Proc. Indian Acad. Sci. (Math. Sci.) Vol. 125, No. 2, May 2015, pp. 187–198.

Sesquilinear uniform vector integral

ION CHIŢESCU[1], LOREDANA IOANA[1], RADU MICULESCU[1,*] and LUCIAN NIŢĂ[2]

[1]Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Academiei Str., 14, 010014, Romania
[2]Technical University of Civil Engineering, Bucharest, Lacul Tei Blvd., 122-124, 020396, Romania
[*]Corresponding author.
E-mail: ionchitescu@yahoo.com; loredana.madalina.ioana@gmail.com; miculesc@yahoo.com; luci6691@yahoo.com

MS received 7 October 2013; accepted 10 January 2014

Abstract. We introduce and study an integral of Hilbert valued functions with respect to Hilbert valued measures. The integral is sesquilinear (bilinear in the real case) and takes scalar values. Basic properties of this integral are studied and some examples are introduced.

Keywords. Hilbert space; vector measure; totally measurable vector function; vector integral; sesquilinear maps.

2010 Mathematics Subject Classification. Primary: 28B05, 46E27, 46E30, 47A07; Secondary: 46A35, 46C05.

1. Introduction

The Cauchy–Riemann integral (quasi unanimously called ‘Riemann integral’ today) is the most natural integral, with clearly definite historical roots. Lebesgue introduced his famous more general integral, together with the principles of modern integration. This theory, together with his integral, dominate contemporary mathematics. Returning to the origin, i.e. to the Cauchy–Riemann integral, A. Denjoy introduced his non absolute integral. J. Kurzweil and R. Henstock introduced, independently and at the same time a non absolute integral which generalizes the Lebesgue integral and is more easy to manipulate. We speak so far about scalar integrals (scalar functions integrated with respect to positive measures-speaking approximately).

The idea of integrating vector functions with respect to scalar measures came naturally on the scene of integration theory in the first part of the twentieth century, in connection with the integral representation of linear (and continuous) operators on function spaces. We must speak about the names of N. Dunford, S. Bochner, B. J. Pettis, O. M. Nikodym, I. M. Gelfand in connection with this idea (see [2], [5], [9, Part I] and [11]). Using an integral of scalar functions with respect to vector measures, Dunford and his school introduced the spectral operators, thus founding the present operator theory (see Part III of [9]). The same problems of integral representations made necessary (and possible) the integration of vector functions with respect to vector measures, the main steps in this directions belonging to Bartle and Dinculeanu (see [1], [6], [7] and [2]). The central idea is practically to use a continuous bilinear map (which may not appear directly). More frequently appears the idea of integrating a vector function with values in E, with respect to a vector measure with values in the vector space of operators $\mathcal{L}(E, F)$, the result of the integration lying in F (see [6] and [7]).

It is worth mentioning that Dinculeanu succeeded in presenting the stochastic integral in this manner, namely he integrated vector functions with respect to measures of bounded semivariation [7]. We mention too, the result in [3], exhibiting the indefinite integral as an integral with respect to a measure valued measure. In a series of papers, beginning with [8], Dobrakov initiated another theory of a bilinear integral.

In the present paper, we consider a Hilbert space X and we integrate X -valued totally measurable functions (i.e. uniform limits of simple functions) with respect to X-valued measures of bounded variation. Our integral is not bilinear, it is sesquilinear (like the scalar product), the origin of this 'defection' being the antilinear isomorphism of X and X' (Riesz–Fréchet theorem). The result of this integration (which can be described as a Bochner-type integration, due to the fact that the measure is of bounded variation) is numerical. We present the basic features of this integral (among them it is worth mentioning the theorem which reduces vector integration to scalar integration). Some illuminating examples are exhibited.

The necessary prerequisites of functional analysis can be found, e.g., in [4] and Part I of [9]. For general topology, see [10].

The authors intend to use the integral introduced in this paper in a series of further papers, dedicated to various topics, such as measure spaces, optimal transport and fractals.

2. Preliminary facts

Throughout the paper, $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$ and the scalar field will be K (either $K = \mathbb{R}$ or $K = \mathbb{C}$). We shall write $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{\infty\}$.

For a non-empty set T and $A \in \mathcal{P}(T) = \{B \mid B \subset T\}$, $\varphi_A : T \to K$ is the characteristic (indicator) function of A. We write $(x_n)_n \subset A$ (or $(x_n)_{n\geq 1} \subset A$) to denote a sequence $(x_n)_n$ whose elements x_n are in A.

For a vector space X over K, a function $f : T \to K$ and a vector $x \in X$, the function $fx : T \to X$ acts as follows: $fx(t) = f(t)x$, for any $t \in T$.

A map $B : X \times Y \to K$ is called sesquilinear in case $B(\alpha x + \beta x', y) = \alpha B(x, y) + \beta B(x', y)$ and $B(x, \alpha y + \beta y^{'}) = \bar{\alpha} B(x, y) + \bar{\beta} B(x, y^{'})$ for any α, β in K, any x, x' in X and any $y, y^{'}$ in Y. A scalar product on X will be denoted by $(x \mid y)$ and the scalar product in K^n is $(x \mid y) = \sum_{p=1}^{n} x_p \overline{y_p}$, $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$.

Given a non empty set T and a normed space $(X, \|.\|)$ for $f : T \to X$, one introduces the function $|f| : T \to \mathbb{R}_+$ acting via $|f|(t) = \|f(t)\|$ for each $t \in T$. Given a non empty set T and a normed space $(X, \|.\|)$, if $(f_n)_{n\in\mathbb{N}}$, f are such that $f_n : T \to X$, $f : T \to X$, we write $f_n \underset{n}{\to} f$ if $f_n(t) \underset{n}{\to} f(t)$ for any $t \in T$ and write $f_n \overset{u}{\underset{n}{\to}} f$ if $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Recall that, for any Hilbert space X, there exists an orthonormal basis $(e_i)_{i\in I} \subset X$ (i.e. $\|e_i\| = 1$, for any $i \in I$, $(e_i \mid e_j) = 0$ if $i \neq j$ and any $x \in X$ can be written in the form $x = S_{i\in I}\ x_i e_i$, where $x_i = (x \mid e_i)$ for any $i \in I$ and $\|x\|^2 = S_{i\in I}\ |x_i|^2$). This leads to the following formula: if $x = S_{i\in I}\ x_i e_i$ and $y = S_{j\in I}\ y_j e_j$ are in X, then $(x \mid y) = S_{i\in I}\ x_i \overline{y_i}$ (summable families).

Let $(T, \sum)$ be a measurable space ($\sum \subset \mathcal{P}(T)$ is σ-algebra) and X be a Banach space. A map $\mu : \sum \to X$ is called a vector measure in case $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$, for any sequence $(A_n)_n \subset \sum$ of mutually disjoint sets. For a vector measure $\mu : \sum \to X$, the variation of μ is the positive (σ-additive) measure $|\mu| : \sum \to \overline{\mathbb{R}_+}$ defined, for any $A \in \sum$ as follows: $|\mu|(A) = \sup\{\sum_{i \in J} \|\mu(A_i)\|\}$ the supremum being computed with respect to all possible partitions π of $A : \pi = (A_i)_{i \in J} = (A_1, A_2, \ldots, A_p)$ (i.e. $A_1, A_2, \ldots, A_p$ are in $\sum$, $\cup_{i=1}^{p} A_i = A$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, hence $J = \{1, 2, \ldots, p\}$). If $|\mu|(T) < \infty$, we say that μ is of bounded variation and we write $\|\mu\| \stackrel{\text{def}}{=} |\mu|(T)$. The vector space

$$\text{cabv}(X) = \{\mu : \sum \to X \mid \mu \text{ is a vector measure of bounded variation}\}$$

becomes a Banach space, when equipped with the norm $\|.\|$ given via $\|\mu\| = |\mu|(T)$. If $\lambda : \mathcal{B} \to \mathbb{R}_+$ is a positive measure, we consider the Banach space $L^2(\lambda)$ equipped with the norm $\|\tilde{f}\| = \|f\|_2$ for any $f \in \tilde{f} \in L^2(\lambda)$ (the elements of $L^2(\lambda)$ are equivalence classes for the λ-almost everywhere equality). We shall abusively write $\|\tilde{f}\|_2 \stackrel{\text{def}}{=} \|\tilde{f}\|$ for any $\tilde{f} \in L^2(\lambda)$. Moreover, $L^2(\lambda)$ is a Hilbert space. Namely, its norm is generated by the scalar product given via $(\tilde{f} \mid \tilde{g}) \stackrel{\text{def}}{=} \int f\bar{g}\mathrm{d}\lambda$ for any $f \in \tilde{f}$ and $g \in \tilde{g}$, where $\bar{g} : T \to K$ is the complex conjugate function of g, namely $\bar{g}(t) = \overline{g(t)}$ for any $t \in T$.

In case $T = \mathbb{N}$ and $\lambda : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}_+}$ is the counting measure, the only negligible set is $\varnothing$ and $L^2(\lambda) \stackrel{\text{def}}{=} l^2 = \{x = (x_n)_{n \geq 1} \mid x_n \in K \text{ and } \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ which is a Banach space when equipped with the norm $\|x\|_2 = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}}$, for $x = (x_n)_{n \geq 1} \in l^2$. Then l^2 is a Hilbert space, its norm being generated by the scalar product $(x \mid y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$ for $x = (x_n)_{n \geq 1}$ and $y = (y_n)_{n \geq 1}$. The canonical orthonormal basis for l^2 is denoted by $(e_n)_{n \geq 1}$.

3. Results

Let us consider a non empty set T, a σ-algebra $\mathcal{B} \subset \mathcal{P}(T)$ and a Hilbert space X. In case (T, d) is a compact metric space, we consider that $\mathcal{B}$ represents the Borel sets of T. The vector space $B(X) = \{f : T \to X \mid f \text{ is bounded}\}$ becomes a Banach space, when equipped with the norm $\|f\|_\infty = \sup\{\|f(t)\| \mid t \in T\}$.

For any partition $(A_1, A_2, \ldots, A_p)$ of T, one can consider a *simple function* (the representation is not unique) $f = \sum_{i=1}^{p} \varphi_{A_i} x_i$, where $x_i \in X$. The vector space of all simple functions is $S(X) = \{f : T \to X \mid f \text{ is simple}\}$. Of course $S(X) \subset B(X)$. The closure of $\mathcal{S}(X)$ in $B(X)$ is the space of *totally measurable functions*: $TM(X) = \overline{S(X)} = \{f : T \to X \mid \text{there exists a sequence } (f_n)_n \subset S(X) \text{ such that } f_n \xrightarrow[n]{u} f\}$.

Another important closed subspace of $B(X)$ is (in case (T, d) is a compact metric space) $C(X) = \{f : T \to X \mid f \text{ is continuous}\}$ which (of course) is Banach when equipped with the norm $\|f\|_\infty$. Actually, we have $C(X) \subset TM(X)$ and, in this respect, we shall construct, for any $f \in C(X)$, *a canonical sequence* $(f_m)_m$ *for* f, as follows. Because $X_1 = f(T)$ is compact, we take an arbitrary $m \in \mathbb{N}$ and obtain the elements $y_1^m = f(t_1^m), y_2^m = f(t_2^m), \ldots, y_{k(m)}^m = f(t_{k(m)}^m)$ in X_1 such that $X_1 \subset \cup_{i=1}^{k(m)} B(y_i^m, \frac{1}{m})$. For any i one has $t_i^m \in A_i^m \stackrel{\text{def}}{=} f^{-1}(B(y_i^m, \frac{1}{m})) \in \mathcal{B}$ and $\cup_{i=1}^{k(m)} A_i^m = T$. We obtain the 'partition' of T, $(B_1^m, B_2^m, \ldots, B_{k(m)}^m)$, where $B_1^m = A_1^m$, $B_2^m = A_2^m \setminus A_1^m, \ldots,$

$B_p^m = A_p^m \setminus (\cup_{i=1}^{p-1} A_i^m), \ldots$ (namely, those B_p^m which are empty are not taken into account; for simplicity reasons, we continue to write abusively $(B_1^m, B_2^m, \ldots, B_{k(m)}^m)$ for this partition which may have less than $k(m)$ sets). We construct the simple function $f_m = \sum_{i=1}^{k(m)} \varphi_{B_i^m} z_i^m$, where z_i^m is arbitrarily taken in $f(B_i^m)$. Take $t \in T$ arbitrarily. For the unique B_i^m such that $t \in B_i^m$, the elements $f(t)$ and z_i^m are in $f(B_i^m) \subset f(A_i^m) \subset B(y_i^m, \frac{1}{m})$, hence $\|f(t) - f_m(t)\| = \|f(t) - z_i^m\| \leq \frac{2}{m}$, and consequently $f_m \underset{m}{\overset{u}{\to}} f$. Notice that $f_m(T) \subset f(T)$ for any m.

DEFINITION 3.1

For any $f \in S(X)$ of the form $f = \sum_{i=1}^m \varphi_{A_i} x_i$ and any $\mu \in cabv(X)$, we define *the integral of f with respect to μ*, which is the number

$$\int f \mathrm{d}\mu \overset{\text{def}}{=} \sum_{i=1}^{m} (x_i \mid \mu(A_i)) \tag{1}$$

(this value does not depend upon the representation of f).

Clearly, in case $X = \mathbb{C}$, one has

$$\int f \mathrm{d}\mu = \sum_{i=1}^{m} x_i \overline{\mu(A_i)}. \tag{2}$$

Because of the obvious relation

$$\left| \int f \mathrm{d}\mu \right| \leq \|\mu\| \, \|f\|_\infty , \tag{3}$$

the linear operation $f \to \int f \mathrm{d}\mu$ is continuous and can be extended (uniform continuity extension) to all of $\overline{S(X)} = TM(X)$.

DEFINITION 3.2

For any $f \in TM(X)$ and for any $\mu \in cabv(X)$, the integral of f with respect to μ is

$$\int f \mathrm{d}\mu = \lim_n \int f_n \mathrm{d}\mu,$$

the limit being the same for any sequence $(f_n)_n \subset S(X)$ such that $f_n \underset{n}{\overset{u}{\to}} f$.

For any $f \in TM(X)$, any $a \in T$ and any $x \in X$, one has $\int f \mathrm{d}(\delta_a x) = (f(a) \mid x)$ (in particular, when $X = K$ and $x = 1$, we write $\delta_a 1 = \delta_a =$ the Dirac measure concentrated at a and one has $\int f \mathrm{d}\delta_a = f(a)$ for any $f \in TM(K)$ and any $a \in T$).

Notice that, in case $X = K$ and $\mu \geq 0$, the integral $\int f \mathrm{d}\mu$ computed here coincides with the standard (classical) integral.

Relation (3) can be improved, namely

$$\left| \int f \mathrm{d}\mu \right| \leq \int |f| \, \mathrm{d}\,|\mu| \leq \|\mu\| \, \|f\|_\infty . \tag{4}$$

The map $H : TM(X) \times cabv(X) \to K$ given via $H(f, \mu) = \int f \mathrm{d}\mu$ is sesquilinear: for f, g in $TM(X)$, μ, ν in $cabv(X)$, and α, β in K we have

$$\int (\alpha f + \beta g)\mathrm{d}\mu = \alpha \int f \mathrm{d}\mu + \beta \int g \mathrm{d}\mu \quad \text{and}$$
$$\int f \mathrm{d}(\alpha\mu + \beta\nu) = \bar{\alpha} \int f \mathrm{d}\mu + \bar{\beta} \int g \mathrm{d}\mu. \tag{5}$$

The map H is also continuous because, using (4), we have $|H(f, \mu)| \le \|\mu\| \, \|f\|_\infty$, hence: ($f_m \underset{m}{\overset{u}{\to}} f$ in $TM(X) \Rightarrow \int f_m \mathrm{d}\mu \underset{m}{\to} \int f \mathrm{d}\mu$) and ($\mu_m \underset{m}{\to} \mu$ in $cabv(X) \Rightarrow \int f \mathrm{d}\mu_m \underset{m}{\to} \int f \mathrm{d}\mu$).

Due to (1), (2) and (5), one can see that, in case $X = K$, one has, for any $f \in TM(K)$ and any $\mu \in cabv(K)$:

$$\int f \mathrm{d}\mu \text{ (present definition)} = \int f \mathrm{d}\bar{\mu} \text{ (standard definition)},$$

where $\bar{\mu} : \mathcal{B} \to K$ is the measure given via $\bar{\mu}(A) = \overline{\mu(A)}$, for each $A \in \mathcal{B}$.

Remark. As we have seen, our integral is sequilinear (see (5)). At the same time, our integral is uniform in the sense that it is obtained via uniform convergence from the integral of the simple functions.

Let $f \in TM(K)$, $\mu \in cabv(K)$ and x, y in X. Then $fx \in TM(X)$, $\mu y \in cabv(X)$ and one has

$$\int (fx)\mathrm{d}(\mu y) = \left(\int f \mathrm{d}\mu \right) \cdot (x \mid y). \tag{6}$$

In particular,

$$\int (fx)\mathrm{d}(\mu x) = \left(\int f \mathrm{d}\mu \right) \|x\|^2 . \tag{6$'$}$$

Using (6), one can reduce the vector integration to scalar integration, in the case $X \equiv K^n$. Namely, for $f = (f_1, f_2, \ldots, f_n) \in TM(K^n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in cabv(K^n)$, where $f_i \in TM(K)$, $\mu_i \in cabv(K)$, for each $i \in \{1, 2, \ldots, n\}$, one has

$$\int f \mathrm{d}\mu = \sum_{i=1}^{n} \int f_i \mathrm{d}\mu_i. \tag{7}$$

We shall illustrate formula (7) as follows.

Example 3.3. Take $T = [0, 1]$, $X = \mathbb{C}^2$ and $\lambda : \mathcal{B} \to \mathbb{R}_+$ the Lebesgue measure on the Borel sets $\mathcal{B}$ of $[0, 1]$. The measure $\mu : \mathcal{B} \to \mathbb{C}^2$ is given via $\mu = (\mu_1, \mu_2)$, where $\mu_1 = \lambda + i\delta_1$ and $\mu_2 = \delta_0 + i\lambda$. The function $f : [0, 1] \to \mathbb{C}^2$ is given via $f = (f_1, f_2)$, where $f(t) = (t, 1 + it)$, i.e. $f_1(t) = t$ and $f_2(t) = 1 + it$. Then $\int f \mathrm{d}\mu = 2 - 2i$.

In order to extend (7) to the infinite dimensional case, we shall need some preliminary facts. We shall consider that (T, d) is a compact metric space. Let us consider a closed subspace $Y \subset X$ and let $\pi_Y : X \to X$ be the corresponding orthogonal projector. For any $f \in C(X)$ and $\mu \in cabv(X)$, one has $\pi_Y \circ f \in C(X)$, $\|\pi_Y \circ f\|_\infty \le \|f\|_\infty$, $\pi_Y \circ \mu \in cabv(X)$ and $\|\pi_Y \circ \mu\| \le \|\mu\|$, because $\|\pi_Y(x)\| \le \|x\|$, for any $x \in X$.

Accepting the previous notations and considerations, we obtain the following.

Theorem 3.4. *Assuming that either $f(T) \subset Y$, or $\mu(\mathcal{B}) \subset Y$, one has*

$$\int f \mathrm{d}\mu = \int (\pi_Y \circ f) \mathrm{d}(\pi_Y \circ \mu). \tag{8}$$

Proof. Indeed, assume first that $f(T) \subset Y$, hence $f = \pi_Y \circ f$. Considering the canonical sequence $(f_m)_m$ for f , one has $f_m(T) \subset f(T) \subset Y$ hence $\pi_Y \circ f_m = f_m$, for any m. We also have $\mu = \pi_Y \circ \mu + \pi_Z \circ \mu$, where Z is the orthogonal complement of Y, hence $\int f_m \mathrm{d}\mu = \int f_m \mathrm{d}(\pi_Y \circ \mu) + \int f_m \mathrm{d}(\pi_Z \circ \mu)$, for any m. For any $y \in f_m(T)$ and any $A \in \mathcal{B}$, one has $(y \mid (\pi_Z \circ \mu)(A)) = 0$, hence $\int f_m \mathrm{d}(\pi_Z \circ \mu) = 0$. It follows that $\int f \mathrm{d}\mu = \lim_m \int f_m \mathrm{d}\mu = \lim_m \int f_m \mathrm{d}(\pi_Y \circ \mu) = \int f \mathrm{d}(\pi_Y \circ \mu) = \int (\pi_Y \circ f) \mathrm{d}(\pi_Y \circ \mu)$. In the case $\mu(\mathcal{B}) \subset Y$, $\mu = \pi_Y \circ \mu$, hence one writes $f = \pi_Y \circ f + \pi_Z \circ f$. Consequently $\int f \mathrm{d}\mu = \int f \mathrm{d}(\pi_Y \circ \mu) = \int (\pi_Y \circ f) \mathrm{d}(\pi_Y \circ \mu) + \int (\pi_Z \circ f) \mathrm{d}(\pi_Y \circ \mu)$. Considering the canonical sequence $(f_m)_m$ for $\pi_Z \circ f$, one can see that $\int f_m \mathrm{d}(\pi_Y \circ \mu) = 0$, for any m (because $f_m(T) \subset Z$), hence $\int (\pi_Z \circ f) \mathrm{d}(\pi_Y \circ \mu) = 0$ and it remains (thus completely proving (8)) $\int f \mathrm{d}\mu = \int (\pi_Y \circ f) \mathrm{d}(\pi_Y \circ \mu)$. □

Remark. If $Y \subset X$ is a closed subspace with orthogonal complement $Z \subset X$ and $f \in C(X)$, $\mu \in cabv(X)$ are such that $f(T) \subset Y$ and $\mu(\mathcal{B}) \subset Z$, and one has

$$\int f \mathrm{d}\mu = 0. \tag{9}$$

We are now in a position to extend formula (7) for $f \in C(X)$, $\mu \in cabv(X)$, when X is an infinite dimensional Hilbert space. In order to state the next result, it will be necessary to introduce some preliminary notations.

Let us consider that the Hilbert space X is infinite dimensional, with orthonormal basis $(e_i)_{i \in I}$. Hence, in view of the fact that, for any $x \in X$, one has $x = \underset{i}{S} x_i e_i$, where $x_i = (x \mid e_i) \in K$ for any $i \in I$, we can write (symbolically): $f = \underset{i}{S} f_i = \underset{i}{S} f_i^1 e_i$ (for any $f \in C(X)$) and $\mu = \underset{i}{S} \mu_i = \underset{i}{S} \mu_i^1 e_i$ (for any $\mu \in cabv(X)$) with the following explanations (for any $i \in I$):

(a) Consider $X_i = Sp(e_i) = \{\alpha e_i \mid \alpha \in K\} \subset X$ and write $\pi_i \overset{\text{def}}{=} \pi_{X_i}$. Hence $\pi_i(x) = (x \mid e_i) e_i$ for any $x \in X$.

(b) For any $t \in T$, one can write $f(t) = \underset{i}{S} f_i^1(t) e_i$, where the Fourier coefficient $f_i^1(t) \in K$ is uniquely determined: $f_i^1(t) = (f(t) \mid e_i)$. We got the function $f_i^1 : T \to K$ and the function $f_i : T \to X$, given via $f_i = f_i^1 e_i = \pi_i \circ f$. For any $A \in \mathcal{B}$, one can write $\mu(A) = \underset{i}{S} \mu_i^1(A) e_i$, the Fourier coefficient $\mu_i^1(A)$ being uniquely determined: $\mu_i^1(A) = (\mu(A) \mid e_i)$. We got the function $\mu_i^1 : \mathcal{B} \to K$ and the function $\mu_i : \mathcal{B} \to X$, given via $\mu_i = \mu_i^1 e_i = \pi_i \circ \mu$.

(c) We have $f_i^1 \in C(K)$, $f_i \in C(X)$ and $\left\| f_i^1 \right\|_\infty = \|f_i\|_\infty \le \|f\|_\infty$. At the same time $\mu_i^1 \in cabv(K)$, $\mu_i \in cabv(X)$ and $\left\| \mu_i^1 \right\| = \|\mu_i\| \le \|\mu\|$.

(d) For a function $f \in C(X)$, we shall consider a sequence $(f^n)_n \subset S(X)$ such that $f^n \underset{n}{\overset{u}{\to}} f$. For any $n \in \mathbb{N}$ and any $i \in I$, we consider the same procedure and write symbolically $f^n = \underset{i}{S} f_i^n = \underset{i}{S} f_i^{n1} e_i$ where $f_i^n = \pi_i \circ f_n = f_i^{n1} e_i$. Here $f_i^{n1} : T \to K$ is uniquely determined and one can see that $f_i^{n1} \in S(K)$.

Theorem 3.5 (Reduction to the scalar integration). *Recall that (T, d) is a compact metric space. With the notation from above: For any $f \in C(X)$ and any $\mu \in cabv(X)$, the family $(\int f_i^1 \mathrm{d}\mu_i^1)_{i \in I}$ is summable and*

$$\int f \mathrm{d}\mu = \underset{i}{S} \int f_i^1 \mathrm{d}\mu_i^1.$$

Proof. Consider again a sequence $(f^n)_n \subset S(X)$ such that $f^n \underset{n}{\overset{u}{\to}} f$. We shall construct a (double) generalized sequence $(g(n, H))_{(n,H) \in \mathbb{N} \times F(I)}$ given by the map $g : \mathbb{N} \times F(I) \to K$ defined as follows:

$$g(n, H) = \sum_{i \in H} \int f_i^{n1} \mathrm{d}\mu_i^1.$$

Here $F(I) = \{\varnothing \neq H \subset I \mid H \text{ is finite}\}$, directed by inclusion.

A. We shall prove that, for any $H \in F(I)$, one has

$$\lim_n g(n, H) = \sum_{i \in H} \int f_i \mathrm{d}\mu \qquad (\alpha)$$

and the limit (α) is uniform with respect to $H \in F(I)$. The proof is as follows. First, we compute the value of the limit in (α). For any $i \in I$: $f^n \underset{n}{\overset{u}{\to}} f \Rightarrow \pi_i \circ f^n \underset{n}{\overset{u}{\to}} \pi_i \circ f \Rightarrow f_i^n \underset{n}{\overset{u}{\to}} f_i \Rightarrow f_i^{n1} \underset{n}{\overset{u}{\to}} f_i^1 \Rightarrow \int f_i^{n1} \mathrm{d}\mu_i^1 \underset{n}{\to} \int f_i^1 \mathrm{d}\mu_i^1$.

Using $(6')$ we get $\int f_i^{n1} \mathrm{d}\mu_i^1 = \int (f_i^{n1} e_i) \mathrm{d}(\mu_i^1 e_i) = \int f_i^n \mathrm{d}\mu_i = \int (\pi_i \circ f^n) \mathrm{d}(\pi_i \circ \mu) = \int (\pi_i \circ f^n) \mathrm{d}\mu = \int f_i^n \mathrm{d}\mu$. The equality $\int (\pi_i \circ f^n) \mathrm{d}(\pi_i \circ \mu) = \int (\pi_i \circ f^n) \mathrm{d}\mu$ is proved noticing that $\pi_i \circ f^n$ is simple, namely it has the form $\pi_i \circ f^n = \sum_p \varphi_{D_p} z_p$ with $z_p \in X_i$, hence $\int (\pi_i \circ f^n) \mathrm{d}\mu = \sum_p (z_p \mid \mu(D_p)) = \sum_p (z_p \mid \pi_i(\mu(D_p))) = \int (\pi_i \circ f^n) \mathrm{d}(\pi_i \circ \mu)$ and (α) is proved because $\lim_n \int f_i^n \mathrm{d}\mu = \int f_i \mathrm{d}\mu$.

As concerns the uniformity with respect to $H \in F(I)$, we take $\varepsilon > 0$ and we shall find $n_\varepsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $n \geq n_\varepsilon$ and any $H \in F(I)$, one has $\left|\sum_{i \in H} \int f_i^{n1} \mathrm{d}\mu_i^1 - \sum_{i \in H} \int f_i \mathrm{d}\mu\right| < \varepsilon$ i.e. $\left|\sum_{i \in H} \int (f_i^n - f_i) \mathrm{d}\mu\right| < \varepsilon$.

Indeed, take $n_\varepsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $n \geq n_\varepsilon$, one has $\|f^n - f\|_\infty < \frac{\varepsilon}{\|\mu\|}$ and take $H \in F(I)$ arbitrarily. For any $t \in T$, one has $(\left|\sum_{i \in H} (f_i^n - f_i)\right|(t))^2 = \left\|\sum_{i \in H} (f_i^n - f_i)(t)\right\|^2 = \left\|\sum_{i \in H} (f_i^{n1} - f_i^1)(t) e_i\right\|^2 = \sum_{i \in H} \left|(f_i^{n1} - f_i^1)(t)\right|^2 \leq \underset{i}{S} \left|(f_i^{n1}(t) - f_i^1(t))\right|^2 = \|f^n(t) - f(t)\|^2 \leq \|f^n - f\|_\infty^2$.

Hence $\left|\sum_{i \in H} \int (f_i^n - f_i) \mathrm{d}\mu\right| \leq \int \left|\sum_{i \in H} (f_i^n - f_i)\right| \mathrm{d}|\mu| \leq \int \|f^n - f\|_\infty \mathrm{d}|\mu| \leq \frac{\varepsilon}{\|\mu\|} |\mu|(T) = \varepsilon$ and the uniformity is proved.

B. We shall prove for fixed $n \in \mathbb{N}$, the existence of the limit

$$\lim_{H \in F(I)} g(n, H) \text{ (generalized sequence)}, \qquad (\beta)$$

i.e. we shall show that the existence of the limit

$$\lim_{H\in F(I)} \sum_{i\in H} \int f_i^n \mathrm{d}\mu \tag{β'}$$

as we have seen.

Reconsidering (β'), it is seen that we must show that the family $(\int f_i^n \mathrm{d}\mu)_{i\in I}$ is summable. Accordingly, we choose arbitrarily $\varepsilon > 0$ and we must show the existence of $H_\varepsilon \in F(I)$ such that, for any $H \in F(I)$, $H \cap H_\varepsilon = \emptyset$, one has

$$\left|\sum_{i\in H} \int f_i^n \mathrm{d}\mu\right| < \varepsilon. \tag{β''}$$

Write $f^n = \sum_{p=1}^{u(n)} \varphi_{A_p^n} x_p^n$ with $A_p^n \in \mathcal{B}$ and $x_p^n \in X$. For any $p \in \{1, 2, \ldots, u(n)\}$, one has $x_p^n = \underset{i}{S} x_{pi}^n e_i$, with $x_{pi}^n = (x_p^n \mid e_i)$. Because the family $(x_{pi}^n e_i)_{i\in I}$ is summable, let us find $H_\varepsilon^p \in F(I)$ with the property: for any $H \in F(I)$, $H \cap H_\varepsilon^p = \emptyset$, one has $\left\|\sum_{i\in H} x_{pi}^n e_i\right\| < \frac{\varepsilon}{\|\mu\|}$.

Write $H_\varepsilon = \cup_{p=1}^{u(n)} H_\varepsilon^p \in F(I)$. Then, let $H \in F(I)$, $H \cap H_\varepsilon = \emptyset$. For any $i \in H$, one has $\int f_i^n \mathrm{d}\mu = \int (\pi_i \circ f^n)\mathrm{d}\mu = \int (\sum_{p=1}^{u(n)} \varphi_{A_p^n} \pi_i(x_p^n))\mathrm{d}\mu = \sum_{p=1}^{u(n)} (\pi_i(x_p^n) \mid \mu(A_p^n))$. But $\pi_i(x_p^n) = x_{pi}^n e_i$, $i \in I$. Hence

$$\begin{aligned}\int f_i^n \mathrm{d}\mu &= \sum_{p=1}^{u(n)} (x_{pi}^n e_i \mid \mu(A_p^n)) = \sum_{p=1}^{u(n)} (x_{pi}^n e_i \mid \underset{j}{S} \mu_j^1(A_p^n) e_j) \\ &= \sum_{p=1}^{u(n)} (x_{pi}^n e_i \mid \mu_i^1(A_p^n) e_i) = \sum_{p=1}^{u(n)} x_{pi}^n \overline{\mu_i^1(A_p^n)}.\end{aligned} \tag{F}$$

Returning to a previous formula:

$$\begin{aligned}\left|\sum_{i\in H} \int f_i^n \mathrm{d}\mu\right| &= \left|\sum_{i\in H} \sum_{p=1}^{u(n)} (x_{pi}^n e_i \mid \mu(A_p^n))\right| = \left|\sum_{p=1}^{u(n)} \sum_{i\in H} (x_{pi}^n e_i \mid \mu(A_p^n))\right| \\ &\le \sum_{p=1}^{u(n)} \left|(\sum_{i\in H} x_{pi}^n e_i \mid \mu(A_p^n))\right| \le \sum_{p=1}^{u(n)} \left\|\sum_{i\in H} x_{pi}^n e_i\right\| \cdot \left\|\mu(A_p^n)\right\| .\end{aligned}$$

For any p, one has $H \cap H_\varepsilon^p = \emptyset$, hence $\left\|\sum_{i\in H} x_{pi}^n e_i\right\| < \frac{\varepsilon}{\|\mu\|}$. We got

$$\left|\sum_{i\in H} \int f_i^n \mathrm{d}\mu\right| \le \frac{\varepsilon}{\|\mu\|} \sum_{p=1}^{u(n)} \left\|\mu(A_p^n)\right\| \le \frac{\varepsilon}{\|\mu\|} |\mu| (T) = \frac{\varepsilon}{\|\mu\|} \|\mu\| = \varepsilon.$$

C. We succeeded in showing that:

- for any $H \in F(I)$, one has

$$\lim_n g(n, H) = \sum_{i\in H} \int f_i \mathrm{d}\mu \tag{α}$$

and the limit is uniform with respect to $H \in F(I)$.

- for any $n \in \mathbb{N}$, there exists

$$\lim_{H \in F(I)} g(n, H). \tag{β}$$

The iterated limit theorem (see Lemma I 7.6 from [9]) says that the iterated limits $\lim_{H \in F(I)} \lim_n g(n, H) = L_{21}$ and $\lim_n \lim_{H \in F(I)} g(n, H) = L_{12}$ exist and $L_{21} = L_{12}$.

Using (α), it is seen that

$$L_{21} = \lim_{H \in F(I)} \sum_{i \in H} \int f_i \mathrm{d}\mu = \underset{i}{S} \int f_i \mathrm{d}\mu.$$

Because $f_i = \pi_i \circ f \in C(X)$ and $f_i(T) \subset X_i$, one has Theorem 3.4: $\int f_i \mathrm{d}\mu = \int (\pi_i \circ f) \mathrm{d}\mu = \int (\pi_i \circ f) \mathrm{d}(\pi_i \circ \mu) = \int f_i \mathrm{d}\mu_i = \int f_i^1 \mathrm{d}\mu_i^1$, hence

$$L_{21} = \underset{i}{S} \int f_i^1 \mathrm{d}\mu_i^1. \tag{I}$$

On the other hand, one has $\int f \mathrm{d}\mu = \lim_n \int f^n \mathrm{d}\mu$. As earlier $f^n = \sum_{p=1}^{u(n)} \varphi_{A_p^n} x_p^n$ and we get $\int f^n \mathrm{d}\mu = \sum_{p=1}^{u(n)} (x_p^n \mid \mu(A_p^n))$. But $x_p^n = \underset{i}{S} x_{pi}^n e_i$ and $\mu(A_p^n) = \underset{j}{S} \mu_j^1(A_p^n) e_j$ as we have seen. Hence

$$\int f^n \mathrm{d}\mu = \sum_{p=1}^{u(n)} (\underset{i}{S} x_{pi}^n e_i \mid \underset{j}{S} \mu_j^1(A_p^n) e_j) = \sum_{p=1}^{u(n)} \underset{i}{S} x_{pi}^n \overline{\mu_i^1(A_p^n)} = \underset{i}{S} \sum_{p=1}^{u(n)} x_{pi}^n \overline{\mu_i^1(A_p^n)}$$
$$= \lim_{H \in F(I)} \sum_{i \in H} \sum_{p=1}^{u(n)} x_{pi}^n \overline{\mu_i^1(A_p^n)} = \lim_{H \in F(I)} \sum_{i \in H} \int f_i^n \mathrm{d}\mu$$

(see (F))$= \lim_{H \in F(I)} \sum_{i \in H} \int f_i^{n1} \mathrm{d}\mu_i^1 = \lim_{H \in F(I)} g(n, H)$.

Consequently $\int f \mathrm{d}\mu = \lim_n \int f^n \mathrm{d}\mu = \lim_n \lim_{H \in F(I)} g(n, H) = L_{12}$, so

$$\int f \mathrm{d}\mu = L_{12}. \tag{II}$$

From (I) and (II), we get $\int f \mathrm{d}\mu = \underset{i}{S} \int f_i^1 \mathrm{d}\mu_i^1$. □

Particular case. If X is separable, it possesses a countable orthonormal basis $(e_n)_{n \in \mathbb{N}}$. Hence, the previous formula becomes

$$\int f \mathrm{d}\mu = \sum_{n=1}^{\infty} \int f_n^1 \mathrm{d}\mu_n^1 \tag{10}$$

(the series converges absolutely, because it converges commutatively).

We finish our general considerations introducing the definition of the integral over a measurable set.

Let us consider $f \in TM(X)$, $\mu \in cabv(X)$ and $A \in \mathcal{B}$. Defining $\varphi_A f : T \to X$ via $(\varphi_A f)(t) = \varphi_A(t) f(t)$ for each $t \in T$, one can see immediately that $\varphi_A f \in TM(X)$

DEFINITION 3.6

The integral

$$\int_A f\mathrm{d}\mu \stackrel{\mathrm{def}}{=} \int (\varphi_A f)\mathrm{d}\mu$$

is called the integral of f with respect to μ over A.

Theorem 3.7. *For any $f \in TM(X)$ and any $\mu \in cabv(X)$, the map $f\mu : \mathcal{B} \to X$ given via*

$$(f\mu)(A) \stackrel{\mathrm{def}}{=} \int (\varphi_A f)\mathrm{d}\mu$$

is countably additive.

Proof. Clearly, $f\mu$ is finitely additive. Now, take a sequence $(A_n)_n \subset \mathcal{B}$ which is decreasing ($A_{n+1} \subset A_n$ for any n) such that $\cap_{n=1}^{\infty} A_n = \emptyset$. We know that, for any $A \in \mathcal{B}$, one has $\left|\int_A f\mathrm{d}\mu\right| \leq \int_A |f|\,\mathrm{d}\,|\mu|$ (which is the same as $|(f\mu)(A)| \leq (|f|\,|\mu|)(A)$). Hence, for any n one has $|(f\mu)(A_n)| \leq (|f|\,|\mu|)(A_n) = \int_{A_n} |f|\,\mathrm{d}\,|\mu|$. Because $|\mu|$ is countably additive, one has $\lim_n \int_{A_n} |f|\,\mathrm{d}\,|\mu| = 0$ and this implies $\lim_n (f\mu)(A_n) = 0$, i.e. $f\mu$ is countably additive. □

DEFINITION 3.8

We call $f\mu$ the measure of basis μ and density f.

Theorem 3.9. *For any $f \in TM(X)$ and $\mu \in cabv(X)$, one has*

$$|f\mu| \leq |f|\,|\mu|\,, \text{ hence } \|f\mu\| \leq \|\,|f|\,|\mu|\,\|\,.$$

Proof. For any $B \in \mathcal{B}$, one has $|(f\mu)(B)| \leq (|f|\,|\mu|)(B)$, hence $\sum_{i=1}^{m} |(f\mu)(A_i)| \leq \sum_{i=1}^{m} (|f|\,|\mu|)(A_i) = (|f|\,|\mu|)(A)$, for any partition $(A_1, A_2, \ldots, A_m)$ of $A \in \mathcal{B}$, a.s.o.□

Remark. The inequality $|f\mu| \leq |f|\,|\mu|$ can be strict (see Example 3.13).

Example 3.10. Take $T = [0, 1]$ and $X = l^2$. Considering the Lebesgue measure $\lambda : \mathcal{B} \to \mathbb{R}_+$ (restricted to the Borel sets $\mathcal{B}$ of $[0, 1]$), we define the measure $\mu : \mathcal{B} \to l^2$ via $\mu(A) = (\frac{\lambda(A)}{n})_{n\geq 1}$ and $\mu \in cabv(l^2)$. Define the continuous function $f : [0, 1] \to l^2$ via $f(t) = (\frac{t}{n})_n$. By using formula (10), we obtain $\int f\mathrm{d}\mu = \frac{\pi^2}{12}$.

Example 3.11. We shall take $T = [0, 1]$, with its Borel sets $\mathcal{B}$ and let $\lambda : \mathcal{B} \to \mathbb{R}_+$ be the Lebesgue measure. Let $X = L^2(\lambda)$. For any $A \in \mathcal{B}$, let $h_A : T \to K$ be the continuous function defined via $h_A(t) = \lambda(A \cap [0, t])$. One can define the countably additive function $\mu_1 : \mathcal{B} \to C(K)$ via $\mu_1(A) = h_A$, with $|\mu_1|\,(T) = 1$. Considering the linear and continuous injection $I : C(K) \to L^2(\lambda)$, given via $I(f) = \tilde{f}$, with $\|I(f)\|_2 = \|\tilde{f}\|_2 \leq$

$\|f\|_\infty$, for any $f \in C(K)$, one can define $\mu : \mathcal{B} \to L^2(\lambda)$, via $\mu = I \circ \mu_1$, i.e. $\mu(A) = \widetilde{h_A}$ for any $A \in \mathcal{B}$, having the property that $\mu \in cabv(L^2(\lambda))$. For $A = [a, b]$ or $A = [a, b)$, where $0 \le a \le b \le 1$, we obtain $\|\mu(A)\| = (b-a)\sqrt{1 - \frac{a+2b}{3}}$, hence $(b-a)\sqrt{1-b} \le \|\mu(A)\| \le (b-a)\sqrt{1-a}$. Using partitions of $[0, 1]$ with intervals of arbitrarily small length and Riemann sums, we finally obtain $\|\mu\| = |\mu|(T) = \frac{2}{3}$.

We shall compute $\int f \mathrm{d}\mu$ for two different functions $f \in C(L^2(\lambda))$.

First computation

Take, for any $t \in [0, 1]$, the function $f_t = \varphi_{[0,t]}$ and define the continuous function $f : [0, 1] \to L^2(\lambda)$ via $f(t) = \tilde{f}_t$. One obtains $\int f \mathrm{d}\mu = 0$.

Second computation

We shall consider the continuous function $f : [0, 1] \to L^2(\lambda)$ given via $f(t) = \tilde{g}_t$. Here $g_t : [0, 1] \to \mathbb{R}$ is given as follows: $g_0(x) = x$ and, for $t \in (0, 1]$, $g_t(x) = 0$ if $0 \le x < t$ and $g_t(x) = x$ if $t \le x \le 1$. Using the partitions of $[0, 1]$ given by $(A_1 = [0, \frac{1}{n}], A_2 = (\frac{1}{n}, \frac{2}{n}], \ldots, A_n = (\frac{n-1}{n}, 1])$, we obtain $\int f \mathrm{d}\mu = \frac{1}{3}$.

Example 3.12. Let us consider $T = \mathbb{N} \cup \{\infty\}$, equipped with the metric d given via $d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|$, with the convention $\frac{1}{\infty} = 0$. Then T is a compact metric space, being the Alexandroff compactification of the separated locally compact space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Namely the topology of T is exactly $P(\mathbb{N}) \cup \mathcal{V}$, where $\mathcal{V} = \{A \subset T \mid \infty \in A$ and there exists $n_A \in \mathbb{N}$ such that $n \in A$ for any $n \ge n_A\}$, hence $\mathcal{B} = \mathcal{P}(T)$, where $\mathcal{B}$ represents the Borel sets of T. A continuous function $f : T \to X$ has the canonical representation $f = \sum_{n=1}^{\infty} f_n^1 e_n$ (see Theorem 3.5) and we shall write $f_n^1(m) = f_{mn}$ and $f_n^1(\infty) = f_{\infty n}$. Similarly, write for any $\mu \in cabv(X)$ the canonical representation $\mu = \sum_{n=1}^{\infty} \mu_n^1 e_n$. To give μ means to give all the values $\mu_n^1(\{m\}) = \mu_{mn}$ and $\mu_n^1(\{\infty\}) = \mu_{\infty n}$. It follows that $\int f \mathrm{d}\mu = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\overline{\mu_{mn}} + \sum_{n=1}^{\infty} f_{\infty n}\overline{\mu_{\infty n}}$.

More concretely, let us take $x = (\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots) \in l^2$, $\mu \in cabv(l^2)$ given via $\mu(\{m\}) = \frac{1}{m^2}x$, $\mu(\{\infty\}) = 0$ and $f \in C(l^2)$ given via $f(m) = \frac{1}{m^2}x$, $f(\infty) = 0$. It follows that $\mu_{mn} = \frac{1}{m^2 n}$, $\mu_{\infty n} = 0$ and $f_{mn} = \frac{1}{m^2 n}$, $f_{\infty n} = 0$. Hence

$$\begin{aligned} \int f d\mu &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\overline{\mu_{mn}} + \sum_{n=1}^{\infty} f_{\infty n}\overline{\mu_{\infty n}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^2 n}\frac{1}{m^2 n} \\ &= (\sum_{m=1}^{\infty} \frac{1}{m^4})(\sum_{n=1}^{\infty} \frac{1}{n^2}) = \frac{\pi^4}{90}\frac{\pi^2}{6} = \frac{\pi^6}{540}. \end{aligned}$$

Example 3.13. We shall consider again the actors from Example 3.11, namely $T = [0, 1]$, the measure μ and the function $f : [0, 1] \to L^2(\lambda)$, given via $f(t) = \tilde{f}_t$, where $f_t = \varphi_{[0,t]}$, for any $t \in [0, 1]$. We shall prove that in this case, the inequality $|f\mu| \le |f|\,|\mu|$ is strict. More precisely, we shall see that $f\mu = 0$, which implies $|f\mu| = 0$, whereas $(|f|\,|\mu|)(T) > 0$.

A. We see first that, for any $t \in [0,1]$, one has $(f\mu)([0,t]) = \int_{[0,t]} f\,\mathrm{d}\mu = 0$. The proof is the same as the proof of the first computation in Example 3.11 except that we take here $[0,t]$ instead of $[0,1]$. Using $(f\mu)(B\setminus A) = (f\mu)(B) - (f\mu)(A)$ for $A \subset B$ in $\mathcal{B}$, we get that, for any $0 \le a \le b \le 1$: $(f\mu)((a,b]) = 0$, $(f\mu)([a,b]) = 0$, because $\mu(\{t\}) = 0$, for any $t \in [0,1]$. Due to the σ-additivity of $f\mu$, this implies $(f\mu)(A) = 0$ for any $A \in \mathcal{B}$. Hence $f\mu = 0$.

B. For any $t \in [0,1]$: $\|f(t)\| = \|\tilde{f}_t\| = (\int \varphi_{[0,t]}^2 \mathrm{d}\lambda)^{\frac{1}{2}} = (\int \varphi_{[0,t]} \mathrm{d}\lambda)^{\frac{1}{2}} = t^{\frac{1}{2}}$. Accepting the fact that $(|f|\,|\mu|)(T) = \int |f|\,\mathrm{d}\,|\mu| = 0$, we shall obtain a contradiction. Indeed, it follows that $|f|\,(t) = t^{\frac{1}{2}} = 0$ $|\mu|$ a.e. Hence $|\mu|\,((0,1]) = 0$, which means $|\mu|\,([0,1]) = 0$, due to the fact that $|\mu|\,(\{0\}) = 0$. This contradicts the previous fact that $|\mu|\,([0,1]) = \frac{2}{3}$.

Our integral describes the dual of $C_X(T)$, where T is a compact metric space and X is a Hilbert space (see also [6]). Namely, any functional x' on $C_X(T)$ acts via $x'(f) = \int f\,\mathrm{d}\mu$, where $\mu \in cabv(X)$. For future developments, we intend to study different norms on $cabv(X)$, defined with the aid of the present integral, thus generalizing the norms used in connection with the optimal transport, according to the theory of G. Monge and L.V. Kantorovich.

References

[1] Bartle R G, A general bilinear vector integral, *Stud. Math.* **15** (1956) 337–352

[2] Chiţescu I, Function Spaces (in Romanian), ed. Şt. Encicl. Bucureşti (1983)

[3] Chiţescu I, The indefinite integral as an integral, *Atti. Sem. Mat. Fis. Univ. Modena* **41** (1993) 349–366

[4] Cristescu R, Functional analysis (in Romanian), Ed. III Ed. Did. Ped., Bucureşti (1979)

[5] Diestel J and Uhl J J Jr, Vector measures (1977) (Providence, Rhode Island: American Mathematical Society)

[6] Dinculeanu N, Vector measures (1966) (Berlin: VEB Deutscher Verlag der Wissenschaften)

[7] Dinculeanu N, Vector Integration and Stochastic Integration in Banach Spaces (2000) (New York, Chichester, Weinheim, Brisbane, Singapore, Toronto: John Wiley & Sons, Inc.)

[8] Dobrakov I, On integration in Banach spaces, I, *Czechoslovak Math. J.* **20(95)** (1970) 511–536

[9] Dunford N and Schwartz J T, Linear Operators. Part I: General Theory (1957); Part III: Spectral Operators (1971) (New York, London, Sydney, Toronto: Interscience Publishers, Inc.)

[10] Kelley J L, General topology (1955) (New York, Toronto, London, Melbourne: American Book-Van Nostrand-Reinhold)

[11] Lukeš J and Maly J, Measure and integral (1995) (Matfyzpress: Publishing House of the Faculty of Mathematics and Physics, Charles University, Prague)

Communicating Editor: B V Rajarama Bhat