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Soliton solutions for a quasilinear Schrödinger equation via Morse theory

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Abstract. In this paper, Morse theory is used to show the existence of nontrivial weak solutions to a class of quasilinear Schrödinger equation of the form

$$-\Delta_p u - \frac{p}{2^{p-1}} u\Delta_p(u^2) = f(x,u)$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary condition.

Keywords. Quasilinear Schrödinger equation; soliton solution; critical point; Morse theory; local linking.

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1. Introduction

In this paper, we deal with the soliton solutions for a quasilinear Schrödinger equation of the form

$$\begin{aligned} -\Delta_p u - \frac{p}{2^{p-1}} u\Delta_p(u^2) &= f(x,u), \text{ in } \Omega, \\ u &= 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$.

When $p = 2$, equation (1.1) is a special case for some physical phenomena (see e.g. [15, 16, 22]). In fact, solutions for the problem (1.1) for $p = 2$ are the existence of standing wave solutions for the following quasilinear Schrödinger equations

$$i\partial_t z = -\Delta z + Wz - f(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z, \tag{1.2}$$

where $W(x)$, $x \in \mathbb{R}^N$ is a given potential, κ is a real constant and f, h are real functions of essentially pure power forms. The semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years (see [1, 6, 27]). Quasilinear equations of the form (1.2) appear more naturally in mathematical physics and have been derived as a model of several physical phenomena corresponding to various types of h (see [2, 8–10, 19, 20, 23–25, 28]).

To the best of our knowledge, very few results are known about equations of the form (1.2) before Liu's research team [15, 16], in which, the existence of positive solution has been proved in [16] by using a constrained minimization argument; the problem (1.2) was transformed into a semilinear one by a change of variables and an Orlicz space framework was used in [15]. Since then several papers appeared in the mathematical literature for the equation defined in the domain $\mathbb{R}^N$ (for example, see [4, 5, 7, 17, 26, 29]). Considering this problem in a bounded domain is a new research field. An example of this topic can be found in [12].

In this paper, we consider soliton solutions for the following quasilinear Schrödinger equations of a much more general form of (1.2) in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary condition,

$$i\partial_t z = -\Delta_p z + Wz - f(|z|^2)z - \kappa\Delta_p h(|z|^2)h'(|z|^2)z,$$

in which $\kappa = \frac{p}{2^{p-1}} > 0$, $h(s) = s$ and $f = f(x, s)$ is a Caratheodory function under some power growth with respect to s; at the same time we assume $W(x) \equiv W$ (a constant) to indicate that the solution stays at a constant potential level. Putting $z(x, t) = \exp(-iWt)u(x)$ we obtain the corresponding equation (1.1) of elliptic type which has a formal variational structure (see §2).

A major difficulty of the problem (1.1) is that the functional corresponding to the equation is not well defined for all $u \in W_0^{1,p}(\Omega)$ if $N \geq p$. We use the method of a change of variables developed in [12] to overcome this difficulty. Then by a standard argument from Morse theory, we develop the existence of nontrivial solutions to our problem in a weak sense.

This paper is organized as follows. In §2, we state our main theorem; in §3, we give some related lemmas to prepare for the proof of the main theorem; and in §4, we prove the main theorems.

2. Main theorem

In this section, we will state our main theorem.

We assume the perturbation $f(x, t)$ is a Caratheodory function satisfying

(F1) $|f(x, t)| \leq C(1 + |t|^{2(q-1)})$ holds for some positive constant C, all $x \in \Omega$ and $t \in \mathbb{R}$, where $1 \leq q < p^* := \frac{Np}{N-p}$.

Firstly we introduce a variational framework of problem (1.1). We observe that (1.1) is the Euler–Lagrange equation associated with the energy functional

$$J(u) := \frac{1}{p}\int_\Omega (1 + p|u|^p)|\nabla u|^p \,\mathrm{d}x - \int_\Omega F(x, u)\,\mathrm{d}x, \tag{2.1}$$

where $F(x, t) = \int_0^t f(x, s)\,\mathrm{d}x$.

It is difficult to apply variational methods to the functional J directly. Unless $N = 1$, the functional J is not well-defined for all $u \in W_0^{1,p}(\Omega)$. To overcome this difficulty, we generalized the changing of variables developed in [4, 29]. That is

$$v := g^{-1}(u),$$

where g is defined by

$$
\begin{aligned}
g'(t) &= \frac{1}{(1+p|g(t)|^p)^{1/p}}, \quad && \forall\, t \in [0, +\infty];\\
g(t) &= -g(-t), && \forall\, t \in (-\infty, 0].
\end{aligned}
$$

We give out the properties of g in the following lemma for the readers convenience. The readers can find the proof of it in [12].

Lemma 2.1. *The function g defined above satisfies the following properties*:

(1) $g(0) = 0$;
(2) g *is uniquely defined,* C^∞ *and invertible*;
(3) $|g'(t)| \le 1$ *for all* $t \in \mathbb{R}$;
(4) $\frac{1}{2} g(t) \le t g'(t) \le g(t)$ *for all* $t > 0$;
(5) $g(t)/t \nearrow 1$, *as* $t \to 0+$;
(6) $|g(t)| \le |t|$ *for all* $t \in \mathbb{R}$;
(7) $g(t)/\sqrt{t} \nearrow K_0 := \sqrt{2} p^{-\frac{1}{2p}}$, *as* $t \to +\infty$;
(8) $|g(t)| \le K_0 |t|^{\frac{1}{2}}$ *for all* $t \in \mathbb{R}$;
(9) $g^2(t) - g(t) g'(t) t \ge 0$ *for all* $t \in \mathbb{R}$;
(10) *there exists a positive constant* C *such that* $|g(t)| \ge C|t|$ *for* $|t| \le 1$ *and* $|g(t)| \ge C|t|^{\frac{1}{2}}$ *for* $|t| \ge 1$;
(11) $|g(t) g'(t)| < K_0^2$ *for all* $t \in \mathbb{R}$.

Under the condition (F1), consider the following functional

$$
\Phi(v) := \frac{1}{p} \int_\Omega |\nabla v|^p \, \mathrm{d}x - \int_\Omega F(x, g(v)) \, \mathrm{d}x. \tag{2.2}
$$

It is easy to see that Φ is well defined on $W_0^{1,p}(\Omega)$ (equipped with the norm $\|v\| := (\int_\Omega |\nabla v|^p \, \mathrm{d}x)^{1/p}$) and $\Phi \in C^1(W_0^{1,p}(\Omega); \mathbb{R})$ by assumption (F1) and Lemma 2.1. Thus for all $w \in W_0^{1,p}(\Omega)$, we have

$$
\langle \Phi(v), w \rangle = \int_\Omega |\nabla v|^{p-2} \nabla v \nabla w \, \mathrm{d}x - \int_\Omega f(x, g(v)) g'(v) w \, \mathrm{d}x.
$$

Then the critical points of Φ are weak solutions (in the usual sense) for the problem

$$
\begin{aligned}
-\Delta_p v &= f(x, g(v)) g'(v), \quad && \text{in } \Omega,\\
v &= 0, && \text{on } \partial\Omega.
\end{aligned} \tag{2.3}
$$

By setting $v = g^{-1}(u)$, it is easy to see that equation (2.3) is equivalent to our problem (1.1), which takes $u = g(v)$ as its solution.

Motivated by the above, we give the following definition of the weak solution for problem (1.1).

DEFINITION 2.1

We say u is a weak solution for problem (1.1), if $v = g^{-1}(u) \in W_0^{1,p}(\Omega)$ is a critical point of the following functional corresponding to problem (2.3):

$$\Phi(v) := \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega F(x, g(v))\,\mathrm{d}x.$$

It is well known (see [11]) that the p-homogeneous boundary value problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u, \quad && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned}$$

has the first eigenvalue $\lambda_1 > 0$, which is simple and has an associated eigenfunction which is positive in Ω. It is also known that λ_1 is an isolated point of $\sigma(-\Delta_p)$, the spectrum of $-\Delta_p$, which contains at least an increasing eigenvalue sequence obtained by Lusternik–Schnirlaman theory.

Let $V = \mathrm{span}\{\phi_1\}$ be the one-dimensional eigenspace associated to λ_1, where $\phi_1 > 0$ in Ω and $\|\phi_1\| := (\int_\Omega |\nabla\phi_1|^p \,\mathrm{d}x)^{\frac{1}{p}} = 1$. Taking one subspace $Y \subset W_0^{1,p}(\Omega)$ completing V such that $W_0^{1,p}(\Omega) = V \oplus Y$, there exists $\lambda_2 > \lambda_1$ such that

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x \geq \lambda_2 \int_\Omega |u|^p \,\mathrm{d}x, \ \ \forall\, u \in Y,$$

where λ_2 is called the second eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$.

We shall also assume the following condition on f:

(F2) There exist $r > 0$, $\hat{\lambda}_1, \hat{\lambda}_2 \in (\lambda_1, \lambda_2)$ such that $\hat{\lambda}_1 < \hat{\lambda}_2$ and $|t| \leq r$ implies

$$\hat{\lambda}_1 |t|^p \leq pF(x,t) \leq \hat{\lambda}_2 |t|^p.$$

Motivated by [14, 18] and [12], our main results are as follows.

Theorem 2.1. *Assume* $(F1)$, $(F2)$ *and the following*,

(F3) $\limsup\limits_{|t|\to\infty} \frac{pK_0^{2p}F(x,t)}{|t|^{2p}} < \lambda_1$

hold, where $K_0 = \sqrt{2}p^{-\frac{1}{2p}}$. *Then problem* (1.1) *has at least two nontrivial weak solutions in the sense of Definition* 2.1.

Theorem 2.2. *Assume* $(F1)$, $(F2)$ *and the following*,

(F4) $\lim\limits_{|t|\to\infty} \frac{pK_0^{2p}F(x,t)}{|t|^{2p}} = \lambda_1$;

(F5) $\lim\limits_{|t|\to\infty} (f(x,t)t - 2pF(x,t)) = +\infty$

hold, where $K_0 = \sqrt{2}p^{-\frac{1}{2p}}$. *Then problem* (1.1) *has at least two nontrivial weak solutions in the sense of Definition* 2.1.

In the next theorem we assume the following condition on f:

(F6) There exists $p^* > \theta > p$, $M > 0$ such that $|t| \geq M$ implies

$$0 < \theta F(x, t) \leq \frac{1}{2} t f(x, t).$$

We can see that function $f(x, t)$ of the following form satisfies assumptions (F1), (F2) and (F6),

$$f(x, t) = \begin{cases} C_1 |t|^{p-2} t, & \text{for small } |t|, \\ C_2 |t|^{2q-3} t, & \text{for big } |t|, \end{cases}$$

in which $p^* > q > p$ and $\hat{\lambda}_2 > C_1 \geq \hat{\lambda}_1$.

Theorem 2.3. *Under assumptions* ($F1$), ($F2$) *and* ($F6$), *the problem* (1.1) *has a nontrivial weak solution in the sense of Definition* 2.1.

3. Preliminaries

For the rest of this paper, we make use of the following notations: X denotes Sobolev space $W_0^{1,p}(\Omega)$ with the norm $\|\cdot\| := (\int_\Omega |\nabla \cdot|^p \,\mathrm{d}x)^{\frac{1}{p}}$; X^* denotes the conjugate space for X; $L^p(\Omega)$ denotes Lebesgue space with the norm $|\cdot|_p$; $\langle \cdot, \cdot \rangle$ is the dual pairing on the space X^* and X; by $\to$ (resp. $\rightharpoonup$) we mean strong (resp. weak) convergence. $|\Omega|$ denotes the Lebesgue measure of the set $\Omega \subset \mathbb{R}^N$; $C, C_1, C_2, \ldots$ denote (possibly different) positive constants.

First let us prove that any bounded Palais–Smale (PS) sequence has a convergent subsequence for the functional Φ.

Lemma 3.1. *Under assumptions* ($F1$), *any bounded sequence* $\{v_n\} \subset X$ *such that* $\Phi'(v_n) \to 0$ *in* X^*, *as* $n \to \infty$, *has a convergent subsequence.*

Proof. Since $\{v_n\}$ is bounded, by the self-reflexive property of X, there exists a subsequence of $\{v_n\}$ (we may also denote it by $\{v_n\}$) and $v \in X$, such that $v_n \rightharpoonup v$ in X and $\|v_n\| \to t_0$ as $n \to \infty$.

Let us now consider the sequence

$$\begin{aligned} P_n \;:=\; & \langle \Phi'(v_n), v_n \rangle + \int_\Omega f(x, g(v_n)) g'(v_n) v_n \,\mathrm{d}x \\ & - \langle \Phi'(v_n), v \rangle - \int_\Omega f(x, g(v_n)) g'(v_n) v \,\mathrm{d}x \\ = \;& \langle \Phi'(v_n), v_n - v \rangle + \int_\Omega f(x, g(v_n)) g'(v_n)(v_n - v) \,\mathrm{d}x \end{aligned}$$

Since $\{v_n\}$ is bounded, from (F1), Lemma 2.1, Hölder inequality and compact Sobolev embedding, we see that

$$\begin{aligned}
&\left|\int_\Omega f(x, g(v_n))g'(v_n)(v_n - v)\,\mathrm{d}x\right| \\
&\quad \le \int_\Omega |f(x, g(v_n))(v_n - v)|\,\mathrm{d}x \\
&\quad \le \left(\int_\Omega |f(x, g(v_n))|^{\frac{q}{q-1}}\,\mathrm{d}x\right)^{\frac{q-1}{q}} \left(\int_\Omega |v_n - v|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\quad \le \left(\int_\Omega C_4(1 + |g(v_n)|)^{2q}\,\mathrm{d}x\right)^{\frac{q-1}{q}} \left(\int_\Omega |v_n - v|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\quad \le \left(\int_\Omega C_5(1 + |v_n|)^q\,\mathrm{d}x\right)^{\frac{q-1}{q}} \left(\int_\Omega |v_n - v|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \to 0, \\
&\qquad\qquad \text{as } n \to +\infty.
\end{aligned} \tag{3.1}$$

By (3.1) and the following

$$|\langle \Phi'(v_n), v_n - v\rangle| \le C\|\Phi'(v_n)\|_{X^*} \to 0,$$

we have $P_n \to 0$.

At the same time, we have

$$P_n = \|v_n\|^p - \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla v\,\mathrm{d}x + o_n(1). \tag{3.2}$$

Moreover, from $\|v_n\|^p \to t_0$ and the weak convergence, we have

$$o_n'(1) = \|v\|^p - \left(\int_\Omega |\nabla v|^{p-2}\nabla v \nabla v_n\,\mathrm{d}x\right) \to 0, \text{ as } n \to +\infty. \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$o_n''(1) + P_n = \int_\Omega \langle |\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v, \nabla v_n - \nabla v\rangle_n \mathrm{d}x,$$

where $\langle \cdot, \cdot\rangle_n$ denotes the scalar product in $\mathbb{R}^n$.

Using the standard inequality in $\mathbb{R}^n$ given by

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle_n \ge \begin{cases} C_p|x - y|^p, & p \ge 2, \\ C_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & p < 2, \end{cases}$$

we have

$$o_n''(1) + P_n \ge C\|v_n - v\|.$$

Thus $v_n \to v$ in X as $n \to +\infty$. □

Lemma 3.2. *Under assumptions* (*F*1) *and* (*F*3) (*or substitute* (*F*4) *and* (*F*5) *for* (*F*3)), *the functional* Φ *is coercive in* X, *that is*, $\Phi(v) \to +\infty$ *as* $\|v\| \to \infty$.

Proof.

(i) Let (F3) hold. From (F1) and (F3) we can see for some small $\epsilon > 0$, there exists a constant $C > 0$ such that

$$F(x,t) \leq \frac{1}{pK_0^{2p}}(\lambda_1 - \epsilon)|t|^{2p} + C, \quad \forall t \in \mathbb{R},\ \text{a.e. } x \in \Omega.$$

So by Sobolev inequality, for $v \in X$,

$$\begin{aligned}
\Phi(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega F(x, g(v))\,\mathrm{d}x \\
&\geq \frac{1}{p}\|v\|^p - \frac{1}{pK_0^{2p}}(\lambda_1 - \epsilon)\int_\Omega |g(v)|^{2p}\,\mathrm{d}x - C|\Omega| \\
&\geq \frac{1}{p}\|v\|^p - \frac{1}{p}(\lambda_1 - \epsilon)\int_\Omega |v|^p\,\mathrm{d}x - C|\Omega| \\
&\geq \frac{1}{p}\left(1 - \frac{\lambda_1 - \epsilon}{\lambda_1}\right)\|v\|^p - C|\Omega| \to +\infty,\ \text{as } \|v\| \to \infty.
\end{aligned}$$

(ii) Let (F4) and (F5) hold. Write $F(x,t) = \frac{1}{pK_0^{2p}}\lambda_1|t|^{2p} + H(x,t)$ and $f(x,t) = \frac{2}{K_0^{2p}}\lambda_1|t|^{2p-2}t + h(x,t)$. Then

$$\lim_{|t|\to\infty} \frac{pK_0^{2p}H(x,t)}{|t|^{2p}} = 0 \quad \text{and} \quad \lim_{|t|\to\infty}(h(x,t)t - 2pH(x,t)) = +\infty.$$

It follows that for any $N > 0$, there is a $R_N > 0$ such that

$$h(x,t)t - 2pH(x,t) \geq N, \quad \forall\, t \in \mathbb{R}, |t| \geq R_N,\ \text{a.e. } x \in \Omega.$$

Intergrating the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{H(x,t)}{|t|^{2p}}\right) = \frac{h(x,t)t - 2pH(x,t)}{|t|^{2p+1}}$$

over the interval $[t, T] \subset [R_N, +\infty)$, we have

$$\frac{H(x,T)}{T^{2p}} - \frac{H(x,t)}{t^{2p}} \geq \frac{N}{2p}\left(\frac{1}{t^{2p}} - \frac{1}{T^{2p}}\right).$$

Letting $T \to +\infty$, we have $H(x,t) \leq -\frac{N}{2p}$, for $t \in \mathbb{R}$, $t \geq R_N$, a.e. $x \in \Omega$. In a similar way, we have $H(x,t) \leq -\frac{N}{2p}$, for $t \in \mathbb{R}$, $t \leq -R_N$, a.e. $x \in \Omega$. So we can see

$$\lim_{|t|\to\infty} H(x,t) \to -\infty \quad \text{a.e. } x \in \Omega. \tag{3.4}$$

We suppose on the contrary, there exists a sequence $\{v_n\} \subset X$ such that $\|v_n\| \to \infty$ as $n \to \infty$, but $\Phi(v_n) \leq C$ for some constant $C \in \mathbb{R}$. Set $w_n = \frac{v_n}{\|v_n\|}$, then up to subsequence, we assume there is some $w_0 \in X$ such that $w_n \rightharpoonup w_0$ in X, $w_n \to w_0$ in $L^2(\Omega)$, and $w_n(x) \to w_0(x)$ for a.e. $x \in \Omega$. Moreover, we have the following:

$$\begin{aligned}
\frac{C}{\|v_n\|^p} &\geq \frac{\Phi(v_n)}{\|v_n\|^p} = \frac{1}{p\|v_n\|^p}\int_\Omega |\nabla v_n|^p\,\mathrm{d}x - \frac{1}{\|v_n\|^p}\int_\Omega F(x, g(v_n))\,\mathrm{d}x \\
&\geq \frac{1}{p}\int_\Omega (|\nabla w_n|^p - \lambda_1 |w_n|^p)\,\mathrm{d}x - \frac{1}{\|v_n\|^p}\int_\Omega H(x, g(v_n))\,\mathrm{d}x \\
&= \frac{1}{p}\int_\Omega (|\nabla w_n|^p - \lambda_1 |w_n|^p)\,\mathrm{d}x - \frac{1}{\|v_n\|^p}\int_{\{|g(v_n)|\geq R_N\}} H(x, g(v_n))\,\mathrm{d}x \\
&\quad -\frac{1}{\|v_n\|^p}\int_{\{|g(v_n)|\leq R_N\}} H(x, g(v_n))\,\mathrm{d}x \\
&\geq \frac{1}{p}\int_\Omega (|\nabla w_n|^p - \lambda_1 |w_n|^p)\,\mathrm{d}x + \frac{N|\Omega|}{2p\|v_n\|^p} \\
&\quad -\frac{1}{\|v_n\|^p}\int_{\{|g(v_n)|\leq R_N\}} H(x, g(v_n))\,\mathrm{d}x \\
&\geq \frac{1}{p}\int_\Omega (|\nabla w_n|^p - \lambda_1 |w_n|^p)\,\mathrm{d}x - \frac{C_1}{\|v_n\|^p},
\end{aligned}$$

which implies

$$\limsup_{n\to\infty}\int_\Omega |\nabla w_n|^2\,\mathrm{d}x \leq \lambda_1 \int_\Omega |w_0|^2\,\mathrm{d}x. \tag{3.5}$$

By the weakly semicontinuous property of the norm and the Sobolev inequality again, we have the converse inequality of (3.5),

$$\begin{aligned}
\lambda_1 \int_\Omega |w_0|^2\,\mathrm{d}x \leq \int_\Omega |\nabla w_0|^2\,\mathrm{d}x &\leq \liminf_{n\to\infty}\int_\Omega |\nabla w_n|^2\,\mathrm{d}x \\
&\leq \limsup_{n\to\infty}\int_\Omega |\nabla w_n|^2\,\mathrm{d}x.
\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), $\int_\Omega |\nabla w_0|^2\,\mathrm{d}x = \lambda_1 \int_\Omega |w_0|^2\,\mathrm{d}x$ and $w_n \to w_0$ in X with $\|w_0\| = 1$. Hence $w_0 = \pm\phi_1$. Take $w_0 = \phi_1$. Then $v_n \to +\infty$ a.e. in Ω, which implies $H(x, g(v_n)) \to -\infty$ by (3.4). So we have

$$\begin{aligned}
C &\geq \Phi(v_n) = \frac{1}{p}\int_\Omega |\nabla v_n|^p\,\mathrm{d}x - \int_\Omega F(x, g(v_n))\,\mathrm{d}x \\
&\geq \frac{1}{p}\int_\Omega (|\nabla v_n|^p - \lambda_1 |v_n|^p)\,\mathrm{d}x - \int_\Omega H(x, g(v_n))\,\mathrm{d}x \\
&\geq -\int_\Omega H(x, g(v_n))\,\mathrm{d}x \to +\infty, \qquad \text{as } n \to \infty,
\end{aligned}$$

which is a contradiction. So we have Φ is coercive in X. □

Lemma 3.3. *Under assumptions* $(F1)$ *and* $(F6)$, *any* (PS) *sequence for the functional* Φ *is bounded.*

Proof. Suppose that $\{v_n\} \subset X$, $|\Phi(v_n)| \leq B$ for some $B \in \mathbb{R}$, and $\Phi'(v_n) \to 0$ in X^* as $n \to \infty$.

After integrating, we obtain from the assumption (F6) that there exists C_1 such that

$$C_1(|t|^{2\theta} - 1) \le F(x,t) \quad \text{for all } x \in \Omega, t \in \mathbb{R}. \tag{3.7}$$

Let $c := \sup_n \Phi(v_n)$ and $\beta \in (\frac{1}{\theta}, \frac{1}{p})$ for large n. From Lemma 2.1(4) and (10), (F6) and the inequality (3.7), we have

$$\begin{aligned}
c + 1 + \|v_n\| &\ge \Phi(v_n) - \beta\langle \Phi'(v_n), v_n\rangle \\
&= \frac{1}{p}\|v_n\|^p - \beta\|v_n\|^p + \int_\Omega \big(\beta f(x, g(v_n))g'(v_n)v_n - F(x, g(v_n))\big)\,\mathrm{d}x \\
&\ge \frac{1}{p}\|v_n\|^p - \beta\|v_n\|^p + \int_\Omega \left(\frac{1}{2}\beta f(x, g(v_n))g(v_n) - F(x, g(v_n))\right)\mathrm{d}x \\
&\ge \left(\frac{1}{p} - \beta\right)\|v_n\|^p + (\theta\beta - 1)\int_\Omega F(x, g(v_n))\,\mathrm{d}x \\
&\ge \left(\frac{1}{p} - \beta\right)\|v_n\|^p + C_1(\theta\beta - 1)\int_\Omega |g(v_n)|^{2\theta}\,\mathrm{d}x - C_3 \\
&\ge \left(\frac{1}{p} - \beta\right)\|v_n\|^p + C_2(\theta\beta - 1)|v_n|_\theta^\theta - C_3, \\
&\ge \left(\frac{1}{p} - \beta\right)\|v_n\|^p - C_3.
\end{aligned}$$

Noticing that $\frac{1}{p} - \beta > 0$ and $\theta\beta - 1 > 0$, we obtain the boundedness of $\{v_n\}$ in X. □

By Lemmas 3.1, 3.2 and 3.3, we have as follows.

Lemma 3.4. *Under assumptions* (*F*1) *and* (*F*3) (*or* (*F*3) *substitute for both* (*F*4) *and* (*F*5)), *the functional* Φ *satisfies the* (*PS*) *condition.*

Lemma 3.5. *Under assumptions* (*F*1) *and* (*F*6), *the functional* Φ *satisfies the* (*PS*) *condition.*

We give the local linking property for our functional Φ.

Lemma 3.6. *Under assumptions* (*F1*) *and* (*F*2), *the functional* Φ *has a local linking at the origin with respect to* $X = V \oplus Y$, *where* V *and* Y *are functional sub-spaces of* $W_0^{1,p}(\Omega)$ *in* §2, *that is, there is a small ball* B_ρ *with the center at* $0 \in X$ *and a small radius* $\rho > 0$ *such that*

(i) $\Phi(v) \le \Phi(0)$, *for* $v \in B_\rho \cap V$;
(ii) $\Phi(y) > \Phi(0)$, *for* $y \in B_\rho \cap Y\backslash\{0\}$.

Proof.

(i) Take $v \in V$, since V is of finite dimension, we can see that $\|v\| \le \rho$ implies $|g(v)| \le r, \forall x \in \Omega$ for $\rho > 0$ small enough. So by (F2), for $\|v\| \le \rho$,

$$\Phi(v) = \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x - \int_\Omega F(x, g(v))\,\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{p}\lambda_1 \int_\Omega |v|^p \,\mathrm{d}x - \int_\Omega F(x, g(v))\,\mathrm{d}x \\
&= \int_{|g(v)|\le r} \left(\frac{1}{p}\lambda_1 |v|^p - F(x, g(v))\right) \mathrm{d}x \\
&\le \int_{|g(v)|\le r} \left(\frac{1}{p}\hat{\lambda}_1 |g(v)|^p - F(x, g(v))\right) \mathrm{d}x \\
&\le 0 = \Phi(0).
\end{aligned}
$$

(ii) Take $y \in Y$. From Lemma 2.1, assumption (F1), (F2), Sobolev embedding and the definition of λ_2 in §2, we have the following

$$
\begin{aligned}
\Phi(y) &= \frac{1}{p}\int_\Omega |\nabla y|^p \,\mathrm{d}x - \int_\Omega F(x, g(y))\,\mathrm{d}x \\
&\ge \frac{1}{p}\int_\Omega (|\nabla y|^p - \hat{\lambda}_2 |y|^p)\,\mathrm{d}x - \int_{\{|g(y)|\le r\}} \left(F(x, g(y)) - \frac{\hat{\lambda}_2}{p}|y|^p\right)\mathrm{d}x \\
&\quad - \int_{\{|g(y)|> r\}} \left(F(x, g(y)) - \frac{\hat{\lambda}_2}{p}|y|^p\right)\mathrm{d}x \\
&\ge \frac{1}{p}\left(1 - \frac{\hat{\lambda}_2}{\lambda_2}\right)\|y\|^p - C_1 \int_{\{|g(y)|> r\}} |g(y)|^{2s}\,\mathrm{d}x \\
&\ge \frac{1}{p}\left(1 - \frac{\hat{\lambda}_2}{\lambda_2}\right)\|y\|^p - C_2 \int_{\{|g(y)|> r\}} |y|^{s}\,\mathrm{d}x \\
&\ge \frac{1}{p}\left(1 - \frac{\hat{\lambda}_2}{\lambda_2}\right)\|y\|^p - C_3 \|y\|^s. \quad (p < s \le p^*)
\end{aligned}
$$

So we can derive when $y \in Y$ and $0 < \|y\| \le \rho$ for $\rho > 0$ small, $\Phi(y) > 0 = \Phi(0)$, which completes the proof. □

Remark 3.1. From the proof of Lemma 3.6, we can get a more stronger result: There exists a $\rho_0 > 0$, such that for any $0 < \rho < \rho_0$, B_ρ satisfies all the conditions required by the definition of local linking. From this point of view, we can conclude $0 \in X$ is the unique critical point of our Φ in a ball small enough.

For an isolated critical point $v \in E$ (E is a given Banach space) of a C_1 functional $f : E \to \mathbb{R}$, we define the critical group of f at v as

$$C_q(f, v) = H_q(f_c \cap B_\rho,\ f_c \cap B_\rho \backslash \{v\})$$

in which $c = f(v)$, $f_c = \{v \in E | f(x) \le c\}$, $\rho > 0$ is small and $H_q(A, B)$ is the q-dimensional homology group for the topological space A relative to its subspace B ($B \subset A$), see in [3].

We are ready to give a result from Morse theory as follows.

Lemma 3.7 (*Theorem* 2.1 *from* [13]). *Let E be a Banach space and $f : E \to \mathbb{R}$ a C^1-functional satisfying the* (*PS*) *condition. Suppose that E has a decomposition $E = W \oplus Z$,* *where W is a finite dimensional subspace, say* $\dim W = m < \infty$. *Suppose there exists a small ball B_ρ with its center at the origin 0 and a small radius $\rho > 0$ such that*

$$\begin{aligned} f(w) &\leq f(0), \quad \text{for } w \in B_\rho \cap W; \\ f(z) &> f(0), \quad \text{for } z \in B_\rho \cap Z\backslash\{0\}. \end{aligned}$$

If $0 \in E$ *is the unique critical point of f in B_ρ, then*

$$C_m(f, 0) = H_m(f_c \cap B_\rho,\ f_c \cap B_\rho\backslash\{0\}) \neq 0.$$

Since $\dim V = 1 < \infty$, by Lemma 3.2, Remark 3.1 and Lemma 3.7, we have as follows.

Lemma 3.8. *Under assumptions* $(F1)$ *and* $(F2)$, *0 is a critical point of Φ and* $C_1(\Phi, 0) \neq 0$.

Remark 3.2. From our Lemma 3.8, and by Theorem 4.2 in [3], Chapter I, there is a $\epsilon > 0$, such that $H_1(\Phi_\epsilon, \Phi_{-\epsilon}) \neq 0$, because it contains a nontrivial group $C_1(\Phi, 0)$ as its subgroup.

In order to give a nontrivial critical point of Φ in Theorem 2.3, we need to compute the topological property of Φ near infinity.

Lemma 3.9. *Under assumptions* $(F1)$ *and* $(F6)$, *there exists a constant $A > 0$ such that*

$$\Phi_a \simeq S^\infty, \quad \text{for } a < -A,$$

in which S^∞ is the unit sphere in X.

Proof. From the assumption of (F6) and Lemma 2.1, there exists a constant $C_1 > 0$ such that

$$F(x, g(t)) \geq C_1|g(t)|^{2\theta} \geq C_2|t|^\theta, \quad \text{for } |t| \geq M.$$

So for $v \in S^\infty$, note that $\theta > 2$ and we have for $t > M$ (the same M in (F6)),

$$\begin{aligned} \Phi(tv) &= \frac{1}{p}\int_\Omega |\nabla tv|^p \,\mathrm{d}x - \int_\Omega F(x, g(tv)) \,\mathrm{d}x \\ &\leq \frac{t^p}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - C_2|t|^\theta \int_\Omega |v|^\theta \,\mathrm{d}x \\ &\leq \frac{1}{p}t^p - C_3(v)t^\theta \to -\infty \quad \text{as } t \to +\infty. \end{aligned}$$

For a given $\epsilon_0 > 0$, set

$$A := \left(1 + \frac{1}{p}\right) M|\Omega| \max_{\overline{\Omega}\times[-M,M]} |f(x, s)| + \epsilon_0.$$

From (F6) and Lemma 2.1,

$$
\begin{aligned}
&\int_\Omega F(x, g(v))\,\mathrm{d}x - \frac{1}{p}\int_\Omega f(x, g(v))g'(v)v\,\mathrm{d}x \\
&\quad = \int_{\{|g(v)|\geq M\}} F(x, g(v))\,\mathrm{d}x + \int_{\{|g(v)|\leq M\}} F(x, g(v))\,\mathrm{d}x \\
&\qquad - \frac{1}{p}\int_{\{|g(v)|\geq M\}} f(x, g(v))g'(v)v\,\mathrm{d}x - \frac{1}{p}\int_{\{|g(v)|\leq M\}} f(x, g(v))g'(v)v\,\mathrm{d}x \\
&\quad \leq \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)\int_{\{|g(v)|\geq M\}} vf(x, g(v))\,\mathrm{d}x + \int_{\{|g(v)|\leq M\}} F(x, g(v))\,\mathrm{d}x \\
&\qquad - \frac{1}{p}\int_{\{|g(v)|\leq M\}} f(x, g(v))g'(v)v\,\mathrm{d}x \\
&\quad \leq \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)\int_{\{|g(v)|\geq M\}} vf(x, g(v))\,\mathrm{d}x + \int_{\{|g(v)|\leq M\}} F(x, g(v))\,\mathrm{d}x \\
&\qquad + \frac{1}{p}\int_{\{|g(v)|\leq M\}} |f(x, g(v))g(v)|\,\mathrm{d}x \\
&\quad \leq \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)\int_{\{|g(v)|\geq M\}} vf(x, g(v))\,\mathrm{d}x \\
&\qquad + \left(1 + \frac{1}{p}\right) M|\Omega| \max_{\overline{\Omega}\times[-M,M]} |f(x, s)| \\
&\quad \leq \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)\int_{\{|g(v)|\geq M\}} vf(x, g(v))\,\mathrm{d}x + A - \epsilon_0.
\end{aligned}
$$

By (F3) and the above inequalities, for $a < -A$, any given $v \in S^\infty$ and $t = t(v) \in \{s \in \mathbb{R}^+ | \Phi(sv) = \frac{1}{p}|s|^p - \int_\Omega F(x, g(sv))\,\mathrm{d}x \leq a\}$, we have

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \Phi(sv) &= \langle \Phi'(tv), v\rangle = |t|^{p-2}t - \int_\Omega f(x, g(tv))g'(tv)v\,\mathrm{d}x \\
&= \frac{p}{t}\left\{\frac{1}{p}|t|^p - \frac{1}{p}\int_\Omega f(x, g(tv))g'(tv)tv\,\mathrm{d}x\right\} \\
&\leq \frac{p}{t}\left\{\left(\int_\Omega F(x, g(tv))\,\mathrm{d}x + a\right)\right. \\
&\qquad \left. - \frac{1}{p}\int_\Omega f(x, g(tv))g'(tv)tv\,\mathrm{d}x\right\} \\
&\leq \frac{p}{t}\left\{a + \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)\int_{\{|g(tv)|\geq M\}} tvf(x, g(tv))\,\mathrm{d}x + A - \epsilon_0\right\} \\
&\leq \frac{p}{t}\left\{\frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)\int_{\{|g(tv)|\geq M\}} g(tv)f(x, g(tv))\,\mathrm{d}x - \epsilon_0\right\} \\
&< 0.
\end{aligned}
$$

By the implicit function theorem, there is an unique $T(v) \in C(S^\infty, \mathbb{R}^+)$ such that

$$\Phi(T(v)v) = a, \quad \forall v \in S^\infty.$$

For $v \neq 0$, set $\hat{T} = \frac{1}{\|v\|}T(\frac{v}{\|v\|})$. Then $\hat{T} \in C(X\backslash\{0\}, \mathbb{R}^+)$ and for all $v \in X\backslash\{0\}$, $\Phi(\hat{T}(v)v) = a$. Moreover, if $\Phi(v) = a$, then $\hat{T}(v) = 1$.

We define a function $\tilde{T} : X\backslash\{0\} \to \mathbb{R}^+$ as

$$\tilde{T}(v) := \begin{cases} \hat{T}(v), & \text{if } \Phi(u) \geq a, \\ 1, & \text{if } \Phi(u) \leq a. \end{cases}$$

Since $\Phi(v) = a$ implies $\hat{T}(v) = 1$, we can say $\tilde{T} \in C(X\backslash\{0\}, \mathbb{R}^+)$.

Finally let $\eta : [0, 1] \times (X\backslash\{0\}) \to X\backslash\{0\}$ as follows:

$$\eta(s, v) = (1 - s)v + s\tilde{T}(v)v.$$

We can see that η is a strong deformation retract from $X\backslash\{0\}$ to Φ_a. Thus $\Phi_a \simeq X\backslash\{0\} \simeq S^\infty$, which completes the proof. □

4. The proof of the main theorem

In order to give the proof of Theorems 2.1 and 2.2, we present the following lemma from [14].

Lemma 4.1. *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ satisfy the (PS) condition and be bounded from below. If Φ has a critical point that is homological nontrivial (here, by homological nontrivial, we mean the critical group at the corresponding critical point is nontrivial) and is not the minimizer of Φ, then Φ has at least three critical points.*

Proof of Theorems 2.1 *and* 2.2

Proof. By Lemmas 3.2 and 3.4, Φ is coercive and satisfies the (PS) condition. Hence Φ is bounded below. By Lemma 3.8, $0 \in X$ is a homological nontrivial critical point of Φ but not a minimizer. Then the conclusion follows from Lemma 4.1. □

In order to give the proof of Theorem 2.3, we present the following theorem from Perera [21].

Lemma 4.2. *Let $B' \subset B \subset A \subset A'$ be topological spaces and $q \in \mathbb{Z}$. If*

$$H_q(A, B) \neq 0 \quad \textit{and} \quad H_q(A', B') = 0,$$

then

$$H_{q+1}(A', A) \neq 0 \quad \textit{or} \quad H_{q-1}(B, B') \neq 0.$$

Proof of Theorem 2.3

Proof. By Remark 3.2, there exists $\epsilon > 0$ such that

$$H_1(\Phi_\epsilon, \Phi_{-\epsilon}) \neq 0.$$

By Lemma 3.9, for $a < -A$, we conclude $\Phi_a \simeq S^\infty$ for $\dim X = +\infty$, we have

$$H_1(X, \Phi_a) = H_1(X, S^\infty) = 0.$$

So from Lemma 4.2,

$$H_2(X, \Phi_\epsilon) \neq 0 \quad \text{or} \quad H_0(\Phi_{-\epsilon}, \Phi_a) \neq 0,$$

which implies by a mini-max argument in [3] driven by the non-trivial homology group $H_2(X, \Phi_\epsilon)$ or $H_0(\Phi_{-\epsilon}, \Phi_a)$ that Φ yields a critical point u such that

$$\Phi(u) > \epsilon \quad \text{or} \quad a < \Phi(u) < -\epsilon.$$

Hence, u is a nontrivial critical point for Φ, which means problem (1.1) has a nontrivial solution. □

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References

[1] Berestycki H and Lions P L, Nonlinear scalar field equations, I: Existence of a ground state, *Arch. Rational Mech. Anal.* **82** (1983) 313–346

[2] Brandi H S, Manus C, Mainfray G and Bonnaud G, Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma, *Phys. Fluids B* **5** (1993) 3539–3550

[3] Chang K Q, Infinite dimensional Morse theory and multiple solution problems (1993) (Boston: Birkhäuser)

[4] Colin M and Jeanjean L, Solutions for a quasilinear Schrödinger equation: a dual approach, *Nonlinear Anal.* **56** (2004) 213–226

[5] Fang X D and Szulkin A, Mutiple solutions for a quasilinear Schrödinger equation, *J. Differential Equations* **254** (2013) 2015–2032

[6] Floer A and Weinstein A, Nonspreading wave packets for the cubic Schödinger with a bounded potential, *J. Funct. Anal.* **69** (1986) 397–408

[7] Gloss E, Existence and concentration of positive solutions for a quasilinear equation in $\mathbb{R}^n$, *J. Math. Anal. Appl.* **371** (2010) 465–484

[8] Hasse R W, A general method for the solution of nonlinear soliton and kink Schrödinger equations, *Z. Physik B* **37** (1980) 83–87

[9] Kurihura S, Large-amplitude quasi-solitons in superfluid films, *J. Phys. Soc. Japan* **50** (1981) 3262–3267

[10] Lange H, Toomire B and Zweifel P F, Time-dependent dissipation in nonlinear Schrödinger systems, *J. Math. Phys.* **36** (1995) 1274–1283

[11] Lindqvist P, On the equation div $(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Proc. Amer. Math. Soc.* **109** (1990) 609–623

[12] Liu D, Soliton solution for a quasilinear Schrödinger equation, *Elec. J. Diff. Equa.* **2013(267)** (2013) 1–13

[13] Liu J Q, The morse index of a saddle point, *Syst. Sci. Math. Sci.* **2** (1989) 32–39

[14] Liu J Q and Su J B, Remarks on multiple nontrivial solutions for quasilinear resonant problems, *J. Math. Anal. Appl.* **258** (2001) 209–222

[15] Liu J Q, Wang Y Q and Wang Z Q, Soliton solutions for quasilinear Schrödinger equations, II, *J. Differential Equations* **187** (2003) 473–493

[16] Liu J Q and Wang Z Q, Soliton solutions for quasilinear Schrödinger equations, i, *Proc. Amer. Math. Soc.* **131** (2003) 441–448
[17] Liu J Q, Wang Z Q and Guo Y X, Multibump solutions for quasilinear Schrödinger equations, *J. Func. Anal.* **262** (2012) 4040–4102
[18] Liu S B, Existence of solutions to a superlinear p-Laplacian equation, *EJDE* **2001** (2001) 1–6
[19] Makhandov V G and Fedyanin V K, Non-linear effects in quasi-one-dimensional models of condensed matter theory, *Physics Reports* **104** (1984) 1–86
[20] Nakamura A, Damping and modification of exciton solitary waves, *J. Phys. Soc. Japan* **42** (1977) 1824–1835
[21] Perera K, Critical groups of critical points produced by local linking with applications, *Abst. Appl. Anal.* **3** (1998) 437–446
[22] Poppenberg M, Schmitt K and Wang Z Q, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Differ. Equ.* **14** (2002) 329–344
[23] Porkolab M and Goldman M V, Upper hybrid solitons and oscillating two-stream instabilities, *Phys. Fluids* **19** (1976) 872–881
[24] Quispel G R W and Capel H W, Equation of motion for the heisenberg spin chain, *Physica* **110 A** (1982) 41–80
[25] Ritchie B, Relativistic self-focusing and channel formation in laser-plasma interactions, *Phys. Rev. E* **50** (1994) 687–689
[26] Silva E A and Vieira G G, Quasilinear asymptotically periodic Schrödinger equations with subcritical growth, *Nonlinear Anal.* **72** (2010) 2935–2949
[27] Strauss W A, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* **55** (1977) 149–162
[28] Takeno S and Homma S, Classical planar heisenberg ferromagnet, complex scalar fields and nonlinear excitation, *Progr. Theoret. Phys.* **65** (1981) 172–189
[29] Wang Y J and Zou W M, Bound states to critical quasilinear Schrödinger equations, *NoDEA Nonlinear Differ. Equ. Appl.* **19** (2012) 19–47

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