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A note on neighborhood total domination in graphs

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Abstract. Let $G = (V, E)$ be a graph without isolated vertices. A dominating set S of G is called a neighborhood total dominating set (or just NTDS) if the induced subgraph $G[N(S)]$ has no isolated vertex. The minimum cardinality of a NTDS of G is called the neighborhood total domination number of G and is denoted by $\gamma_{\rm nt}(G)$. In this paper, we obtain sharp bounds for the neighborhood total domination number of a tree. We also prove that the neighborhood total domination number is equal to the domination number in several classes of graphs including grid graphs.

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1. Introduction

For notation and graph theory terminology in general, we follow [3]. Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = \bigcup_{v\in S} N[v]$. The degree $\deg(x)$ of a vertex x denotes the number of neighbors of x in G. The diameter $\mathrm{diam}(G)$ of G is the maximum distance between any pair of vertices of G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A set of vertices S in G is a dominating set, if $N[S] = V(G)$, while it is a total dominating set (TDS), if $N(S) = V(G)$. The domination number, $\gamma(G)$ of G is the minimum cardinality of a dominating set of G, and the total domination number, $\gamma_{\rm t}(G)$ of G is the minimum cardinality of a TDS of G. For references on domination theory, see, for example, [2, 3, 6, 7].

Arumugam and Sivagnanam in [1] introduced the concept of neighborhood domination in graphs. A dominating set S of a graph G is called a neighborhood total dominating set (NTDS) if the induced subgraph $G[N(S)]$ contains no isolated vertex. A NTDS S is said to be minimal if no proper subset of S is a NTDS. The minimum cardinality of a NTDS of G is called the neighborhood total domination number of G and is denoted by $\gamma_{\rm nt}(G)$. If S is a NTDS and $|S| = \gamma_{\rm nt}(G)$, then we call S a $\gamma_{\rm nt}(G)$-set. This concept has been further studied by Henning and Jafari Rad [4].

The cartesian product of two graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G)\times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if either (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.

In this paper, we obtain sharp bounds for the neighborhood total domination number of a tree. We also prove that the neighborhood total domination number is equal to the domination number in several classes of graphs including grid graphs.

We recall that a leaf in a graph G is a vertex v with $\deg(v) = 1$, and a support vertex is a vertex which is adjacent to a leaf. A double star is a tree with precisely two vertices of degree more than one. We refer a path, a cycle and a complete graph of order n with P_n, C_n and K_n, respectively. We make use of the following.

Theorem 1.1 [5].

(1) $\gamma(P_2\Box P_n) = \lfloor \frac{n+2}{2} \rfloor$,
(2) $\gamma(P_3\Box P_n) = \lfloor \frac{3n+4}{4} \rfloor$,
(3) $\gamma(P_4\Box P_n) = n + 1$ *for* $n = 2, 3, 5, 6, 9$ *and* $\gamma(P_4\Box P_n) = n$ *otherwise.*

2. Trees

In this section, we present sharp bounds for the neighborhood total domination number of a tree.

Theorem 2.1. *If T is a tree of order $n \geq 2$ with l leaves and s support vertices, then*

$$\gamma_{\rm nt}(T) \geq \frac{n+s-l+2}{3}$$

and this bound is sharp.

Proof. The proof is by induction on n. The result is obvious for $n = 2$. Now, let $n > 2$. Then $\mathrm{diam}(T) \geq 2$. If $\mathrm{diam}(T) = 2$, then $T = K_{1,n-1}$ and so $\gamma_{\rm nt}(T) = 2$, while $\frac{n+s-l+2}{3} = \frac{4}{3}$. These are sufficient as the basic step of the induction. Assume the result is correct for any tree T' of order $n' < n$. If $\mathrm{diam}(T) = 3$ then T is a double star $S(a, b)$. It can be easily seen that $\gamma_{\rm nt}(T) = \frac{n+s-l+2}{3} = 2$. Thus $\mathrm{diam}(T) \geq 4$. Let x and y be two leaves with $d(x, y) = \mathrm{diam}(T)$. We root T at x. Let z be the parent of y, w be the parent of z, and u be the parent of w.

Assume that $\deg(z) \geq 3$. Let $y_1 \neq y$ be a child of z. Assume that $y \in S$. If $z \in S$, then we may assume that $y_1 \notin S$, since otherwise $S - \{y_1\}$ is a NTDS for T, a contradiction. Then S is a NTDS for $T - y_1$. By the inductive hypothesis,

$$|S| \geq \frac{n-1+s-(l-1)+2}{3} = \frac{n+s-l+2}{3}.$$

Thus $z \notin S$. This implies that $y_1 \in S$. Then $S-\{y_1\}$ is a NTDS for $T-y_1$. By the inductive hypothesis, $|S| - 1 \geq \frac{n-1+s-(l-1)+2}{3}$ and so $|S| > \frac{n+s-l+2}{3}$. Thus we assume that $y \notin S$. Then S contains no child of z, and so S is a NTDS for $T - y_1$. By the inductive hypothesis,

$$|S| \geq \frac{n-1+s-(l-1)+2}{3} = \frac{n+s-l+2}{3}.$$

We thus may assume that $\deg(z) = 2$. Then any child of w is either a leaf or a support vertex of degree two.

Assume that w has a child $z_1 \neq y$ with $\deg(z_1) = 2$. Let y_1 be the child of z_1. Assume that $w \in S$. Clearly $\{y, z\} \cap S \neq \emptyset$ and $\{y_1, z_1\} \cap S \neq \emptyset$. Then $(S - \{z, y, y_1\}) \cup \{z_1\}$ is a

NTDS for $T - \{y, z\}$. By the inductive hypothesis, $|S| - 1 \geq \frac{n-2+(s-1)-(l-1)+2}{3}$ and thus $|S| > \frac{n+s-l+2}{3}$. Thus $w \notin S$. If $z \in S$, then $y \in S$ and then we consider the $\gamma_{\mathrm{nt}}(T)$-set $(S - \{y\}) \cup \{w\}$ containing w which we discussed this case earlier. Thus we assume that $z \notin S$ and similarly $z_1 \notin S$. Then $\{y, y_1\} \subseteq S$. Now $S - \{y\}$ is a NTDS for $T - \{y, z\}$. By the inductive hypothesis, $|S| - 1 \geq \frac{n-2+(s-1)-(l-1)+2}{3}$ and thus $|S| > \frac{n+s-l+2}{3}$.

We thus assume that z is the only child of w which is a support vertex, and so any other child of w is a leaf. We consider the following cases:

Case 1. $\deg(w) \geq 3$. Let a_1 be a leaf adjacent to w. We have two subcases.

Subcase 1.1. $w \in S$. We may assume that $z \in S$, since otherwise $y \in S$ and then we consider the $\gamma_{\mathrm{nt}}(T)$-set $(S - \{y\}) \cup \{z\}$. Now clearly $a_1 \notin S$. If $\deg(w) > 3$ then S is a NTDS for $T - a_1$. By the inductive hypothesis.

$$|S| \geq \frac{n - 1 + s - (l - 1) + 2}{3} \geq \frac{n + s - l + 2}{3}.$$

Thus we assume that $\deg(w) = 3$. If $u \in S$ then $S - \{z\}$ is a NTDS for $T - \{y, z\}$. By the inductive hypothesis, $|S| - 1 \geq \frac{n-2+s-1-(l-1)+2}{3}$ and thus $|S| > \frac{n+s-l+2}{3}$. Thus we assume that $u \notin S$. Assume that $N(u) \cap S = \{w\}$. Then clearly u is not a support vertex. Then $S - \{z\}$ is a NTDS for $T - \{a_1, y, z\}$. By the inductive hypothesis, $|S| - 1 \geq \frac{n-3+s-2-(l-2)+2}{3}$ if $\deg(u) \geq 3$ and $|S| - 1 \geq \frac{n-3+s-2-(l-1)+2}{3}$ if $\deg(u) = 2$. In each case, $|S| \geq \frac{n+s-l+2}{3}$. Thus $N(u) \cap S \neq \{w\}$. If u is a support vertex and u_1 is a leaf adjacent to u, then $u_1 \in S$ and we consider the $\gamma_{\mathrm{nt}}(T)$-set $(S - \{u_1\}) \cup \{u\}$ containing u which we have discussed this case earlier. Thus we assume that u is not a support vertex.

Assume that $\deg(u) \geq 3$. Let w_1 be a child of u. Since u is not a support vertex, $\deg(w_1) \geq 2$. Let w_2 be a child of w_1. If $\deg(w_2) \geq 2$, then we consider a child w_3 of w_2. Since $d(x, w_3) = \mathrm{diam}(T)$, and w_3 plays the same role of y, we may assume that $\deg(w_2) = 2$. Then similar to the case w we can see that w_2 is the only child of w_1 of degree more than one, and thus any child of w_1 different from w_2 is a leaf. Then $|S \cap N[w_1]| \geq 2$, and we may assume that $w_1, w_2 \in S$. Then $S - \{w, z\}$ is a NTDS for $T - \{a_1, w, z, y\}$. By the inductive hypothesis, $|S| - 2 \geq \frac{n-4+s-2-(l-2)+2}{3}$ and thus $|S| \geq \frac{n+s-l+2}{3}$. We thus assume that $\deg(w_2) = 1$. Then we may assume that $w_1, w_2 \in S$. Then $S - \{w, z\}$ is a NTDS for $T - \{a_1, w, z, y\}$. By the inductive hypothesis, $|S| - 2 \geq \frac{n-4+s-2-(l-2)+2}{3}$ and thus $|S| \geq \frac{n+s-l+2}{3}$.

Thus $\deg(u) = 2$. Let v be the parent of u. If $v = x$, then we can easily see that $\gamma_{\mathrm{nt}}(T) = 3$ and $\frac{n+s-l+2}{3} = \frac{8}{3}$. Thus $v \neq x$ and so $\mathrm{diam}(T) \geq 5$. If $N(u) \cap S \neq \emptyset$, then $S - \{w, z\}$ is a NTDS for $T - \{a_1, w, z, y\}$. By the inductive hypothesis, $|S| - 2 \geq \frac{n-4+s-2-(l-2)+2}{3}$ and thus $|S| \geq \frac{n+s-l+2}{3}$. Thus $N(u) \cap S = \emptyset$. In particular, v is not a support vertex. Now $S - \{w, z\}$ is a NTDS for $T - \{u, a_1, w, z, y\}$. By the inductive hypothesis, $|S| - 2 \geq \frac{n-5+s-2-(l-2)+2}{3}$ if $\deg(v) \geq 3$ and $|S| - 2 \geq \frac{n-5+s-2-(l-1)+2}{3}$ if $\deg(v) = 2$. In each case, $|S| \geq \frac{n+s-l+2}{3}$.

Subcase 1.2. $w \notin S$. Then $a_1 \in S$. If w has another child $a_2 \neq a_1$ with $\deg(a_2) = 1$, then $a_2 \in S$ and we consider the $\gamma_{\mathrm{nt}}(T)$-set $(S - \{a_1\}) \cup \{w\}$ containing w which we discussed earlier. Thus we assume that $\deg(w) = 3$. If $z \in S$, then $y \in S$ and $(S - \{a_1, y\}) \cup \{w\}$ is a NTDS for T, a contradiction. Therefore $z \notin S$, and so $y \in S$. If $u \in S$, then $S - \{a_1\}$ is a NTDS for $T - \{a_1\}$. By the inductive hypothesis, $|S| - 1 \geq \frac{n+s-1-(l-1)+2}{3}$ and thus

$|S| > \frac{n+s-l+2}{3}$. Thus $u \notin S$. Now $S-\{y\}$ is a NTDS for $T-\{y,z\}$. By the inductive hypothesis, $|S|-1 \geq \frac{n-2+s-(l-1)+2}{3}$ and thus $|S| > \frac{n+s-l+2}{3}$.

Case 2. $\deg(w) = 2$. Assume that $w \in S$. Clearly we may assume that $z \in S$, since otherwise $y \in S$ and we consider the $\gamma_{\rm nt}(T)$-set $(S-\{y\}) \cup \{z\}$. If $u \in S$, then $S-\{z\}$ is a NTDS for $T-\{y,z\}$. By the inductive hypothesis, $|S|-1 \geq \frac{n-2+s-(l-1)+2}{3}$ and thus $|S| > \frac{n+s-l+2}{3}$. Thus we assume that $u \notin S$. If $N(u) \subseteq S$, then $S-\{w\}$ is a NTDS for $T-\{y\}$. By the inductive hypothesis, $|S|-1 \geq \frac{n-1+s-(l-1)+2}{3}$ and thus $|S| > \frac{n+s-l+2}{3}$. Thus we assume that $N(u) \nsubseteq S$. Then $S-\{z\}$ is a NTDS for $T-\{y,z\}$. By the inductive hypothesis, $|S|-1 \geq \frac{n-2+s-l+2}{2}$ and thus $|S| > \frac{n+s-l+2}{3}$.

We thus assume that $w \notin S$. If $z \in S$, then $y \in S$ and we consider the $\gamma_{\rm nt}(T)$-set $(S-\{y\}) \cup \{w\}$ containing w which has been discussed earlier. Thus $z \notin S$, and so $y \in S$. Then $u \in S$ dominate w. Then $S-\{y\}$ is a NTDS for $T-\{w,z,y\}$. If $\deg(u) \geq 3$, then by the inductive hypothesis, $|S|-1 \geq \frac{n-3+s-1-(l-1)+2}{3}$ and if $\deg(u) = 2$, then by the inductive hypothesis, $|S|-1 \geq \frac{n-3+s-l+2}{3}$. In each case, $|S| \geq \frac{n+s-l+2}{3}$.

To see the sharpness, consider double stars or paths of order $3n+1$. □

Theorem 2.2. *If T is a tree of order $n \geq 3$ with l leaves and s support vertices, then*

$$\gamma_{\rm nt}(T) \leq \frac{n+l-s}{2}$$

and this bound is sharp.

Proof. The proof is by induction on n. The result is obvious for $n = 3$. Thus $n \geq 4$. If $\mathrm{diam}(T) = 2$, then $T = K_{1,n-1}$, and so $\gamma_{\rm nt}(T) = 2$ and $\frac{n+l-s}{\geq}2$, since $n \geq 4$. These are sufficient as the basic step of the induction. Assume the result is correct for any tree T' of order $n' < n$. If $\mathrm{diam}(T) = 3$, then T is a double star $S(a,b)$. It can be easily seen that $\gamma_{\rm nt}(T) = 2$ and $\frac{n+s-l+2}{2} = n-2 \geq 4$. Thus $\mathrm{diam}(T) \geq 4$. Let x and y be two leaves with $d(x,y) = \mathrm{diam}(T)$. We root T at x. Let z be the parent of y and w be the parent of z. First, assume that $\deg(z) \geq 3$. Let $T_1 = T-y$ and let S_1 be a $\gamma_{\rm nt}(T_1)$-set. By the induction hypothesis, $|S_1| \leq \frac{n-1+l-1-s}{2}$. Then $S_1 \cup \{y\}$ is a NTDS for T, and $\gamma_{\rm nt}(T) \leq |S_1 \cup \{y\}| \leq \frac{n+l-s}{2}$. Next assume that $\deg(z) = 2$. Let $T_2 = T-\{y,z\}$, and let S_2 be a $\gamma_{\rm nt}(T_2)$-set. By the induction hypothesis, if $\deg(w) \geq 3$, then $|S_2| \leq \frac{n-2+l-1-(s-1)}{2}$ and if $\deg(w) = 2$ then $|S_2| \leq \frac{n-2+l-s}{2}$. Then $S_2 \cup \{y\}$ is a NTDS for T and $\gamma_{\rm nt}(T) \leq |S_2 \cup \{y\}| \leq \frac{n+l-s}{2}$.

To see the sharpness consider a path P_6. □

3. On graphs G with $\gamma_{\rm nt}(G) = \gamma(G)$

Arumugam *et al.* [1] presented some classes of graphs including paths and cycles with equal domination number and neighborhood total domination number. In this section, we prove that the neighborhood total domination number is equal to the domination number in several classes of graphs including grid graphs. We begin with the following observation.

Observation 3.1. For a graph G, $\gamma_{\rm nt}(G) = \gamma(G)$ if and only if there is a $\gamma(G)$-set S such that $G[N(S)]$ has no isolated vertex.

Recall that a graph is *claw-free* if it does not contain a star of order four as an induced subgraph.

COROLLARY 3.2

If G is a claw-free graph with $\delta(G) \geq 3$, then $\gamma_{\mathrm{nt}}(G) = \gamma(G)$.

PROPOSITION 3.3

If G is a graph with no isolated vertex, $\gamma_{\mathrm{nt}}(G) > 2$ and $\mathrm{diam}(G) \geq 3$, then $\gamma_{\mathrm{nt}}(\bar{G}) = \gamma(\bar{G})$.

Proof. Let x, y be two vertices with $d(x, y) = \mathrm{diam}(G)$. Then $\{x, y\}$ is a dominating set for $\bar{G}$. If $\bar{G}[N_{\bar{G}}(\{x, y\})]$ has an isolated vertex, say a, then $\deg_G(a) = n - 2$. Without loss of generality, assume that $x \in N_G(a)$. Then $\{a, b\}$ is a NTDS for G, where $b \in N_G(a) \cap N_G(y)$, a contradiction. □

It is straightforward to obtain the following.

PROPOSITION 3.4

(1) *If $1 \leq n_1 \leq n_2 \leq \cdots \leq n_m$, then $\gamma_{\mathrm{nt}}(K_{n_1,n_2,\ldots,n_m}) = \gamma(K_{n_1,n_2,\ldots,n_m})$ if and only if $n_1 \geq 2$.*
(2) *If $2 \leq m \leq n$, then $\gamma_{\mathrm{nt}}(K_m \Box K_n) = \gamma(K_m \Box K_n)$.*

Theorem 3.5. *For $2 \leq m \leq n$, $\gamma_{\mathrm{nt}}(P_m \Box P_n) = \gamma(P_m \Box P_n)$.*

Proof. Let $G = P_m \Box P_n$ and $V(G) = \{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$, where for each $i = 1, 2, \ldots, m$, v_{ij} is adjacent to $v_{i(j+1)}$ for $j = 1, 2, \ldots, n-1$, and for each $j = 1, 2, \ldots, n$, v_{ij} is adjacent to $v_{(i+1)j}$ for $i = 1, 2, \ldots, m-1$.

The result can be easily checked for small values of m, n using Theorem 1.1, thus assuming that m, n are at least 5.

We prove by induction on the number of isolated vertices of $G[N(S)]$ for a $\gamma(G)$-set S to show that there is a $\gamma(G)$-set S' obtained from S such that S' is a neighborhood total dominating set for G.

Let S be a $\gamma(G)$-set. If the number of isolated vertices of $G[N(S)]$ is zero, then S is clearly a neighborhood total dominating set for G. This provides the base step. Thus assume that the result is correct if the number of isolated vertices of $G[N(S)]$ is less than k. Now assume that the number of isolated vertices of $G[N(S)]$ is k. Let v_{ij} be an isolated vertex in $G[N(S)]$. Assume that $i \neq 1, m$ and $j \neq 1, n$. Thus $N(v_{ij}) \subseteq S$. It is obvious that $N(x) \cap S = \emptyset$ for each $x \in \{v_{i(j-1)}, v_{i(j+1)}, v_{(i-1)j}, v_{(i+1)j}\}$. If $N(v_{i(j+2)}) \cap (S - \{v_{i(j+1)}\}) \neq \emptyset$, then $S - \{v_{i(j+1)}\}$ is a dominating set G, a contradiction. Thus

$$N(v_{i(j+2)}) \cap (S - \{v_{i(j+1)}\}) = \emptyset. \tag{1}$$

Similarly,

$$N(v_{i(j-2)}) \cap (S - \{v_{i(j-1)}\}) = \emptyset, \tag{2}$$

$$N(v_{(i-2)j}) \cap (S - \{v_{(i-1)j}\}) = \emptyset, \quad (3)$$
$$N(v_{(i+2)j}) \cap (S - \{v_{(i+1)j}\}) = \emptyset. \quad (4)$$

Note that if m or n is small, then at least one of $v_{i(j+2)}$, $v_{i(j-2)}$, $v_{(i-2)j}$, and $v_{(i+2)j}$ exists. Then $S_1 = (S - \{v_{i(j+1)}\}) \cup \{v_{i(j+2)}\}$ is a $\gamma(G)$-set in which the number of isolated vertices of $G[N(S_1)]$ is $k - 1$. Now the inductive hypothesis and Observation 3.1 imply the result. Next, assume that $i = 1$. Then $N(x) \cap S = \emptyset$ for each $x \in \{v_{i(j-1)}, v_{i(j+1)}, v_{(i+1)j}\}$. If $N(v_{i(j+2)}) \cap (S - \{v_{i(j+1)}\}) \neq \emptyset$, then $S - \{v_{i(j+1)}\}$ is a dominating set G, a contradiction. Thus (1) and similarly (2) and (4) hold. Then $S_1 = (S - \{v_{i(j+1)}\}) \cup \{v_{i(j+2)}\}$ is a $\gamma(G)$-set in which the number of isolated vertices of $G[N(S_1)]$ is $k - 1$. Now the inductive hypothesis and Observation 3.1 imply the result. If $j = 1$, then the proof is similarly verified. □

Similarly the following is verified.

Theorem 3.6.

(1) *For* $2 \leq m \leq n$, $\gamma_{\text{nt}}(P_m \Box C_n) = \gamma(P_m \Box C_n)$.
(2) *For* $3 \leq m \leq n$, $\gamma_{\text{nt}}(C_m \Box C_n) = \gamma(C_m \Box C_n)$.

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