graded algebra Γ, there exists a finite quiver, and an ideal $I=(\rho)$ of the path algebra kQ, such that $\Gamma\cong kQ/(\rho)$, where ρ is the relation set. And we call (Q,ρ) the bound quiver of Γ. A path in Q is called a *bound path* if its image in Γ is nonzero. A bound quiver (Q,ρ) is called *homogeneous* provided that each of the paths appearing in a given linear combination of ρ has the same length.

Fix an integer $p\geq 1$, a homogeneous bound quiver Q is said to be *stable of Loewy length* $p+1$ if it satisfies the following conditions:

(1) A permutation τ is defined on the vertex set of Q;
(2) The maximal bound paths of Q have the same length p;
(3) For each vertex i, there is a maximal bound path from $\tau(i)$ to i, and there is no bound path of length p from $\tau(i)$ to j for any $j\neq i$;
(4) Any two maximal bound paths starting at the same vertex are linearly dependent.

τ is called the *Nakayama translation* of the stable bound quiver Q.

Let A be a graded self-injective algebra of Loewy length $p+1$, then its bound quiver (Q,ρ) is a stable bound quiver of Loewy length $p+1$, and the Nakayama translation on Q_0 is induced by a Nakayama automorphism τ of A [3]. Denote by x_{hl}^{s} an arrow ending at the vertex l and starting at the vertex h, then $\{x_{hl}^{s}\mid s=1,\dots,n_{hl}=\dim_k e_hA_1e_l\}$ is a basis of the vector space $e_hA_1e_l$. For each vertex i, all the maximal bound paths ending at the vertex i start at the vertex $\tau(i)$ and they are pairwise linear dependent. Write $u_{\tau(i)i}$ for the basic element which is a maximal bound path from $\tau(i)$ to i. Then $\{u_{\tau(i)i}\mid i\in Q_0\}$ is a basis of the vector A_p.

Furthermore, assume that A is a connected left (p,q)-Koszul self-injective algebra with $p,q\geq 2$. By Lemma 2.4, $A^!$ is a left (q,p)-Koszul self-injective algebra. Denote by (Q,ρ) the bound quiver of A, then $(Q^{op},\rho^{\perp})$ is the bound quiver of $A^!$ [4]. Then by the above argument $(Q^{op},\rho^{\perp})$ is a stable bound quiver of Loewy length of $q+1$. Let $\{e_h\mid h\in Q_0\}$ be a complete set of primitive orthogonal idempotents in A, and denote by $\{e_h^*\mid h\in Q_0^{op}\}$ the dual basis of Q_0^{op}. Whenever there is no danger of confusion, we also identify e_h^* with e_h. Let $\{x_{hl}^{s}\mid s=1,\dots,n_{hl}=\dim_k e_hA_1e_l\}$ be the arrow set which consists of all the arrows ending at l and starting at h in Q_1, denote $\{x_{lh}^{s*}\mid s=1,\dots,n_{hl}\}$ the dual basis of $e_lA_1^!e_h$ with suffixes reversed. For simplicity, we also denote x_{lh}^{s*} by y_{lh}^{s}, so $\{y_{lh}^{s}\mid s=1,\dots,n_{hl}\}$ is a basis of $e_lA_1^!e_h$. Let ν be the Nakayama translation of $(Q^{op},\rho^{\perp})$, and write $\omega_{\nu(i)i}$ for the basic element which is a maximal bound path from $\nu(i)$ to i, then $\{\omega_{\nu(i)i}\mid i\in Q_0^{op}\}$ is a basis of $A_q^!$. Thus we have a Nakayma automorphism associated to $\{\omega_{\nu(i)i}\mid i\in Q_0^{op}\}$ (Lemma 3.2 of [5]), also denoted by ν. And the basis elements of $A_q^{!*}$ dual to the basis element $\omega_{\nu(i)i}$ of $A_q^!$ will be denoted by $\omega_{i\nu(i)}^{*}$ with suffixes reversed.

Theorem 3.1. *Let A be a connected left (p,q)-Koszul self-injective algebra with $p,q \geq 2$. Then $E = \mathrm{Ext}_A^{\bullet}(S,S)$ is isomorphic to the twisted polynomial algebra $A^![t;\overline{\tau^{-1}\nu^{-1}}]$ in one indeterminate t of degree $q+1$, in which the grading of $A^!$ is the length grading, ν is a Nakayama automorphism of $A^!$, τ is a Nakayama automorphism of A, and $tb = \overline{\tau^{-1}\nu^{-1}}(b)t$ for each $t \in A^!$.*

Proof. Firstly, let us take an element t' in $E^{q+1} = \mathrm{Ext}_A^{q+1}(S,S)$ corresponding to the generator t of degree $q+1$ in $A^![t;\overline{\tau^{-1}\nu^{-1}}]$. We choose t' to be the exact sequence

$$0 \longrightarrow A_p \otimes A_q^{!*} \longrightarrow P^q \longrightarrow \cdots \longrightarrow P^0 \longrightarrow S \longrightarrow 0, \tag{1}$$

obtained from the left Koszul complex of A. Then by Lemma 2.2, $P^r \cong A \otimes A_r^{!*}$, $0 \leq r \leq q$. By Lemma 2.4, $A_p \otimes A_q^{!*} \cong A_0$. Thus, t' is an S-module generator of $\mathrm{Ext}_A^{q+1}(S,S)$, and we choose t to be t'.

To obtain generators of E^1, we use formulae

$$E^1 \cong \mathrm{Ext}_A^1(S,S) \cong \mathrm{Hom}_A(P_1,S) \cong \mathrm{Hom}_S(DA_1^!,S) \cong A_1^!$$

and recall that $e_h A_1^! e_l$ has y_{hl}^s as a basis. Let $[y_{hl}^s]$ denote the short exact sequence

$$[y_{hl}^s] : 0 \longrightarrow S \longrightarrow Y_{lh}^s \longrightarrow S \longrightarrow 0$$

obtained by pushout of the exact sequence

$$P^1 = A \otimes A_1^{!*} \longrightarrow P^0 = A \otimes A_0^{!*} \longrightarrow S \longrightarrow 0 \tag{2}$$

along the map $\overline{y_{hl}^s} : P^1 \longrightarrow S$ given by

$$1 \otimes y_{ji}^{k*} \longmapsto \begin{cases} 0, & \text{if}(i,j,k) \neq (h,l,s), \\ e_l, & \text{if}(i,j,k) = (h,l,s). \end{cases}$$

In the following, we will show that for any $y_{hl}^s \in e_h A_1^! e_l$, the following formulae holds:

$$t'[y_{hl}^s] = [\bar{\tau}\nu^{-1}(y_{hl}^s)]t'. \tag{3}$$

These formulae relate elements of E^{q+2}, so we need to extend the projective resolution (1) by two further terms to

$$P^{q+2} \xrightarrow{d^{q+2}} P^{q+1} \xrightarrow{d^{q+1}} P^q \xrightarrow{d^q} \cdots \longrightarrow P^0 \longrightarrow S \longrightarrow 0.$$

Since $\ker d^q = A_p \otimes A_q^{!*}$, we may choose $P^{q+1} = A \otimes A_p \otimes A_q^{!*}$. It has A-module generators $1 \otimes u_{\tau(j)j} \otimes \omega_{j\nu(j)}^*$ and d^{q+1} is the map $1 \otimes u_{\tau(j)j} \otimes \omega_{j\nu(j)}^* \longmapsto u_{\tau(j)j} \otimes \omega_{j\nu(j)}^*$.

We next choose $P^{q+2} = A \otimes A_1^{!*} \otimes A_p \otimes A_q^{!*}$. It has A-module generators $1 \otimes y_{\tau(i)\tau(j)}^{k*} \otimes u_{\tau(j)j} \otimes \omega_{j\nu(j)}^*$ and the map d^{q+2} is given by

$$d^{q+2}(1 \otimes y_{\tau(i)\tau(j)}^{k*} \otimes u_{\tau(j)j} \otimes \omega_{j\nu(j)}^*) = x_{\tau(i)\tau(j)}^k \otimes u_{\tau(j)j} \otimes \omega_{j\nu(j)}^*.$$

For the proof of the formulae (3) we exhibit maps $\theta, \phi : P^{q+2} \to S$ representing the elements $t'[y_{hl}^s]$, $[\overline{\tau^{-1}\nu^{-1}}(y_{hl}^s)]t'$,respectively.

Let $\zeta : P^{q+1} \to A_p \otimes A_q^{!*} \cong S$, $1 \otimes u_{\tau(j)j} \otimes \omega^*_{j\nu(j)} \mapsto u_{\tau(j)j} \otimes \omega^*_{j\nu(j)}$ be the map which induces t', and it lifts to give a commutative diagram

$$\begin{array}{ccccccc} P^{q+2} & \xrightarrow{d^{q+2}} & P^{q+1} & & & & \\ \downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle \zeta_0} & \searrow{\scriptstyle \zeta} & & & \\ P^1 & \xrightarrow{d^1} & P^0 & \xrightarrow{\varepsilon} & S & \longrightarrow & 0, \end{array}$$

in which ζ_0, ζ_1 are given by

$$\zeta_0(1 \otimes u_{\tau(j)j} \otimes \omega^*_{j\nu(j)}) = 1 \otimes e_{\tau(j)},$$
$$\zeta_1(1 \otimes y^{k*}_{\tau(i)\tau(j)} \otimes u_{\tau(j)j} \otimes \omega^*_{j\nu(j)}) = 1 \otimes y^{k*}_{\tau(i)\tau(j)}.$$

For simplicity, denote n_{hl} by n, then $\{y^s_{hl} | s = 1, \dots, n\}$ is a basis of the vector space $e_h A^!_1 e_l$. In the following, we will first calculate $\overline{\tau^{-1}}\nu^{-1}(y^s_{hl})$, and we need more information.

By Lemma 2.4, $A^!$ is a graded self-injective algebra. Then we may choose a basis $\{\beta_s | s = 1, \dots, n\}$ of the vector space $e_l A^!_{q-1} e_{\nu^{-1}(h)}$ such that $y^s_{hl}\beta_t = \delta_{st}\omega_{h\nu^{-1}(h)}$. Furthermore, take a basis $\{\gamma_s | s = 1, \dots, n\}$ of $e_{\nu^{-1}(h)} A^!_1 e_{\nu^{-1}(l)}$ such that $\beta_t\gamma_s = \delta_{ts}\omega_{l\nu^{-1}(l)}$. And we know $\nu^{-1}(y^s_{hl}) = \gamma_s$, $s = 1, \dots, n$.

The basis elements of $e_{\nu^{-1}(l)} A_1 e_{\nu^{-1}(h)}$ dual to the basis elements γ_s will be denoted by γ^*_s. Since $(A^!)^! \cong A$, $\{\gamma_s = \gamma^{**}_s | s = 1, \dots, n\}$ is the dual basis of $\{\gamma^*_s | s = 1, \dots, n\}$. Noting that $\{\gamma^*_s | s = 1, \dots, n\}$ is a basis of the vector space $e_{\nu^{-1}(l)} A_1 e_{\nu^{-1}(h)}$, we may choose a basis $\{t_s | s = 1, \dots, n\}$ of the vector space $e_{\tau\nu^{-1}(h)} A_{p-1} e_{\nu^{-1}(l)}$ such that $t_s\gamma^*_t = \delta_{st} u_{\tau\nu^{-1}(h),\nu^{-1}(h)}$. Furthermore, we may choose a basis $\{\alpha_s | s = 1, \dots, n\}$ of the vector space $e_{\tau\nu^{-1}(l)} A_1 e_{\tau\nu^{-1}(h)}$ such that $\alpha_k t_s = \delta_{ks} u_{\tau\nu^{-1}(l),\nu^{-1}(l)}$. It follows that $\tau^{-1}(\alpha_s) = \gamma^*_s$, $s = 1, \dots, n$. Let $\{\alpha^*_s | s = 1, \dots, n\}$ denote the dual basis of $e_{\tau\nu^{-1}}(h) A^!_1 e_{\tau\nu^{-1}(l)}$ associated to the basis $\{\alpha_s | s = 1, \dots, n\}$ of $e_{\tau\nu^{-1}(h)} A_1 e_{\tau\nu^{-1}(h)}$. Then by the definition of $\overline{\tau^{-1}}$, we have $\overline{\tau^{-1}}(\gamma_s) = \alpha^*_s$, $s = 1, \dots, n$. We may assume that $\alpha^*_s = \Sigma^n_{k=1} t_{sk} y^k_{\tau\nu^{-1}(h),\tau\nu^{-1}(l)}$, $s = 1, \dots, n$. Thus $\overline{\alpha^*_s} = \Sigma^n_{k=1} t_{sk} \overline{y^k_{\tau\nu^{-1}(h),\tau\nu^{-1}(l)}}$, $s = 1, \dots, n$. We take ϕ to the composite function $\phi = \overline{\alpha^*_s}\zeta_1$. It is given on the generators of P^{q+2} by

$$\phi(1 \otimes y^{k*}_{\tau(i)\tau(j)} \otimes u_{\tau(j)j} \otimes \omega^*_{j\nu(j)}) = \delta_{j,\nu^{-1}(h)}\delta_{i,\nu^{-1}(l)} t_{sk} e_{\tau\nu^{-1}(l)}. \tag{4}$$

In the following, we will calculate θ. By definition, $[y^s_{hl}]$ is the pushout of (2) by $\overline{y^s_{hl}} : P^1 \to S$. Since the right multiplication map $\cdot y^s_{hl}$ on the left Koszul complex of A commutes with the differentials, $\overline{y^s_{hl}}$ lifts to a morphism of complexes

$$\begin{array}{ccccccccccccc} P^{q+2} & \xrightarrow{d^{q+2}} & P^{q+1} & \xrightarrow{d^{q+1}} & P^q & \xrightarrow{d^q} & \cdots & \xrightarrow{d^2} & P^1 & & & & \\ \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \cdot y^s_{hl}} & & & & \downarrow{\scriptstyle \cdot y^s_{hl}} & \searrow{\scriptstyle \overline{y^s_{hl}}} & & & \\ P^{q+1} & \xrightarrow{d^{q+1}} & P^q & \xrightarrow{d^q} & P^{q-1} & \xrightarrow{d^{q-1}} & \cdots & \xrightarrow{d^1} & P^0 & \xrightarrow{\varepsilon} & S & \longrightarrow & 0. \end{array}$$

We claim that η_1 satisfies the following formulae:

$$\eta_1(1 \otimes u_{\tau(j)j} \otimes \omega^*_{j\nu(j)}) = \delta_{j,\nu^{-1}(h)} t_s \otimes \omega^*_{\nu^{-1}(l)l}.$$

We need to verify that η_1 satisfies the equation $d^q\eta_1 = \cdot y^s_{hl}d^{q+1}$.

$$\begin{aligned} d^q\eta_1(1\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)}) &= \delta_{j,\nu^{-1}(h)}d^q(t_s\otimes\omega^*_{\nu^{-1}(l)l})\\ &= \delta_{j,\nu^{-1}(h)}\Sigma^n_{i=1}(t_s\gamma^*_i)\otimes\gamma^{**}_i\cdot\omega^*_{\nu^{-1}(l)l}\\ &= \delta_{j,\nu^{-1}(h)}u_{\tau\nu^{-1}(h),\nu^{-1}(h)}\otimes\gamma_s\cdot\omega^*_{\nu^{-1}(l)l}\\ &= \delta_{j,\nu^{-1}(h)}u_{\tau\nu^{-1}(h),\nu^{-1}(h)}\otimes\beta^*_s\\ &= \delta_{j,\nu^{-1}(h)}u_{\tau\nu^{-1}(h),\nu^{-1}(h)}\otimes\omega^*_{\nu^{-1}(l)l}\cdot y^s_{hl}\\ &= \cdot y^s_{hl}d^{q+1}(1\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)}). \end{aligned}$$

The dual $A^{!*}$ is a natural $A^!-A^!$-bimodule, here $\gamma_s\cdot\omega_{\nu^{-1}(l)l}$ means the left action of γ_s on $A^{!*}$, and $\omega^*_{\nu^{-1}(l)l}\cdot y^s_{hl}$ means the right action of y^s_{hl} on $A^{!*}$.

Furthermore, we give the action of η_2. Take the generators $1\otimes y^{k*}_{\tau(i)\tau(j)}\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)}\in P^{q+2}=A\otimes A^{!*}_1\otimes A_p\otimes A^{!*}_q$, then there exist $\lambda_{ks}\in k$ such that $y^{k*}_{\tau(i)\tau(j)}t_s = x^k_{\tau\nu^{-1}(l)\tau\nu^{-1}(h)}t_s=\lambda_{ks}u_{\tau\nu^{-1}(l),\nu^{-1}(l)}$, $1\le k,s\le n$. Note that we have the following facts.

$$\begin{aligned} &\eta_1d^{q+2}(1\otimes y^k_{\tau(i)\tau(j)}\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)})\\ &= \eta_1d^{q+2}(1\otimes x^k_{\tau(i)\tau(j)}\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)})\\ &= \eta_1(x^k_{\tau(i)\tau(j)}\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)})\\ &= x^k_{\tau(i)\tau(j)}\eta_1(1\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)})\\ &= \delta_{j,\nu^{-1}(h)}(x^k_{\tau(i)\tau(\nu^{-1}(h))}t_s\otimes\omega^*_{\nu^{-1}(l)l})\\ &= \delta_{j,\nu^{-1}(h)}\delta_{i,\nu^{-1}(l)}\lambda_{ks}(u_{\tau\nu^{-1}(l),\nu^{-1}(l)}\otimes\omega^*_{\nu^{-1}(l)l}). \end{aligned}$$

Let $\eta_2(1\otimes y^k_{\tau(i)\tau(j)}\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)}) = \delta_{j,\nu^{-1}(h)}\delta_{i,\nu^{-1}(l)}\lambda_{ks}(1\otimes u_{\tau\nu^{-1}(l),\nu^{-1}(l)}\otimes\omega^*_{\nu^{-1}(l)l})$, then η_2 satisfies $d^{q+2}\eta_2=\eta_1d^{q+2}$. We take θ to be the composite function $\theta=\zeta\eta_2$. We will show that $\theta=\phi$, and hence the formulae (3) follows. This will complete the proof of the theorem by Proposition 2.3. Take the generators $1\otimes y^{k*}_{\tau(i)\tau(j)}\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)}$ of P^{q+2}, then

$$\begin{aligned} \theta(1\otimes y^{k*}_{\tau(i)\tau(j)}\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)}) &= \delta_{j,\nu^{-1}(h)}\delta_{i,\nu^{-1}(l)}\lambda_{ks}e_{\tau\nu^{-1}(l)},\\ \phi(1\otimes y^{k*}_{\tau(i)\tau(j)}\otimes u_{\tau(j)j}\otimes\omega^*_{j\nu(j)}) &= \delta_{j,\nu^{-1}(h)}\delta_{i,\nu^{-1}(l)}t_{sk}e_{\tau\nu^{-1}(l)}. \end{aligned}$$

To show $\theta=\phi$, we only need to show for any $1\le k,s\le n$, $t_{sk}=\lambda_{ks}$. Since $\{\alpha_s|s=1,\dots,n\}$ is a basis of the vector space $e_{\tau\nu^{-1}(l)}A_1e_{\tau\nu^{-1}(h)}$, we may assume that $\alpha_s=\Sigma^n_{k=1}m_{sk}x^k_{\tau\nu^{-1}(l),\tau\nu^{-1}(h)}$, $m_{sk}\in k$, $s=1,\dots,n$. Write $M=(m_{ij})_{n\times n}$, $\Lambda=(\lambda_{ij})_{n\times n}$, $T=(t_{ij})_{n\times n}$. Then by the formulae $\overline{\tau^{-1}}(\gamma_s)=\alpha^*_s=\Sigma^n_{k=1}t_{sk}y^k_{\tau\nu^{-1}(h),\tau\nu^{-1}(l)}$, we have that $T=M^{-1}$. On the other hand, by the formulae $x^k_{\tau\nu^{-1}(l),\tau\nu^{-1}(h)}t_s=\lambda_{ks}u_{\tau\nu^{-1}(l)\nu^{-1}(l)}$ and $\alpha_st_k=\delta_{sk}u_{\tau\nu^{-1}(l)\nu^{-1}(l)}$, we have $M'\Lambda=E$. Thus $T=\Lambda'$. It follows that $t_{sk}=\lambda_{ks}$, $1\le s,k\le n$. □

Remark 3.2. Note that our t in the proof of Theorem 3.1 is not identical with the t in Theorem 5.3 of [2]. Let τ' and ν' be the other two Nakayama automorphisms of A and $A^!$. The foregoing proof of Theorem 3.1 does not depend on the choice of Nakayama automorphisms. Therefore, we still have the isomorphism $\mathrm{Ext}^\bullet_A(S,S)\cong A^![t;\overline{\tau'^{-1}}\nu'^{-1}]$.

Recall that the trivial extension $A \ltimes M$ of an algebra A by an A-bimodule M is the algebra defined on the vector spaces $A \oplus M$ with multiplication defined by

$$(a, x)(b, y) = (ab, ay + xb)$$

for $a, b \in A$ and $x, y \in M$.

In the case we take the bimodule M as DA, the algebra $T(A) = A \ltimes DA$ is called the trivial extension of A.

Now let us apply our results to a special case discussed in [2]. For the definition of preprojective algebras, we follow the definition in section 4.1 of [2]. Let $Q = (Q_0, Q_1)$ be a Dynkin quiver. The preprojective algebra $B(Q)$ is the quotient $k\bar{Q}/(\rho)$. The quiver $\bar{Q}$ has vertex set Q_0 and arrows $y_{ij} : i \to j$ and $y_{ji} : j \to i$ corresponding to each arrow $i \to j$ in Q_1. The relation set consists of the quadratic relations $\rho_i = \Sigma_{j \in \mathcal{N}(i)} \epsilon_{ij} y_{ij} y_{ji}, i \in Q_0$,where $\mathcal{N}(i)$ is the set of neighbours of i in Q and

$$\epsilon_{ij} = \begin{cases} 1, & \text{if } i \to j \in Q_1, \\ -1, & \text{if } j \to i \in Q_1. \end{cases}$$

In the following, we assume that Q is in bipartite orientation with more than two vertices. Then the trivial extension $A := T(kQ)$ of path algebra kQ is a left $(2, h_Q - 2)$-Koszul self-injective algebra [2], where h_Q is the coxeter number of the underlying Dynkin graph. It was shown in section 5.1 of [2] that $A = A_0 \oplus A_1 \oplus A_2$, where A_0 has a basis $\{e_i | i \in Q_0\}$, A_1 has a basis $\{x_{ij}, x_{ji} | i \to j \in Q_1\}$ and A_2 has a basis $\{f_i | i \in Q_0\}$, where $f_i = x_{ij} x_{ji}$ for each neighbour j of i. Thus the quadratic dual $A^!$ is actually the preprojective algebra $B(Q)$. We give the descriptions of Nakayama automorphisms of $A^! = B(Q)$ and A as follows.

Let $S_i, i \in Q_0$, be the simple module of $B(Q)$ corresponding to the vertex i. The Nakayama functor ν defines a permutation ν on the vertices, which is given by $\nu(S_i) = S_{\nu(i)}$. For the preprojective algebra $B(Q)$, the algebra automorphism β defined below gives a Nakayama automorphism of $B(Q)$ (Theorem 4.8 of [2]). It is defined on generators by

$$\beta(e_i) = e_{\nu(i)}, \beta(y_{ij}) = \begin{cases} y_{\nu(i)\nu(j)}, & \text{if } \epsilon_{ij} = 1, \\ \epsilon_{\nu(i)\nu(j)} y_{\nu(i)\nu(j)}, & \text{if } \epsilon_{ij} = -1. \end{cases}$$

Since A is a symmetric algebra, we may choose the Nakayama automorphism τ in Theorem 3.1 to be trivial, and hence $\overline{\tau^{-1}}$ is also trivial. Noticing that $\beta^2 = 1$, we recover Theorem 5.3 of [2] by Theorem 3.1 and Remark 3.2 as follows.

COROLLARY 3.3

For A as above, $\mathrm{Ext}^{\bullet}_A(S, S)$ *is isomorphic to the twisted polynomial algebra $A^![t; \beta]$ in one indeterminate t of degree $q + 1$, in which the grading of $A^! = B(Q)$ is the length grading*, *β is the Nakayama automorphism of $A^!$ described as above*, *and $tb = \beta(b)t$ for each $b \in A^!$.*

Acknowledgements

This work is supported by Natural Science Foundation of China (#11271119 and #11201220) and by Hunan Provincial Innovation Foundation for Postgraduate (#CX2012B199).

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Communicating Editor: Parameswaran Sankaran