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Morozov-type discrepancy principle for nonlinear ill-posed problems under η-condition

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Abstract. For proving the existence of a regularization parameter under a Morozov-type discrepancy principle for Tikhonov regularization of nonlinear ill-posed problems, it is required to impose additional nonlinearity assumptions on the forward operator. Lipschitz continuity of the Freéchet derivative and requirement of the Lipschitz constant to depend on a source condition is one such restriction (Ramlau P, *Numer. Funct. Anal. Optim.* **23**(**1&22**) (2003) 147–172). Another nonlinearity condition considered by Scherzer (*Computing*, **51** (1993) 45–60) was by requiring the forward operator to be close to a linear operator in a restricted sense. A seemingly natural nonlinear assumption which appears in many applications which attracted attention in various contexts of the study of nonlinear problems is the so-called η-condition. However, a Morozov-type discrepancy principle together with η-condition does not seem to have been studied, except in a recent paper by the author (*Bull. Aust. Math. Soc.* **79** (2009) 337–342), where error estimates under a general source condition is derived, by assuming the existence of the parameter. In this paper, the existence of the parameter satisfying a Morozov-type discrepancy principle is proved under the η-condition on the forward operator, by assuming the source condition as in the papers of Scherzer (*Computing*, **51** (1993) 45–60) and Ramlau (*Numer. Funct. Anal. Optim.* **23**(**1&22**) (2003) 147–172). This source condition is, in fact, a special case of the source condition in the author's paper (*Bull. Aust. Math. Soc.* **79** (2009) 337–342).

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1. Introduction and preliminaries

Let X and Y be Hilbert spaces, and $F : D \subseteq X \to Y$ be a nonlinear operator. For $y \in Y$, consider the equation

$$F(x) = y. \tag{1.1}$$

We assume that $y \in R(F)$ and D is an open set, and the above equation is ill-posed, in the sense that a unique solution which depends continuously on the data y does not exist. In such a situation, it is necessary to have a stable approximation procedure, usually called a regularization method, to obtain approximations for a sought for solution.

In the sequel, we shall look for approximations for a solution $x^\dagger$ which is closest to a pre-assigned $\bar{x} \in X$, i.e., an $x^\dagger \in D$ such that $F(x^\dagger) = y$ and

$$\|x^\dagger - \bar{x}\| = \inf_{x \in D} \{\|x - \bar{x}\| : F(x) = y\}.$$

Such a solution exists if, for example, F is *weakly closed*, for then, it can be seen that the set S_y of all solutions is a weakly closed subset of X. Further, uniqueness of such an $x^\dagger$ can be guaranteed, if in addition, the set S_y is convex (see [2]).

For the purpose of regularization of (1.1), we use Tikhonov regularization, in which a regularized solution is obtained as a minimizer of the *Tikhonov functional* $J_\alpha(\cdot, y)$ defined by

$$J_\alpha(x, y) = \|F(x) - y\|^2 + \alpha\|x - \bar{x}\|^2, \quad x \in D,$$

where α is a positive parameter. That is, for $\alpha > 0$, we look for $x_\alpha \in D$ such that

$$J_\alpha(x_\alpha, y) = \inf_{x \in D} J_\alpha(x, y).$$

Existence of such an x_α is guaranteed, if for example, F is *weakly closed*. Under this assumption of weak closedness of F, a regularized solution x_α is, in some sense, stable. In fact, the following fact is known (see [2]).

> If F is weakly closed, and if (y_n) is a sequence in Y such that $y_n \to y$ as $n \to \infty$, and for each $n \in \mathbb{N}$, if $x_n \in D$ is such that
>
> $$J_\alpha(x_n, y_n) = \inf_{u \in D} J_\alpha(u, y_n),$$
>
> then (x_n) has a convergent subsequence such that its limit minimizes $J_\alpha(\cdot, y)$. Moreover, limit of every convergent subsequence of (x_n) minimizes $J_\alpha(\cdot, y)$.

In practice, the data y may not be known exactly. Let $y^\delta \in Y$ be the noisy data with noise level $\delta > 0$, that is,

$$\|y - y^\delta\| \leq \delta. \tag{1.2}$$

In such a case, in place of x_α, we are having x_α^δ which is a minimizer of $J_\alpha(\cdot, y^\delta)$, that is,

$$J_\alpha(x_\alpha^\delta, y^\delta) = \inf_{x \in D} J_\alpha(x, y^\delta).$$

It is known that, due to ill-posedness of equation (1.1), the set

$$D_\delta := \{x_\alpha^\delta : 0 < \alpha \leq \alpha_0\}$$

need not be bounded for any $\alpha_0 > 0$ and $\delta > 0$ (see [6]). Therefore, to have an approximation of $x^\dagger$ from the set D_δ, it is necessary to choose the parameter α depending on δ and possibly y^δ, say $\alpha_\delta := \alpha(\delta, y^\delta)$, such that

$$\|x_{\alpha_\delta}^\delta - x^\dagger\| \to 0 \quad \text{as } \delta \to 0.$$

The above requirement is often met when F satisfies certain nonlinear conditions and when the unknown solution $x^\dagger$ has some additional *smoothness properties*. In this regard. Engl *et al.* [3] proved the following result.

Theorem 1.1 [3]. *Suppose F is differentiable in a neigbourhood V_0 of $x^\dagger$ and F' is Lipschitz at $x^\dagger$ with Lipschitz constant L, i.e.,*

$$\|F'(x) - F'(x^\dagger)\| \le L\|x - x^\dagger\| \qquad \forall x \in V_0. \tag{1.3}$$

If

$$x^\dagger - \bar{x} = F'(x^\dagger)^* w \tag{1.4}$$

for some $w \in Y$ and

$$\|Lw\| < 1, \tag{1.5}$$

then

$$\|x^\dagger - x_\alpha^\delta\| \le \frac{1}{\sqrt{1 - L\|w\|}} \left(\frac{\delta}{\sqrt{\alpha}} + \sqrt{\alpha}\|w\| \right).$$

The above theorem shows that, under the conditions (1.3), (1.4) and (1.5), if $\alpha := \alpha_\delta$ is chosen such that

$$\alpha_\delta \to 0 \quad \text{and} \quad \frac{\delta}{\sqrt{\alpha_\delta}} \to 0 \quad \text{as } \delta \to 0,$$

then

$$\|x_{\alpha_\delta}^\delta - x^\dagger\| \to 0 \quad \text{as } \delta \to 0.$$

Further, if $\alpha_\delta \sim \delta$, then

$$\|x_{\alpha_\delta}^\delta - x^\dagger\| = O(\sqrt{\delta}).$$

In [8], Ramlau considered a Morozov-type discrepancy principle for choosing the regularization parameter α. He proved the following result:

Theorem 1.2 [8]. *Assume the conditions* (1.3), (1.4) *and* (1.5) *on F and $x^\dagger - \bar{x}$. Further assume that $\|x^\dagger - \bar{x}\| \ge \rho$ for some $\rho > 0$. Then for δ sufficiently small and*

$$c > 1 + \left(1 + \frac{2\|F'(\bar{x})\|}{L}\right) \max\left\{\frac{4}{\rho^2}, 1\right\},$$

there exists α satisfying

$$\delta \le \|F(x_\alpha^\delta) - y^\delta\| < c\delta$$

and

$$\|x_\alpha^\delta - x^\dagger\| \le \left[\frac{2(1+c)\|w\|}{1 - L\|w\|}\right]^{1/2} \sqrt{\delta}.$$

An apparent disadvantage of Theorems 1.1 and 1.2 is the restriction (1.5) which links the nonlinearity of F with the source condition on $x^\dagger - \bar{x}$. In [9], Scherzer proved the existence of a regularization parameter under a Morozov-type discrepancy principle without linking the nonlinearity of F with any source condition on $x^\dagger$, but the nonlinearity assumption is seemingly stronger. The main theorem in Scherzer [9] is the following.

Theorem 1.3 [9]. *Suppose F is differentiable on D and there exists $k_0 > 0$ and function $k(x, z, v)$ such that*

$$[F'(x) - F'(z)]v = F'(z)k(x, z, v) \quad \text{with } \|k(x, z, v)\| \le k_0\|x - z\|\,\|v\| \tag{1.6}$$

for all $x, y, z \in D$, where D is assumed to be convex and $x^\dagger$ is an interior point of D. Assume further that

$$\|F(\bar{x}) - y^\delta\| > c\delta^2 \quad \text{for } c := 1 + \rho^2$$

with ρ satisfying $k_0\left(1 + \frac{1}{\rho}\right)\|x_0 - \bar{x}\| < 1$. Then there exists $\alpha \in (0, \alpha_0]$ with $\alpha_0 := \rho^2\delta^2/\|x^\dagger - \bar{x}\|^2$ such that

$$\|F(x_\alpha^\delta) - y^\delta\|^2 = c\,\delta^2.$$

Our purpose, in this paper is twofold. First consider a nonlinearity condition on F, which is different from (1.3) and (1.6), namely,

$$\|F(x) - F(v) - F'(v)(x - v)\| \le \eta\|F(x) - F(v)\| \quad \forall x, v \in D \tag{1.7}$$

for some nonnegative constant η, which yield error estimates under the source condition

$$x^\dagger - \bar{x} = F'(x^\dagger)^* w$$

for some $w \in Y$, without any dependence of η on $\|w\|$. Secondly, to show the existence of a regularization parameter under a Morozov-type discrepancy principle which guarantee an order optimal error estimate with respect to the assumed source condition.

We may recall that, in the literature, condition (1.7) is called an *η-condition* or *tangential cone condition*. This condition is satisfied for many of the inverse problems that appear in applications (see e.g. [1, 4, 5]).

2. Error estimates under η-condition

Throughout the paper we assume that for every $\tilde{y} \in Y$ and $\alpha > 0$, the Tikhonov functional

$$J_\alpha(x, \tilde{y}) = \|F(x) - y\|^2 + \alpha\|x - \bar{x}\|^2, \quad x \in D,$$

attains infimum at some $\tilde{x}_\alpha \in D$, i.e.,

$$J_\alpha(\tilde{x}_\alpha, \tilde{y}) = \inf_{x \in D} J_\alpha(x, \tilde{y}).$$

Further, we assume the following:

Assumptions:

(1) y is in the range of F,
(2) $y^\delta \in Y$ is a noisy data satisfying (1.2),
(3) F is Fréchet differentiable in D and satisfies the η-condition (1.7), i.e.,

$$\|F(x) - F(v) - F'(v)(x - v)\| \le \eta\|F(x) - F(v)\| \quad \forall x, v \in D$$

for some non-negative constant η,

(4) $x^\dagger \in D$ with $F(x^\dagger) = y$ and $\bar{x} \in D$ satisfy the source condition (1.4), i.e.,

$$x^\dagger - \bar{x} = F'(x^\dagger)^* w,$$

for some $w \in Y$,

(5) There exists $\rho > 0$ and $\omega > 0$ such that

$$\|x^\dagger - \bar{x}\| \geq \rho, \quad \|w\| \leq \omega.$$

Let x_α and x_α^δ be in D which are minimizers of the Tikhonov functional $J_\alpha(\cdot, \tilde{y})$ with y and y^δ in place of $\tilde{y}$, i.e.,

$$J_\alpha(x_\alpha, y) = \inf_{x \in D} J_\alpha(x, y), \quad J_\alpha(x_\alpha^\delta, y^\delta) = \inf_{x \in D} J_\alpha(x, y^\delta).$$

We observe that Assumption (3) implies

$$\|F'(v)(x - v)\| \leq (1 + \eta)\|F(x) - F(v)\| \quad \forall\, x, v \in D. \tag{2.1}$$

We first obtain an estimate for the error $\|x_\alpha - x^\dagger\|$ which, in particular, lead to the convergence $x_\alpha \to x^\dagger$ as $\alpha \to 0$.

Theorem 2.1. *For every* $\alpha > 0$,

$$\|x_\alpha - x^\dagger\| \leq 2(1 + \eta)\omega\sqrt{\alpha}.$$

In particular,

$$\|x_\alpha - x^\dagger\| \to 0 \quad as \ \alpha \to 0.$$

Proof. Since $J_\alpha(x_\alpha, y) = \inf_{x \in D} J_\alpha(x, y)$,

$$\|F(x_\alpha) - y\|^2 + \alpha\|x_\alpha - \bar{x}\|^2 \leq \|F(x^\dagger) - y\|^2 + \alpha\|x^\dagger - \bar{x}\|^2 = \alpha\|x^\dagger - \bar{x}\|^2.$$

But,

$$\|x_\alpha - \bar{x}\|^2 = \|x_\alpha - x^\dagger\|^2 + \|x^\dagger - \bar{x}\|^2 + 2\langle x_\alpha^\delta - x^\dagger,\ x^\dagger - \bar{x}\rangle$$

Hence, we obtain

$$\|F(x_\alpha) - y\|^2 + \alpha\|x_\alpha - x^\dagger\|^2 \leq 2\alpha\langle x^\dagger - x_\alpha,\ x^\dagger - \bar{x}\rangle.$$

By Assumptions (4) and (5), and the relation (2.1),

$$\begin{aligned}
|\langle x^\dagger - x_\alpha,\ x^\dagger - \bar{x}\rangle| &= |\langle x^\dagger - x_\alpha,\ F'(x^\dagger)^* w\rangle| \\
&= |\langle F'(x^\dagger)(x^\dagger - x_\alpha),\ w\rangle| \\
&= \|F'(x^\dagger)(x^\dagger - x_\alpha)\|\, \omega \\
&\leq (1 + \eta)\|F(x^\dagger) - F(x_\alpha)\|\, \omega \\
&= (1 + \eta)\omega\|y - F(x_\alpha)\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|F(x_\alpha) - y\|^2 + \alpha\|x_\alpha - x^\dagger\|^2 &\leq 2\alpha|\langle x^\dagger - x_\alpha,\ x^\dagger - \bar{x}\rangle| \\
&\leq 2\alpha(1 + \eta)\omega\|y - F(x_\alpha)\|.
\end{aligned}$$

From this we have

$$\|F(x_\alpha) - y\| \le 2\alpha(1+\eta)\omega$$

and

$$\alpha\|x_\alpha - x^\dagger\|^2 \le 2\alpha(1+\eta)\omega\|y - F(x_\alpha)\| \le (2\alpha(1+\eta)\omega)^2.$$

Hence,

$$\|x_\alpha - x^\dagger\| \le 2(1+\eta)\omega\sqrt{\alpha}.$$

In particular, $\|x_\alpha - x^\dagger\| \to 0$ as $\alpha \to 0$. □

The following lemma is crucial in deriving an estimate for the error $\|x_\alpha^\delta - x^\dagger\|$ and also for establishing the existence of the regularization parameter in the next section.

Lemma 2.2. *For every* $\alpha > 0$ *and* $\delta > 0$,

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \le \delta^2 + 2\alpha\omega\,(1+\eta)(\|F(x_\alpha^\delta) - y^\delta\| + \delta).$$

Proof. By the definition of x_α^δ, we have

$$\|F(x_\alpha^\delta)-y^\delta\|^2+\alpha\|x_\alpha^\delta-\bar{x}\|^2 \le \|F(x^\dagger)-y^\delta\|^2+\alpha\|x^\dagger-\bar{x}\|^2 \le \delta^2+\alpha\|x^\dagger-\bar{x}\|^2.$$

Thus,

$$\|F(x_\alpha^\delta)-y^\delta\|^2+\alpha\|x_\alpha^\delta-x^\dagger\|^2 \le \delta^2+\alpha(\|x^\dagger-\bar{x}\|^2+\|x_\alpha^\delta-x^\dagger\|^2-\|x_\alpha^\delta-\bar{x}\|^2).$$

But,

$$\|x_\alpha^\delta - \bar{x}\|^2 = \|x_\alpha^\delta - x^\dagger\|^2 + \|x^\dagger - \bar{x}\|^2 + 2\langle x_\alpha^\delta - x^\dagger,\ x^\dagger - \bar{x}\rangle$$

so that

$$\|x^\dagger - \bar{x}\|^2 + \|x_\alpha^\delta - x^\dagger\|^2 - \|x_\alpha^\delta - \bar{x}\|^2 = 2\langle x^\dagger - x_\alpha^\delta,\ x^\dagger - \bar{x}\rangle.$$

Therefore, we have

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \le \delta^2 - 2\alpha\langle x_\alpha^\delta - x^\dagger,\ x^\dagger - \bar{x}\rangle.$$

Now, under the Assumption (4),

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \le \delta^2 - 2\alpha\langle F'(x^\dagger)(x_\alpha^\delta - x^\dagger),\ u\rangle.$$

Using relation (2.1), we have

$$\begin{aligned}\|F'(x^\dagger)(x_\alpha^\delta - x^\dagger)\| &\le (1+\eta)\|F(x_\alpha^\delta) - F(x^\dagger)\|\\ &\le (1+\eta)[\|[F(x_\alpha^\delta) - y^\delta] + (y^\delta - y)\|\\ &\le (1+\eta)(\|F(x_\alpha^\delta) - y^\delta\| + \delta).\end{aligned}$$

Hence,

$$\begin{aligned}\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 &\le \delta^2 + 2\alpha\|w\|\,\|F'(x^\dagger)(x_\alpha^\delta - x^\dagger)\|\\ &\le \delta^2 + 2\alpha\|w\|\,(1+\eta)(\|F(x_\alpha^\delta) - y^\delta\| + \delta).\end{aligned}$$

This completes the proof. □

COROLLARY 2.3

Let $c \geq 1$ and $\delta > 0$. Suppose that there exists $\alpha > 0$ such that

$$\delta \leq \|F(x_\alpha^\delta) - y^\delta\| \leq c\,\delta.$$

Then, for such α,

$$\|x_\alpha^\delta - x^\dagger\|^2 \leq 2(1+\eta)\omega(1+c)\delta.$$

Proof. By Lemma 2.2, we have

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \leq \delta^2 + 2\alpha\omega\,(1+\eta)(\|F(x_\alpha^\delta) - y^\delta\| + \delta).$$

Hence, we obtain

$$\begin{aligned} \delta^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 &\leq \delta^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \\ &\leq \|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \\ &\leq \delta^2 + 2\alpha\omega\,(1+\eta)(\|F(x_\alpha^\delta) - y^\delta\| + \delta) \\ &\leq \delta^2 + 2\alpha\omega\,(1+\eta)(c\delta + \delta). \end{aligned}$$

Thus,

$$\|x_\alpha^\delta - x^\dagger\|^2 \leq 2\omega\,(1+\eta)(1+c)\delta$$

completing the proof. □

Theorem 2.4. *For every $\alpha > 0$ and $\delta > 0$,*

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x^\dagger - x_\alpha^\delta\|^2 \leq (d\alpha + \delta)^2,$$

where $d := 2(1+\eta)\omega$. In particular,

$$\|F(x_\alpha^\delta) - y^\delta\| \leq d\alpha + \delta \quad \text{and} \quad \|x^\dagger - x_\alpha^\delta\| \leq \frac{d\alpha + \delta}{\sqrt{\alpha}}.$$

Further, if $\alpha = c_0\delta$ for some $c_0 > 0$, then

$$\|x^\dagger - x_\alpha^\delta\| \leq \tilde{c}_0\sqrt{\delta},$$

where $\tilde{c}_0 := [2(1+\eta)\omega + 1]/\sqrt{c_0}$.

Proof. Let $\lambda := \|F(x_\alpha^\delta) - y^\delta\|$. Then, the conclusion in Lemma 2.2 takes the form

$$\lambda^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \leq \delta^2 + 2\alpha(1+\eta)\omega(\lambda + \delta) = b\lambda + c, \tag{2.2}$$

where

$$b = d\alpha = 2\alpha(1+\eta)\omega, \qquad c := \delta^2 + 2\alpha(1+\eta)\omega\delta = \delta^2 + b\delta.$$

In particular, $\lambda^2 \leq b\lambda + c$ so that

$$\lambda \leq \frac{1}{2}(b + \sqrt{b^2 + 4c}\,).$$

But

$$b^2 + 4c = b^2 + 4(\delta^2 + b\delta) = (b + 2\delta)^2.$$

Hence

$$\lambda \leq \frac{1}{2}[b + (b + 2\delta)] = b + \delta.$$

Therefore,

$$b\lambda + c \leq b(b + \delta) + \delta^2 + b\delta = (b + \delta)^2.$$

Hence, from (2.2),

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x^\dagger - x_\alpha^\delta\|^2 \leq (b + \delta)^2.$$

The particular cases are obvious from the above inequality. □

COROLLARY 2.5

Let $\delta > 0$ be given. Suppose there exists $\alpha > 0$ such that

$$\|F(x_\alpha^\delta) - y^\delta\| \geq \delta^2.$$

Then for such α,

$$\|x^\dagger - x_\alpha^\delta\| \leq \sqrt{d(d\alpha + 2\delta)},$$

where $d := 2(1 + \eta)\omega$.

Proof. From Theorem 2.4,

$$\delta^2 + \alpha\|x^\dagger - x_\alpha^\delta\|^2 \leq (b + \delta)^2 = \delta^2 + b(b + 2\delta).$$

Hence

$$\alpha\|x^\dagger - x_\alpha^\delta\|^2 \leq b(b + 2\delta) = d\alpha(d\alpha + 2\delta),$$

where $d := 2\omega(1 + \eta)$. Thus, $\|x^\dagger - x_\alpha^\delta\| \leq \sqrt{d(d\alpha + 2\delta)}$. □

3. Morozov-type discrepancy principle

In this section, we prove under the Assumption (1)–(5) in Section 2, the existence of a parameter α satisfying the Morozov-type discrepancy principle,

$$\delta \leq \|F(x_\alpha^\delta) - y^\delta\| \leq c\,\delta \tag{3.1}$$

for some $c \geq 1$, and obtain the corresponding order optimal rate, provided F is *completely continuous*, that is, for every bounded sequence (x_n) in D which converges weakly to some $x \in D$, the sequence $((F(x_n))$ converges to $F(x)$.

We shall make use of the following result from Ramlau [8].

PROPOSITION 3.1

Let F be completely continuous. Let $\delta > 0$ and $c \geq 1$ be such that $\|y^\delta - F(\bar{x})\| \geq c\,\delta$. Then, only one of the following alternatives hold:

(i) *There exists $\alpha > 0$ satisfying* (3.1).

(ii) *For every* $\alpha > 0$, *there exist minimizers* $x_\alpha^{\delta,1}$ *and* $x_\alpha^{\delta,2}$ *of* $J_\alpha(\cdot\,, y^\delta)$ *such that*

$$\|F(x_\alpha^{\delta,1}) - y^\delta\| \le \delta \le c\delta \le \|F(x_\alpha^{\delta,2}) - y^\delta\|.$$

In view of the above lemma, in addition to the Assumptions (1)–(5) in §2, we shall also assume that F is completely continuous.

PROPOSITION 3.2

Suppose $x_\alpha^{\delta,1}$ *and* $x_\alpha^{\delta,2}$ *are minimizers of* $J_\alpha(\cdot\,, y^\delta)$ *such that*

$$\|F(x_\alpha^{\delta,1}) - y^\delta\| \le \delta \le \|F(x_\alpha^{\delta,2}) - y^\delta\|.$$

Then

$$\alpha\|x_\alpha^{\delta,1} - \bar{x}\|^2 \le \kappa\delta \quad \textit{and} \quad \|F(x_\alpha^{\delta,2}) - y^\delta\|^2 \le \delta^2 + \kappa\delta,$$

where $\kappa := (1+\eta)(2\delta + \|F(x^\dagger) - F(\bar{x})\|)$.

Proof. It is known that (see [2]) if x_α^δ is a minimizer of $J_\alpha(\cdot, y^\delta)$, then

$$F'(x_\alpha^\delta)^*[F(x_\alpha^\delta) - y^\delta] + \alpha(x_\alpha^\delta - \bar{x}) = 0.$$

Hence,

$$\alpha(x_\alpha^\delta - \bar{x}) = F'(x_\alpha^\delta)^*[y^\delta - F(x_\alpha^\delta)]$$

so that

$$\begin{aligned}\alpha\|x_\alpha^\delta - \bar{x}\|^2 &= \langle \alpha(x_\alpha^\delta - \bar{x}),\ x_\alpha^\delta - \bar{x}\rangle \\ &= \langle F'(x_\alpha^\delta)^*[y^\delta - F(x_\alpha^\delta)],\ x_\alpha^\delta - \bar{x}\rangle \\ &= \langle y^\delta - F(x_\alpha^\delta),\ F'(x_\alpha^\delta)(x_\alpha^\delta - \bar{x})\rangle \\ &\le \|y^\delta - F(x_\alpha^\delta)\|\,\|F'(x_\alpha^\delta)(x_\alpha^\delta - \bar{x})\|.\end{aligned}$$

Thus, using (1.7)

$$\alpha\|x_\alpha^\delta - \bar{x}\|^2 \le \|y^\delta - F(x_\alpha^\delta)\|\,(1+\eta)\|F(x_\alpha^\delta) - F(\bar{x})\|. \tag{3.2}$$

Since

$$\begin{aligned}\|F(x_\alpha^{\delta,1}) - F(\bar{x})\| &\le \|F(x_\alpha^{\delta,1}) - y^\delta\| + \|y^\delta - y\| + \|F(x^\dagger) - F(\bar{x})\| \\ &\le 2\delta + \|F(x^\dagger) - F(\bar{x})\|,\end{aligned}$$

from (3.2) we have

$$\alpha\|x_\alpha^{\delta,1} - \bar{x}\|^2 \le \delta(1+\eta)(2\delta + \|F(x^\dagger) - F(\bar{x})\|) = \kappa\delta.$$

Hence,

$$\|F(x_\alpha^{\delta,2}) - y^\delta\|^2 \le J_\alpha(x_\alpha^{\delta,2}, y^\delta) = J_\alpha(x_\alpha^{\delta,1}, y^\delta) \le \delta^2 + \alpha\|x_\alpha^{\delta,1} - \bar{x}\|^2 \le \delta^2 + \kappa\delta.$$

□

Theorem 3.3. *Let* $\delta_0 > 0$ *be small enough such that*

$$2(1+\eta)\omega(\sqrt{\delta_0^2 + \kappa_0\delta_0} + \delta_0) \leq \rho^2/4,$$

where $\kappa_0 := (1+\eta)(2\delta_0 + \|F(x^\dagger) - F(\bar{x})\|)$. *Suppose that for* $0 < \delta \leq \delta_0$ *and* $\alpha > 0$, *the assumptions in Proposition* 3.2 *are satisfied. Then,*

$$\|F(x_\alpha^{\delta,2}) - y^\delta\| \leq \left(\frac{4d\kappa}{\rho^2} + 1\right)\delta,$$

where $d := 2(1+\eta)\omega$ *and* $\kappa := (1+\eta)(2\delta + \|F(x^\dagger) - F(\bar{x})\|)$.

Proof. By Proposition 3.2,

$$\|F(x_\alpha^{\delta,1}) - y^\delta\|^2 \leq \delta^2 \leq \|F(x_\alpha^{\delta,2}) - y^\delta\|^2$$

and $J_\alpha(x_\alpha^{\delta,1}, y^\delta) = J_\alpha(x_\alpha^{\delta,2}, y^\delta)$. Hence, we have

$$\|\bar{x} - x_\alpha^{\delta,2}\| \leq \|\bar{x} - x_\alpha^{\delta,1}\|. \tag{3.3}$$

Recall from Lemma 2.2, that

$$\alpha\|x_\alpha^\delta - x^\dagger\|^2 \leq (\delta^2 - \|F(x_\alpha^\delta) - y^\delta\|^2) + 2\alpha(1+\eta)\omega(\|F(x_\alpha^\delta) - y^\delta\| + \delta).$$

Hence, by Proposition 3.2,

$$\begin{aligned}\alpha\|x_\alpha^{\delta,2} - x^\dagger\|^2 &\leq (\delta^2 - \|F(x_\alpha^{\delta,2}) - y^\delta\|^2) + 2\alpha(1+\eta)\omega(\|F(x_\alpha^{\delta,2}) - y^\delta\|6 + \delta)\\ &\leq 2\alpha(1+\eta)\omega(\sqrt{\delta^2 + \kappa\delta} + \delta).\end{aligned}$$

Therefore,

$$\|x_\alpha^{\delta,2} - x^\dagger\|^2 \leq 2(1+\eta)\omega(\sqrt{\delta^2 + \kappa\delta} + \delta) \leq \tilde{\kappa}\sqrt{\delta},$$

where

$$\tilde{\kappa} := 2(1+\eta)\omega(\sqrt{\delta + \kappa} + \sqrt{\delta}).$$

Let $\delta_0 > 0$ be small enough such that

$$2(1+\eta)\omega(\sqrt{\delta_0^2 + \kappa_0\delta_0} + \delta_0) \leq \rho^2/4,$$

where $\kappa_0 := (1+\eta)(2\delta_0 + \|F(x^\dagger) - F(\bar{x})\|)$. Then, for $0 < \delta \leq \delta_0$, we have $\tilde{\kappa}\sqrt{\delta} \leq \rho^2/4$. Hence,

$$\|x_\alpha^{\delta,2} - x^\dagger\|^2 \leq \tilde{\kappa}\sqrt{\delta} \leq \rho^2/4$$

and

$$\|x_\alpha^{\delta,2} - \bar{x}\| \geq \|\bar{x} - x^\dagger\| - \|x_\alpha^{\delta,2} - x^\dagger\| \geq \rho - \sqrt{\tilde{\kappa}\sqrt{\delta}} \geq \rho/2.$$

Therefore, by (3.3) and Proposition 3.2,

$$\alpha\rho^2/4 \leq \alpha\|x_\alpha^{\delta,2} - \bar{x}\|^2 \leq \alpha\|x_\alpha^{\delta,1} - \bar{x}\|^2 \leq \kappa\delta.$$

Thus,

$$\alpha \leq (4\kappa/\rho^2)\delta.$$

Now, recall from Theorem 2.4 that $\|F(x_\alpha^\delta)-y^\delta\| \leq d\alpha+\delta$, where $d := 2(1+\eta)\omega$. Hence,

$$\|F(x_\alpha^{\delta,2}) - y^\delta\| \leq d\alpha + \delta \leq [(4d\kappa/\rho^2) + 1]\delta.$$

□

Theorem 3.4. *Let* $\delta_0 > 0$ *be as in Theorem* 3.3 *and*

$$c > (4d\kappa/\rho) + 1$$

for $0 < \delta \leq \delta_0$, *where* $d := 2(1+\eta)\omega$ *and* $\kappa := (1+\eta)(2\delta + \|F(x^\dagger) - F(\bar{x})\|)$. *Then, there exists* α *satisfying the Morozov-type discrepancy principle* (3.1). *Further,*

$$\|x_\alpha^\delta - x^\dagger\|^2 \leq 2(1+\eta)(1+c)\omega\delta.$$

Proof. Suppose there does not exist α satisfying (3.1), Then, by Proposition 3.1, there exist $x_\alpha^{\delta,1}$ and $x_\alpha^{\delta,2}$ which are minimizers of $J_\alpha(\cdot\,, y^\delta)$ such that

$$\|F(x_\alpha^{\delta,1}) - y^\delta\| \leq \delta \leq c\delta \leq \|F(x_\alpha^{\delta,2}) - y^\delta\|.$$

This is in contradiction with Theorem 3.3, since $c > (4d\kappa/\rho^2)+1$. Now, the error estimate is obtained by Corollary 2.3. □

Remark 3.5. We must keep in mind that the conditions Assumptions (4) and (5) in §2 are *a priori*, and hence the estimate given in Theorem 3.4 for the quantity c that appears in the discrepancy principle (3.1) is not verifiable. However, if estimate β for the quantity ω/ρ and estimate $\hat{\kappa}$ for κ are available in the form

$$\beta \geq \frac{\omega}{\rho}, \quad \hat{\kappa} \geq \kappa,$$

then the condition for c can be given as

$$c > 2(1+\eta)\beta\hat{\kappa} + 1.$$

It is to be remarked that one of the purposes of the paper is to overcome some apparent disadvantage of Theorems 1.1 and 1.2 which involves the condition (1.5) linking the nonlinearity of F with the source condition.

Remark 3.6. It is to be remarked that in [7], the author obtained order optimal error estimate for the Tikohnov regularized solution x_α^δ under the η-condition (1.7) and a general source condition,

$$x^\dagger - \bar{x} = \sqrt{\varphi(A^*A)}w, \qquad A := F'(x^\dagger) \tag{3.4}$$

by choosing the regularization parameter under a Morozov-type discrepancy principle without proving its existence. In view of the results of this paper, one may investigate the possibility of extending the results by replacing the source condition (1.4) by the general source condition (3.4).

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