Proc. Indian Acad. Sci. (Math. Sci.) Vol. 125, No. 2, May 2015, pp. 209–219.

Volume sums of polar Blaschke–Minkowski homomorphisms

CHANG-JIAN ZHAO

Department of Mathematics, China Jiliang University, Hangzhou 310018, People's Republic of China
E-mail: chjzhao@163.com; chjzhao@cjlu.edu.cn

MS received 30 June 2013; revised 7 June 2014

Abstract. In this article, we establish Minkowski and Aleksandrov–Fenchel type inequalities for the volume sum of polars of Blaschke–Minkowski homomorphisms.

Keywords. Blaschke–Minkowski homomorphism; volume differences; volume sum; projection body operator.

2010 Mathematics Subject Classification. 52A40, 52A30.

1. Introduction and statement of main results

The well-known classical Minkowski and Brunn–Minkowski inequalities can be stated as follows:

If K and L are convex bodies in $\mathbb{R}^n$, then (see, e.g., [29])

$$V_1(K, L)^n \geq V(K)^{n-1} V(L)$$

and

$$V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}.$$

In each case, equality holds if and only if K and L are homothetic. Here, $+$ denotes the usual Minkowski sum and $V_1(K, L)$ denotes the mixed volume of the convex bodies K and L defined by

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) \mathrm{d}S(K, u),$$

where $h(L, u) = \max\{u \cdot x : x \in L\}$ is the support function of L and $S(K, u)$ is the surface area measure of K (see, e.g., [29]).

Let K and L be star bodies in $\mathbb{R}^n$, then the dual Minkowski and Brunn–Minkowski inequalities state that (see [21])

$$\tilde{V}_1(K, L)^n \leq V(K)^{n-1} V(L)$$

and

$$V(K \tilde{+} L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n}.$$

In each case, equality holds if and only if K and L are dilates. Here, $\tilde{+}$ denotes the radial sum and $\tilde{V}_1(K, L)$ denotes the dual mixed volume of the star bodies K and L, defined by

$$\tilde{V}_1(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-1} \rho(L, u) \mathrm{d}S(u),$$

where $\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}$ is the radial function of K and $S(u)$ is the spherical Lebesgue measure (see [7]).

In 2004, Leng [17] defined the volume difference function of the compact domains D and K, where $D \subseteq K$, by

$$D_V(K, D) = V(K) - V(D).$$

The following Minkowski and Brunn–Minkowski type inequalities for volume difference functions were also established by Leng [17].

Theorem 1.1. *If K, L, D and D' are compact domains, $D \subseteq K$, $D' \subseteq L$, and D' is a homothetic copy of D, then*

$$(V_1(K, L) - V_1(D, D'))^n \geq (V(K) - V(D))^{n-1}(V(L) - V(D')) \tag{1.1}$$

and

$$(V(K + L) - V(D + D'))^{1/n} \geq (V(K) - V(D))^{1/n} + (V(L) - V(D'))^{1/n}. \tag{1.2}$$

In each case, equality holds if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$ for some constant μ.

Recently, Lv [24] introduced the dual volume difference function for star bodies and established the following dual Minkowski and Brunn–Minkowski type inequalities for them.

Theorem 1.2. *If K, L, D and D' are star bodies in $\mathbb{R}^n$, and $D \subseteq K$, $D' \subseteq L$, and L is a dilate of K, then*

$$(\tilde{V}_1(K, L) - (\tilde{V}_1(D, D'))^n \leq (V(K) - V(D))^{n-1}(V(L) - V(D')), \tag{1.3}$$

with equality if and only if D and D' are dilates and $(K, D) = \mu(L, D')$ for some constant μ, and

$$(V(K \tilde{+} L) - (V(D \tilde{+} D'))^{1/n} \leq (V(K) - V(D))^{1/n} + (V(L) - V(D'))^{1/n}, \tag{1.4}$$

with equality if and only if D and D' are dilates and $(V(K), V(D)) = \mu(V(L), V(D'))$ for some constant μ.

In fact, more general versions on these types of inequalities were proved in [17] and [24], respectively. Moreover, inequalities for p-quermassintegral difference functions were established in [44].

Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$, i.e. compact, convex subsets of $\mathbb{R}^n$ with non-empty interiors. The topology on $\mathcal{K}^n$ is induced by the Hausdorff metric.

DEFINITION 1.1 [30]

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Blaschke–Minkowski homomorphism if it satisfies the following conditions:

(a) Φ is continuous.
(b) For all $K, L \in \mathcal{K}^n$,

$$\Phi(K \sharp L) = \Phi(K) + \Phi(L),$$

where $\sharp$ denotes the Blaschke sum (see e.g. [8] and [13]) of the convex bodies K and L.
(c) For all $K, L \in \mathcal{K}^n$ and every $\vartheta \in SO(n)$,

$$\Phi(\vartheta K) = \vartheta \Phi(K),$$

where $SO(n)$ denotes the group of rotations in n dimensions.

Blaschke–Minkowski homomorphisms are an important notion in the theory of convex body valued valuations (see, e.g., [1–4, 9, 11, 14, 16, 18–20, 23, 25–28, 31, 33–36, 40]). Their natural dual, radial Blaschke–Minkowski homomorphisms were introduced by Schuster [30] and further investigated in [32].

If Φ is a Blaschke–Minkowski homomorphism, then we denote by $\Phi(K_1, \dots, K_{n-1})$ the induced mixed Blaschke–Minkowski homomorphism of the convex bodies $K_1, \dots, K_{n-1}$ (see §2). The convex body $\Phi(K_1, \dots, K_{n-1})$ contains the origin in its interior, as was shown in [30].

If K is a convex body that contains the origin in its interior, the polar body of K is defined by

$$K^* := \{x \in \mathbb{R}^n | x \cdot y \le 1, y \in K\}.$$

In particular, the polar bodies $(\Phi(K_1, \dots, K_{n-1}))^*$ and $(\Phi K)^*$ are well defined. We will simply write $\Phi^*(K_1, \dots, K_{n-1})$ and $\Phi^* K$ rather than $(\Phi(K_1, \dots, K_{n-1}))^*$ and $(\Phi K)^*$. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$, then we write $\Phi_i^* K$ for $\Phi^*(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B, \dots, B}_{i})$, and we write $\Phi_i^*(K, L)$ for $\Phi(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{L, \dots, L}_{i})$. Note that $\Phi_0^* K \equiv \Phi^* K$.

In 2006, Schuster [30] established the following Minkowski and Aleksandrov–Fenchel type inequalities for polar Blaschke–Minkowski homomorphisms.

Theorem 1.3. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke–Minkowski homomorphism. If K, L are convex bodies in $\mathbb{R}^n$, then*

$$V(\Phi_1^*(K, L))^{n-1} \le V(\Phi^* K)^{n-2} V(\Phi^* L) \tag{1.5}$$

with equality if and only if K and L are homothetic.

Theorem 1.4. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke–Minkowski homomorphism. If K_i $(1 \le i \le n-1)$ are convex bodies in $\mathbb{R}^n$, and $1 \le r \le n-1$, then*

$$V(\Phi^*(K_1, \dots, K_{n-1}))^r \le \prod_{j=1}^{r} \Phi^*(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_{n-1}). \tag{1.6}$$

Motivated by the work of Leng and Lv, we give the following definition.

DEFINITION 1.2

The volume sum function for a polar Blaschke–Minkowski homomorphism $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ and convex bodies K and D, $S_V(\Phi^* K, \Phi^* D)$, is defined by

$$S_V(\Phi^* K, \Phi^* D) = V(\Phi^* K) + V(\Phi^* D).$$

In 2005, the volume sum function was first introduced in [46] as follows: If K and D are star bodies, then the dual quermassintegral sum function of the star bodies K and D, $S_{\tilde{W}_i}(K, D)$, is defined by

$$S_{\tilde{W}_i}(K, D) = \tilde{W}_i(K) + \tilde{W}_i(D),$$

where $0 \le i \le n-1$. When $i = 0$, we have

$$S_V(K, D) = V(K) + V(D)$$

which is called the volume sum function of the star bodies K and L. The Minkowski inequality for the dual quermassintegral sum of mixed intersection bodies (see [22]) was also established.

In 2007, the L_p-dual quermassintegral sum function of star bodies K and D was introduced in [41] and Minkowski and Aleksandrov–Fenchel type inequalities for the L_p-dual quermassintegral sum of polar projection bodies were also established. In 2009, the Minkowski inequality for the L_p-dual quermassintegral sum of mixed intersection bodies was established in [47]. In 2010, an Aleksandrov–Fenchel type inequality for the dual quermassintegral sum of L_p-mixed intersection bodies was established in [45].

The aim of this paper is to establish the following Minkowski and Aleksandrov–Fenchel type inequalities for the volume sum of polar Blaschke–Minkowski homomorphisms.

Theorem 1.5. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke–Minkowski homomorphism. Let D, D', K and L be convex bodies in $\mathbb{R}^n$ and let D be a dilate of D'. If $1 \le j < n-1$, then*

$$S_V(\Phi_j^*(K, L), \Phi_j^*(D, D'))^{n-1} \le S_V(\Phi^* K, \Phi^* D)^{n-j-1} S_V(\Phi^* L, \Phi^* D')^j \tag{1.7}$$

with equality if and only if K and L are homothetic and $(V(\Phi^* K), V(\Phi^* L)) = \mu(V(\Phi^* D), V(\Phi^* D'))$ for some constant μ.

Theorem 1.6. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke–Minkowski homomorphism. If K_i and D_i $(1 \le i \le n-1)$ are convex bodies in $\mathbb{R}^n$, and the bodies D_j $(j = 1, \ldots, r)$ are homothetic copies of each other, then for every $1 \le r \le n-1$,*

$$\begin{aligned} & S_V(\Phi^*(K_1, \ldots, K_{n-1}), \quad \Phi^*(D_1, \ldots, D_{n-1}))^r \\ & \le \prod_{j=1}^{r} S_V\Big(\Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \\ & \quad \Phi^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})\Big). \end{aligned}$$

2. Definitions and preliminaries

The setting for this paper is an n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{K}^n$ denote the set of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. Let u denote the unit vectors, and B denote the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then we write $v(K_1^u, \ldots, K_{n-1}^u)$ for the mixed volume (see below) of $K_1^u, \ldots, K_{n-1}^u$ in the space E_u. If $K_1 = \cdots = K_{n-1} = K$, then we write $v(K^u)$ for $v(K^u, \ldots, K^u)$.

We use $V(K)$ for the n-dimensional volume of a convex body K. Let $h(K, \cdot) : S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$; i.e. for $u \in S^{n-1}$,

$$h(K, u) = \max\{u \cdot x : x \in K\},$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$ (see e.g. [38]).

2.1 *Mixed volumes*

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial in λ_i given by

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (2.1), it is called the mixed volume of $K_i, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is usually written as $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the i-th projection measure (quermassintegral) of K and is written as $W_i(K)$.

2.2 *Projection bodies and mixed projection bodies*

If $K \in \mathcal{K}^n$, then the projection body of a convex body K will be denoted by $\mathbf{\Pi} K$ and is defined as the convex body whose support function is given by

$$h(\mathbf{\Pi} K, u) = v(K^u), \quad u \in S^{n-1}. \tag{2.2}$$

If $K_1, \ldots, K_r \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, then the projection body of the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathcal{K}^n$ can be written as a symmetric homogeneous polynomial of degree $(n-1)$ in λ_i (see [23]):

$$\mathbf{\Pi}(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum \lambda_{i_1} \ldots \lambda_{i_{n-1}} \mathbf{\Pi}_{i_1 \cdots i_{n-1}}, \tag{2.3}$$

where the sum is a Minkowski sum taken over all $(n-1)$-tuples $(i_1, \ldots, i_{n-1})$ of positive integers not exceeding r. The body $\mathbf{\Pi}_{i_1 \ldots i_{n-1}}$ depends only on the bodies

$K_{i_1}, \ldots, K_{i_{n-1}}$, and is uniquely determined by (2.3), it is called *the mixed projection body* of $K_{i_1}, \ldots, K_{i_{n-1}}$, and is written as $\mathbf{\Pi}(K_i, \ldots, K_{i_{n-1}})$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{\Pi}(K_{i_1}, \ldots, K_{i_{n-1}})$ will be written as $\mathbf{\Pi}_i(K, L)$. If $L = B$, then $\mathbf{\Pi}_i(K, L)$ is denoted by $\mathbf{\Pi}_i K$ and when $i = 0$, $\mathbf{\Pi}_i K$ is just $\mathbf{\Pi} K$.

The support function of the mixed projection body of $K_1, \ldots, K_{n-1}$ is given by

$$h(\mathbf{\Pi}(K_1, \ldots, K_{n-1}), u) = v(K_1^u, \ldots, K_{n-1}^u). \tag{2.4}$$

2.3 *Mixed Blaschke–Minkowski homomorphisms*

Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke–Minkowski homomorphism. There is a continuous operator (see [30])

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

symmetric in its arguments such that for $K_1, \ldots, K_r$ and $\lambda_1, \ldots, \lambda_r \geq 0$,

$$\Phi(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{n-1} \Phi(K_{i_1}, \ldots, K_{i_{n-1}}). \tag{2.5}$$

Clearly, the above continuous operator generalizes the notion of Blaschke–Minkowski homomorphism. We call

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$$

the mixed Blaschke–Minkowski homomorphism induced by Φ. Mixed Blaschke–Minkowski homomorphisms were first studied in detail in [30]. If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$, we write $\Phi_i K$ for $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$ and call Φ_i the mixed Blaschke–Minkowski homomorphism of order i. For $0 \leq i \leq n$, we write $\Phi_i(K, L)$ for $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{L, \ldots, L}_{i})$. Note that $\Phi_0 K \equiv \Phi K$.

3. Auxiliary results

The following results will be required to prove our main theorems.

Lemma 3.1 [30]. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke–Minkowski homomorphism. If $K, L \in \mathcal{K}^n$ and $0 \leq j \leq n-2$, then*

$$V(\Phi_j^*(K, L))^{1/(n-1)} \leq V(\Phi^* K)^{n-j-1} V(\Phi^* L)^j \tag{3.1}$$

with equality if and only if K and L are homothetic.

Lemma 3.2 [46]. *If $a, b \geq 0$ and $c, d > 0$, then for $0 < p < 1$,*

$$a^p c^{p-1} + b^p d^{1-p} \leq (a+b)^p (c+d)^{1-p}, \tag{3.2}$$

with equality if and only if $a/b = c/d$.

Lemma 3.3 [5, 15]. *If* $a_1, b_1, \ldots, l_1 \geq 0$, $a_2, b_2, \ldots, l_2 > 0$ *and* $\alpha + \beta + \cdots + \lambda = 1$, *then*

$$a_1^{\alpha} b_1^{\beta} \cdots l_1^{\lambda} + a_2^{\alpha} b_2^{\beta} \cdots l_2^{\lambda} \leq (a_1 + a_2)^{\alpha} (b_1 + b_2)^{\beta} \cdots (l_1 + l_2)^{\lambda}, \tag{3.3}$$

with equality if and only if $a_1/a_2 = b_1/b_2 = \cdots = l_1/l_2$.

4. Inequalities for polar Blaschke–Minkowski homomorphisms

4.1 *Minkowski-type inequalities*

Theorem 4.1. *Let* $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ *be an even Blaschke–Minkowski homomorphism. Let* D, D', K *and* L *be convex bodies in* $\mathbb{R}^n$ *and let* D *be a dilate of* D'. *If* $1 \leq j < n-1$, *then*

$$S_V(\Phi_j^*(K, L), \Phi_j^*(D, D'))^{n-1} \leq S_V(\Phi^* K, \Phi^* D)^{n-j-1} S_V(\Phi^* L, \Phi^* D')^j \tag{4.1}$$

with equality if and only if K *and* L *are homothetic and* $(V(\Phi^* K), V(\Phi^* L)) = \mu(V(\Phi^* D), V(\Phi^* D'))$ *for some constant* μ.

Proof. By Lemma 3.1, we have

$$V(\Phi_j^*(K, L))^{n-1} \leq V(\Phi^* K)^{n-j-1} V(\Phi^* L)^j$$

with equality if and only if K and L are homothetic. Since D is a homothetic copy of D', we have

$$V(\Phi_j^*(D, D'))^{n-1} = V(\Phi^* D)^{n-j-1} V(\Phi^* D')^j.$$

Therefore, since $\frac{n-j-1}{n-1} + \frac{j}{n-1} = 1$, it follows from Lemma 3.2 that

$$\begin{aligned} &S_V(\Phi_j^*(K, L), \Phi_j^*(D, D')) \\ &\quad \leq V(\Phi^* K)^{(n-j-1)/(n-1)} V(\Phi^* L)^{j/(n-1)} \\ &\qquad + V(\Phi^* D)^{(n-j-1)/(n-1)} V(\Phi^* D')^{j/(n-1)} \\ &\quad \leq (V(\Phi^* K) + V(\Phi^* D))^{(n-j-1)/(n-1)} (V(\Phi^* L) + V(\Phi^* D'))^{j/(n-1)}. \\ &\quad = S_V(\Phi^* K, \Phi^* D)^{(n-j-1)/(n-1)} S_V(\Phi^* L, \Phi^* D')^{j/(n-1)}. \end{aligned}$$

By the equality conditions of Lemma 3.1 and (3.4), equality holds if and only if K and L are homothetic and $(V(\Phi^* K), V(\Phi^* L)) = \mu(V(\Phi^* D), V(\Phi^* D'))$ for some constant μ. □

If we take the projection body operator $\mathbf{\Pi}$ as the Blaschke–Minkowski homomorphism in Theorem 4.1, then we obtain the following.

COROLLARY 4.1

Let D, D', K *and* L *be convex bodies in* $\mathbb{R}^n$ *and let* D *be a dilate of* D'. *If* $1 \leq j < n-1$, *then*

$$S_V(\mathbf{\Pi}_j^*(K, L), \mathbf{\Pi}_j^*(D, D'))^{n-1} \leq S_V(\mathbf{\Pi}^* K, \mathbf{\Pi}^* D)^{n-j-1} S_V(\mathbf{\Pi}^* L, \mathbf{\Pi}^* D')^j \tag{4.2}$$

with equality if and only if K *and* L *are homothetic and* $(V(\mathbf{\Pi}^* K), V(\mathbf{\Pi}^* L)) = \mu(V(\mathbf{\Pi}^* D), V(\mathbf{\Pi}^* D'))$ *for some constant* μ.

4.2 *Aleksandrov–Fenchel-type inequalities*

Theorem 4.2. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke–Minkowski homomorphism. If K_i and D_i $(1 \le i \le n-1)$ are convex bodies in $\mathbb{R}^n$, and the bodies D_j $(j = 1, \ldots, r)$ are homothetic copies of each other, then for every $1 \le r \le n-1$,*

$$
\begin{aligned}
&[V(\Phi^*(K_1, \ldots, K_{n-1})) + V(\Phi^*(D_1, \ldots, D_{n-1}))]^r \\
&\quad \le \prod_{j=1}^{r} S_V(\Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \\
&\qquad \Phi^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})).
\end{aligned} \tag{4.3}
$$

Proof. By Theorem 1.4, we have

$$
V(\Phi^*(K_1, \ldots, K_{n-1}))^r \le \prod_{j=1}^{r} V(\Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})).
$$

Since the bodies $D_j (j = 1, \ldots, r)$ are homothetic copies of each other, we have

$$
V(\Phi^*(D_1, \ldots, D_{n-1}))^r = \prod_{j=1}^{r} V(\Phi^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})).
$$

Hence

$$
\begin{aligned}
&V(\Phi^*(K_1, \ldots, K_{n-1})) + V(\Phi^*(D_1, \ldots, D_{n-1})) \\
&\quad \le \left(\prod_{j=1}^{r} V(\Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})) \right)^{1/r} \\
&\qquad + \left(\prod_{j=1}^{r} V(\Phi^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})) \right)^{1/r}.
\end{aligned} \tag{4.4}
$$

Now, using the special case $\alpha = \beta = \cdots = \lambda = 1/r$ of Lemma 3.3, we obtain

$$
\begin{aligned}
&S_V(\Phi^*(K_1, \ldots, K_{n-1}), \Phi^*(D_1, \ldots, D_{n-1})) \\
&\le \left(\prod_{j=1}^{r} \left[V(\Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})) \right.\right. \\
&\qquad \left.\left. + V(\Phi^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})) \right] \right)^{1/r} \\
&= \prod_{j=1}^{r} S_V(\Phi^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \\
&\qquad \Phi^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1}))^{1/r}.
\end{aligned}
$$

□

If we take the projection body operator $\mathbf{\Pi}$ as the Blaschke–Minkowski homomorphism in Theorem 4.2, we obtain the following.

COROLLARY 4.2

If K_i and D_i, $1 \leq i \leq n-1$ are convex bodies in $\mathbb{R}^n$, the bodies D_j $(j = 1, \ldots, r)$ are homothetic copies of each other, then for $1 \leq r \leq n-1$,

$$(V(\mathbf{\Pi}^*(K_1, \ldots, K_{n-1})) + V(\mathbf{\Pi}^*(D_1, \ldots, D_{n-1})))^r \leq \prod_{j=1}^{r} S_V(\mathbf{\Pi}^*(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \mathbf{\Pi}^*(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})). \tag{4.5}$$

We finally remark that inequalities for volume difference of polar Blaschke–Minkowski homomorphisms were established in [42], inequalities for L_p-intersection bodies were established in [6, 10, 12, 37, 39, 43, 48], and for L_p-mixed intersection bodies in [39].

Acknowledgements

The author is grateful to the referee for his excellent suggestions and comments. This research is supported by the National Natural Sciences Foundation of China (11371334).

References

[1] Abardia J, Difference bodies in complex vector spaces, *J. Funct. Anal.* **263** (2012) 3588–3603
[2] Abardia J, Minkowski valuations in a 2-dimensional complex vector space, *Int. Math. Res. Not.* doi:10.1093/imrn/rnt251
[3] Abardia J and Bernig A, Projection bodies in complex vector spaces, *Adv. Math.* **227** (2011) 830–846
[4] Alesker S, Bernig A and Schuster F E, Harmonic analysis of translation invariant valuations, *Geom. Funct. Anal.* **21** (2011) 751–773
[5] Beckenbach E F and Bellman R, Inequalities, 2nd edition (1965) (Springer-Verlag)
[6] Berck G, Convexity of L_p-intersection bodies, *Adv. Math.* **222** (2009) 920–936
[7] Gardner R J, Geometric Tomography (1996) (New York: Cambridge Univ. Press)
[8] Gardner R J, Parapatits L and Schuster F E, A characterization of Blaschke addition, *Adv. Math.* **254** (2014) 396–418
[9] Haberl C, Star body valued valuations, *Indiana Univ. Math. J.* **58** (2009) 2253–2276
[10] Haberl C, L_p-intersection bodies, *Adv. Math.* **217** (2008) 2599–2624
[11] Haberl C, Minkowski valuations intertwining with the special linear group, *J. Eur. Math. Soc.* **14** (2012) 1565–1597
[12] Haberl C and Ludwig M, A characterization of L_p intersection bodies, *Int. Math. Res. Not.* **2006** (2006) Article ID 10548, 29 pages
[13] Haberl C and Parapatits L, Valuations and surface area measures, *J. Reine Angew. Math.* **687** (2014) 225–245
[14] Haberl C and Schuster F E, General L_p affine isoperimetric inequalities, *J. Diff. Geom.* **83** (2009) 1–26

[15] Hardy G H, Littlewood J E and Pólya G, Inequalities (1934) (Cambridge: Cambridge Univ. Press)

[16] Kiderlen M, Blaschke and Minkowski-Endomorphisms of convex bodies, *Trans. Amer. Math. Soc.* **358** (2006) 5539–5564

[17] Leng G S, The Brunn–Minkowski inequality for volume differences, *Adv. Appl. Math.* **32** (2004) 615–624

[18] Ludwig M, Minkowski valuations, *Trans. Amer. Math. Soc.* **357(10)** (2005) 4191–4213

[19] Ludwig M, Projection bodies and valuations, *Adv. Math.* **172** (2002) 158–168

[20] Ludwig M, Minkowski areas and valuations, *J. Diff. Geom.* **86** (2010) 133–161

[21] Lutwak E, Dual mixed volumes, *Pacific J. Math.* **58** (1975) 531–538

[22] Lutwak E, Intersection bodies and dual mixed volumes, *Adv. Math.* **71** (1988) 232–261

[23] Lutwak E, Mixed projection inequalities, *Trans. Amer. Math. Soc.* **287(1)** (1985) 91–105

[24] Lv S J, Dual Brunn–Minkowski inequality for volume differences, *Geom. Dedicata* **145** (2010) 169–180

[25] Parapatits L, $SL(n)$-covariant L_p-Minkowski valuations, *J. London Math. Soc.* **89** (2014) 397–414

[26] Parapatits L, $SL(n)$-contravariant L_p-Minkowski valuations, *Trans. Amer. Math. Soc.* **366** (2014) 1195–1211

[27] Parapatits L and Schuster F E, The Steiner formula for Minkowski valuations, *Adv. Math.* **230** (2012) 978–994

[28] Parapatits L and Wannerer T, On the inverse Klain map, *Duke Math. J.* **162** (2013) 1895–1922

[29] Schneider R, Convex Bodies: The Brunn–Minkowski Theory (1993): Cambridge Univ. Press)

[30] Schuster F E, Volume inequalities and additive maps of convex bodies, *Mathematika* **53** (2006) 211–234

[31] Schuster F E, Convolutions and Multiplier Transformations of Convex Bodies, *Trans. Amer. Math. Soc.* **359** (2007) 5567–5591

[32] Schuster F E, Valuations and Busemann-Petty type problems, *Adv. Math.* **219** (2008) 344–368

[33] Schuster F E, Crofton measures and Minkowski valuations, *Duke Math. J.* **154** (2010) 1–30

[34] Schuster F E and Wannerer T, $GL(n)$ contravariant Minkowski valuations, *Trans. Amer. Math. Soc.* **364** (2012) 815–826

[35] Wannerer T, $GL(n)$ equivariant Minkowski valuations, *Indiana Univ. Math. J.* **60** (2011) 1655–1672

[36] Weberndorfer M, Shadow systems of asymmetric L_p zonotopes, *Adv. Math.* **240** (2013) 613–635

[37] Yuan J and Cheung W, L_p-intersection bodies, *J. Math. Anal. Appl.* **338** (2008) 1431–1439

[38] Zhang G Y, The affine Sobolev inequality, *J. Diff. Geom.* **53** (1999) 183–202

[39] Zhao C J, L_p-mixed intersection bodies, *Sci. China* **51** (2008) 2172–2188

[40] Zhao C J, On Blaschke–Minkowski homomorphisms, *Geom. Dedicata* **149** (2010) 373–378

[41] Zhao C J, L_p-dual Quermassintegral sums, *Sci. China* **50** (2007) 1347–1360

[42] Zhao C J, On polars of Blaschke–Minkowski homomorphisms, *Math. Scand.* **111** (2012) 147–160

[43] Zhao C J and Cheung W, L_p-Brunn–Minkowski inequality, *Indag. Mathem.* **20** (2009) 179–190

[44] Zhao C J and Cheung W, On p-quermassintegrals differences function, *Proc. Indian Acad. Sci.* **116** (2006) 221–231

[45] Zhao C J and Cheung W, L_p-mixed intersetion bodies and star duality, *Proc. Indian Acad. Sci.* **120** (2010) 429–440

[46] Zhao C J and Leng G, Inequalities for dual quermassintegrals of mixed intersetion bodies, *Proc. Indian Acad. Sci.* **115** (2005) 79–91

[47] Zhao C J and Mihály B, L_p-Minkowski and Aleksandrov–Fenchel type inequalities, *Balkan J. Geom. Appl.* **14** (2009) 34–43

[48] Zhu X and Leng G, On the L_p-intersection body, *Appl. Math. Mech.* **28** (2007) 1669–1678

Communicating Editor: B V Rajarama Bhat