Proc. Indian Acad. Sci. (Math. Sci.) Vol. 125, No. 2, May 2015, pp. 161–165.
© Indian Academy of Sciences

# Smoothness of limit functors

BENEDICTUS MARGAUX

Laboratoire de Recherche “Princess Stephanie", Monte Carlo 51840, Monaco
E-mail: benedictus.margaux@gmail.com

MS received 11 December 2013; revised 21 April 2014

**Abstract.** Let $S$ be a scheme. Assume that we are given an action of the one dimensional split torus $\mathbb{G}_{m,S}$ on a smooth affine $S$-scheme $\mathfrak{X}$. We consider the limit (also called attractor) subfunctor $\mathfrak{X}_\lambda$ consisting of points whose orbit under the given action ‘admits a limit at 0’. We show that $\mathfrak{X}_\lambda$ is representable by a smooth closed subscheme of $\mathfrak{X}$. This result generalizes a theorem of Conrad *et al.* (Pseudo-reductive groups (2010) Cambridge Univ. Press) where the case when $\mathfrak{X}$ is an affine smooth group and $\mathbb{G}_{m,S}$ acts as a group automorphisms of $\mathfrak{X}$ is considered. It also occurs as a special case of a recent result by Drinfeld on the action of $\mathbb{G}_{m,S}$ on algebraic spaces (Proposition 1.4.20 of Drinfeld V, On algebraic spaces with an action of $\mathfrak{G}_m$, preprint 2013) in case $S$ is of finite type over a field.

**Keywords.** Group schemes; torus action; limit subscheme.

**Mathematics Subject Classification.** 14L30, 14L15.

## 1. Introduction

Let $S$ be a scheme and denote by $\mathbb{A}^1_S = \mathrm{Spec}(\mathcal{O}_S[t])$ the affine line over $S$ and by $\mathbb{G}_{m,S} = \mathrm{Spec}(\mathcal{O}_S[t, t^{-1}])$ the one dimensional split torus over $S$. Let $\mathfrak{X}$ be a scheme over $S$ equipped with an action $\lambda : \mathbb{G}_{m,S} \to \mathrm{Aut}(\mathfrak{X})$ in the sense of group schemes [4].

Let $e : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$ be the identity map. Given a scheme $T$ over $S$ we can precompose $e$ with the canonical map $\mathbb{G}_{m,T} \to \mathbb{G}_{m,S}$ to obtain an element $e_T \in \mathbb{G}_{m,S}(\mathbb{G}_{m,T})$. We define the $S$-subfunctor $\mathfrak{X}_\lambda$ of $\mathfrak{X}$ as follows: for a scheme $T$ over $S$,

$$\mathfrak{X}_\lambda(T) = \{x \in \mathfrak{X}(T) \mid \lambda(e_T) \,.\, x_{\mathbb{G}_{m,T}} \in \mathrm{Im}(\mathfrak{X}(\mathbb{A}^1_T) \to \mathfrak{X}(\mathbb{G}_{m,T}))\}.$$

The map $\mathfrak{X}(\mathbb{A}^1_T) \to \mathfrak{X}(\mathbb{G}_{m,T})$ is the restriction map and is injective whenever $\mathfrak{X}$ is separated over $S$ (in particular, whenever $\mathfrak{X}$ is affine over $S$; this will be the case in our main result).

$\mathfrak{X}_\lambda$ is called the limit (called also attractor) functor of $\mathfrak{X}$ with respect to $\lambda$. This terminology is justified since, roughly speaking, $\mathfrak{X}_\lambda(T)$ is comprised of elements $x \in \mathfrak{X}(T)$ such that $\lambda(e_T) \cdot x_{\mathbb{G}_{m,T}}$ has a limit when $t \mapsto 0$ (where $t$ is the coordinate of $\mathbb{G}_{m,T} = \mathrm{Spec}(\mathcal{O}_T[t, t^{-1}]) = \mathbb{G}_{m,S} \times_S T$). More precisely, the orbit map $\mathbb{G}_{m,S} \to \mathfrak{X}$, $g \to \lambda(g) \cdot x$

This paper was written before [5] was announced. The two works are independent and the approaches to the problem different. We believe that both methods shed light into an interesting question.

extends to a morphism $h_x : \mathbb{A}^1_S \to \mathfrak{X}$. The evaluation at 0 of this extension is called the limit point map (denoted by $q$ in what follows and studied in Lemma 2.2(2)). Our main result is the following smoothness fact.

**Theorem 1.1.** *Assume that $\mathfrak{X}$ is a smooth affine scheme over a scheme $S$. If $\lambda : \mathbb{G}_{m,S} \to Aut(\mathfrak{X})$ is an action of $\mathbb{G}_{m,S}$ on $\mathfrak{X}$, then $\mathfrak{X}_\lambda$ is representable by a smooth closed subscheme of $\mathfrak{X}$.*

*Remark* 1.2.

(a) The same holds for the fixed point subscheme $\mathfrak{X}^\lambda$ (Proposition A.8.10 of [3]). Our methods provide another proof of this result (see Remark 3.3 below).
(b) In the case when $S$ is the spectrum of a field, the statement is due to Bialynicki-Birula (§4 of [1]).

As mentioned in the Abstract, Theorem 1.1 generalizes the special important case of an action (as group automorphisms) of $\mathbb{G}_{m,S}$ on an affine smooth group scheme $\mathfrak{G}$ over $S$ due to Conrad *et al.* (see Proposition 2.18 and Remark 2.11 of [3]). It turns out that this fact is crucial in Conrad's 'dynamic method' for constructing coroots and parabolic subgroups of reductive group schemes (§4.1 of [2]). If we specialize only to the group case, one sees that our proof is more elementary since it uses only the characterization of smoothness in terms of completions of local rings and avoids having to appeal to unipotent group scheme considerations.

It also occurs as a special case of a result by Drinfeld on the action of $\mathbb{G}_{m,S}$ on algebraic spaces (Proposition 1.4.20 of [5]) in the case when $S$ is of finite type over a field.

## 2. Representability and first properties

Let $S$ be a scheme. In this section, $\mathfrak{X}$ stands for an arbitrary affine scheme over $S$ equipped with an action $\lambda$ of $\mathbb{G}_{m,S}$. We denote by $\mathfrak{X}^\lambda$ the functor of fixed points

$$\mathfrak{X}^\lambda(T) = \{x \in \mathfrak{X}(T) \mid \lambda(e_T) \,.\, x_{\mathbb{G}_{m,T}} = x_{\mathbb{G}_{m,T}} \in \mathfrak{X}(\mathbb{G}_{m,T})\}.$$

It is a $S$-subfunctor of $\mathfrak{X}_\lambda$. We denote by $c_\lambda : \mathcal{O}_\mathfrak{X} \to \mathcal{O}_\mathfrak{X}[t^{\pm 1}]$ the coaction associated to $\lambda$ (§4.7 of [8]). It induces a decomposition into eigenspaces

$$\mathcal{O}_\mathfrak{X} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{O}_\mathfrak{X})_n.$$

*Lemma* 2.1. *The $S$-functor $\mathfrak{X}_\lambda$ (resp. $\mathfrak{X}^\lambda$) is representable by a closed $S$-subscheme of $\mathfrak{X}$ whose sheaf of quasi-coherent ideals is generated by $(\mathcal{O}_\mathfrak{X})_n$ for $n$ running over the negative integers (resp. non zero integers).*

*Proof.* Similar to Lemma 2.1.4 of [3].

For each scheme $T$ over $S$ and each $x \in \mathfrak{X}_\lambda(T)$, the orbit map $\mathbb{G}_{m,T} \to \mathfrak{X}_T$, $t \mapsto \lambda(t) \,.\, x_{\mathbb{G}_{m,T}}$ extends to a morphism $h_x : \mathbb{A}^1_T \to \mathfrak{X}_T$. Applying this to the 'universal point' $id : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ of $\mathfrak{X}_\lambda$, we get a natural map

$$h : \mathbb{A}^1_S \times_S \mathfrak{X}_\lambda \to \mathfrak{X}.$$

Since $h\big(\mathbb{G}_{m,S} \times_S \mathfrak{X}_\lambda\big) \subset \mathfrak{X}_\lambda$, it follows that $h$ factors through the closed subscheme $\mathfrak{X}_\lambda$ of $\mathfrak{X}$. We still denote this factorization by $h : \mathbb{A}^1_S \times_S \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$. The specialization at 0 induces an $S$-map $q^\dagger : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$.

*Lemma* 2.2.

(1) $\mathfrak{X}^\lambda = \mathfrak{X}_\lambda \times_{\mathfrak{X}} \mathfrak{X}_{-\lambda}$.
(2) *The map $q^\dagger$ factors through $\mathfrak{X}^\lambda$. It thus defines an $S$-scheme morphism $q : \mathfrak{X}_\lambda \to \mathfrak{X}^\lambda$ such that the composite $\mathfrak{X}^\lambda \to \mathfrak{X}_\lambda \xrightarrow{q} \mathfrak{X}^\lambda$ is the identity.*

*Proof.* Let $T$ be an $S$-scheme.

(1) If $x \in \mathfrak{X}_\lambda(T) \cap \mathfrak{X}_{-\lambda}(T)$, then the orbit map extends to a map $\mathbb{P}^1_T \to \mathfrak{X}_T$, where $\mathbb{P}^1_T$ stands for the projective line over $T$. Since $\mathfrak{X}$ is affine, the orbit map is then constant. Hence $\lambda(t) \,.\, x_{\mathbb{G}_{m,T}} = x_{\mathbb{G}_{m,T}}$ and $x \in \mathfrak{X}^\lambda(T)$.
(2) Let $x \in \mathfrak{X}_\lambda(T)$ and let $x_0 = q^\dagger(x) \in \mathfrak{X}(T)$. For each scheme $T'$ over $T$ and each $a \in H^0(T', \mathbb{G}_m)$, we have $\lambda(at) \,.\, x_{\mathbb{G}_{m,T'}} = \lambda(a) \,.\, \big(\lambda(t) \,.\, x_{\mathbb{G}_{m,T'}}\big)$. By letting $t \mapsto 0$, we get that $x_{0,T'} = \lambda(a) \,.\, x_{0,T'}$. Hence $H^0(T', \mathbb{G}_m).x_{0,T'} = x_{0,T'}$. So we conclude that $x_0 \in \mathfrak{X}^\lambda(T)$.

## 3. Smoothness

Throughout, we assume that $\mathfrak{X}$ is smooth and affine over $S$. In this section, we establish Theorem 1.1. Without loss of generality, we can assume that $S$ is affine. We start by the following classical reduction to the noetherian case as explained in Section $\mathrm{VI}_B$.10.2 of [8].

*Lemma* 3.1. *To establish Theorem* 1.1, *there is no loss of generality in assuming that $S = Spec(A)$, where $A$ is a noetherian ring.*

*Proof.* Since the statement is local on $S$, we can obviously assume that $S$ is affine. Let $S = \mathrm{Spec}(A)$. We write $A = \varinjlim_{\alpha \in I} A_\alpha$ as the direct limit of its finitely generated where the partial order on $I$ is defined by the inclusion. More precisely, we have $\alpha \leq \beta$ iff $A_\alpha \subseteq A_\beta$. We consider now the coordinate ring $A[\mathfrak{X}]$ of $\mathfrak{X}$. Since $\mathfrak{X}$ is smooth, it is a finitely presented algebra over $A$, so there exists $\alpha_0 \in I$ and a finitely presented affine $A_{\alpha_0}$-scheme $\mathfrak{X}_{\alpha_0}$ such that $A[\mathfrak{X}] \cong A \otimes_{A_{\alpha_0}} [\mathfrak{X}_{\alpha_0}]$. We put $\mathfrak{X}_\alpha = \mathfrak{X}_{\alpha_0} \times_{\mathrm{Spec}(A_{\alpha_0})} \mathrm{Spec}(A_\alpha)$ for each $\alpha \geq \alpha_0$.

On the same vein, the coaction $c_\lambda : A[\mathfrak{X}] \to A[\mathfrak{X}][t^{\pm 1}]$ is defined 'at a finite level', that is, there exists $\alpha_1 \geq \alpha_0$ and a $A_{\alpha_1}$-morphism $c_{\alpha_1} : A_{\alpha_1}[\mathfrak{X}_{\alpha_1}] \to A_{\alpha_1}[\mathfrak{X}][t^{\pm 1}]$ which gives $c_\lambda$ by base change to $A$. Since $A_{\alpha_1}$ is a subring of $A$, $c_{\alpha_1}$ induces a right $A_{\alpha_1}[\mathfrak{X}][t^{\pm 1}]$-comodule structure on $A_{\alpha_1}[\mathfrak{X}_{\alpha_1}]$ (the comodule properties hold when passing to $A$). In other words, $c_{\alpha_1}$ defines an action of $\mathbb{G}_{m,\mathrm{Spec}(A_{\alpha_1})}$ which gives rise to $\lambda$ after the base change $A_{\alpha_1} \to A$. Possibly $\mathfrak{X}_{\alpha_1}$ is not smooth over $\mathrm{Spec}(A_{\alpha_1})$; by Proposition 17.7.8.(ii) of [6], there exists however $\alpha_2 \geq \alpha_1$ such that $\mathfrak{X}_{\alpha_2}$ is smooth over $\mathrm{Spec}(A_{\alpha_2})$. To summarize, we have defined a smooth affine scheme $\mathfrak{X}_{\alpha_2}$ over $\mathrm{Spec}(A_{\alpha_2})$ equipped with an action $\lambda_{\alpha_2}$ of $\mathbb{G}_{m,\mathrm{Spec}(A_{\alpha_2})}$ such that $\mathfrak{X}$ together with the action are extended from $\mathfrak{X}_{\alpha_2}$ and $\lambda_{\alpha_2}$ with respect to the base change of rings $A_{\alpha_2} \to A$. We have then an isomorphism of schemes $(\mathfrak{X}_{\alpha_2})_{\lambda_{\alpha_2}} \times_{\mathrm{Spec}(A_{\alpha_2})} \mathrm{Spec}(A) \xrightarrow{\sim} \mathfrak{X}_\lambda$. Since $A_{\alpha_2}$ is finitely generated over $\mathbb{Z}$, it is a noetherian ring. Smoothness being stable by arbitrary base change, it follows that is it enough to prove the smoothness of the scheme $(\mathfrak{X}_{\alpha_2})_{\lambda_{\alpha_2}}$ over $\mathrm{Spec}(A_{\alpha_2})$.

Henceforth we assume that $S$ is the spectrum of a noetherian ring. We start by looking at smoothness at a fixed point.

*Lemma* 3.2. *Let $x \in \mathfrak{X}^\lambda$. Then $\mathfrak{X}_\lambda$ is smooth at $x$.*

*Proof.* We denote by $s \in S$ the image of $x$ under the structure map $\mathfrak{X} \to S$.

*First case.* $\kappa(s) \xrightarrow{\sim} \kappa(x)$. We denote by $A$ (resp. $B$, $C$) the coordinate ring of $S$ (resp. $\mathfrak{X}$, $\mathfrak{X}_\lambda$). We denote by $\mathfrak{m}_A$ (resp. $\mathfrak{m}_B$, $\mathfrak{m}_C$) the maximal ideal of $A$ (resp. $B$, $C$) at $s$ (resp. $x$, $x$) and by $\hat{A}$ (resp. $\hat{B}$, $\hat{C}$) the respective completions. According to Proposition 17.5.3.(d") of [6], there exists an isomorphism $\hat{B} \simeq \hat{A}[[t_1, \ldots, t_d]]$. Our first objective is to make our choice of local parameters precise.

The cotangent space $V = \mathfrak{m}_B/\mathfrak{m}_B^2$ of $\mathfrak{X}$ at $x$ is a $\kappa(x)$-representation of dimension $d$ of $\mathbb{G}_{m,\kappa(x)}$. We diagonalize it as $V = \oplus_{n \in M} V_n$, where $M \subset \mathbb{Z}$ stands for the set of weights. Let $(\overline{b}_1, \ldots, \overline{b}_d)$ be a $\kappa(x)$-basis of $V$ consisting of eigenvectors of respective weights $n_1 \leq n_2 \leq \cdots \leq n_d$. Since the surjective map $B \to B/\mathfrak{m}_B$ is $\mathbb{Z}$-homogeneous we can lift each $\overline{b}_i$ to some homogeneous $b_i \in B_{n_i}$. From Proposition 17.5.3 of [6], there exists an isomorphism

$$f : \hat{A}[[t_1, \ldots, t_d]] \simeq \hat{B}, \quad t_i \mapsto b_i.$$

Next we observe that $\hat{C}$ is also the completion of the $B$-module $C = B/J$ with respect to the $\mathfrak{m}_B$-adic topology. By Theorem 8.11 of [7], this completion is isomorphic to $\hat{B}/J\hat{B}$, hence the map $\hat{B}/J\hat{B} \to \hat{C}$ is an isomorphism. From Lemma 2.1, it follows that $J\hat{B}$ is generated by the $t_i$'s such that $n_i < 0$. Since $\hat{B}/J \otimes_B \hat{B} \xrightarrow{\sim} \hat{C}$, we conclude that $\hat{C} \simeq \hat{A}[[t_r, \cdots t_d]]$, hence that $\mathfrak{X}$ is smooth at $x$ by the completion criterion (Proposition 17.5.3 of [6]).

*General case.* We use the 'diagonal trick' that results from applying the base change $g : S' = \mathfrak{X} \to S$, which is smooth, so, in particular, flat and locally of finite presentation. Denote by $\mathfrak{U}$ (resp. $\mathfrak{U}'$) the smooth locus of $f : \mathfrak{X}_\lambda \to S$ (resp. of $f' : \mathfrak{X}'_\lambda = \mathfrak{X}_\lambda \times_S S' \to S'$). According to Proposition 17.7.4.(v) of [6], we have $\mathfrak{U}' = g^{-1}(\mathfrak{U})$. Now consider the point $s' = x$ of $S'$ and the diagonal point $x' = (x, s')$ of $\mathfrak{X}'$ which belongs to $\mathfrak{X}'_\lambda = \mathfrak{X}'_\lambda \times_S S'$. By the first case, $x' \in \mathfrak{U}'$, hence $x \in \mathfrak{U}$. We conclude that $\mathfrak{X}_\lambda \to S$ is smooth at $x$.

*Remark* 3.3. In a similar fashion one can show that $\mathfrak{X}^\lambda$ is smooth over $S$ and also that $q : \mathfrak{X}_\lambda \to \mathfrak{X}^\lambda$ is smooth.

For handling the general case, we denote by $\mathfrak{U} \subset \mathfrak{X}_\lambda$ the smooth locus of the morphism $\mathfrak{X}_\lambda \to S$. It is an open subset which is $\mathbb{G}_{m,S}$-stable. By Lemma 2.2, we can consider the morphism $h : \mathbb{A}^1 \times_S \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$ and form the following cartesian square

$$\begin{array}{ccc} \mathfrak{V} & \longrightarrow & \mathfrak{U} \\ \big\downarrow & & \big\downarrow \\ \mathbb{A}^1 \times_S \mathfrak{X}_\lambda & \xrightarrow{h} & \mathfrak{X}_\lambda \end{array} .$$

The bottom map comes equipped with an action of $\mathbb{G}_{m,S} \times_S \mathbb{G}_{m,S}$ which acts on each factor on the left-hand side and by $\lambda(t_1 t_2)$ on the right-hand side. Since $\mathfrak{U} \subset \mathfrak{X}_\lambda$ is preserved under this action, it follows that the square is actually $\mathbb{G}_{m,S}^2$-equivariant. We keep this fact in mind. The second observation is that by Lemma 2.2(2), $h(\{0\} \times_S \mathfrak{X}_\lambda) \subset \mathfrak{X}^\lambda$, hence $\{0\} \times_S \mathfrak{X}_\lambda \subset \mathfrak{V}$.

*Claim* 3.4. For each geometric point $s : \mathrm{Spec}(k) \to S$ ($k$ algebraically closed), we have $\mathfrak{V}_s = \mathbb{A}_k^1 \times_k (\mathfrak{X}_\lambda)_s$.

The fiber $\mathfrak{V}_s$ at $s$ is an open subset of the affine $k$-variety $\mathbb{A}_k^1 \times_k (\mathfrak{X}_\lambda)_s$ which is $\mathbb{G}_{m,k}^2$-equivariant and contains $\{0\} \times_k (\mathfrak{X}_\lambda)_k$. If $w = (u, x) \in k^\times \times (\mathfrak{X}_\lambda)_s(k)$, we have $(k^\times, 1) \cdot w \cap \mathfrak{V}_s(k) \neq \emptyset$ so that $w \in \mathfrak{V}_s(k)$. Therefore $\mathfrak{V}_s(k) = k \times (\mathfrak{X}_\lambda)_s(k)$ and we conclude that $\mathfrak{V}_s = \mathbb{A}_k^1 \times_k (\mathfrak{X}_\lambda)_s$.

The claim is thus proven and this shows that $\mathfrak{V}$ contains all points of $\mathbb{A}^1 \times_S \mathfrak{X}_\lambda$. Thus $\mathfrak{V} = \mathbb{A}^1 \times_S \mathfrak{X}_\lambda$, $\mathfrak{U} = \mathfrak{X}_\lambda$ and $\mathfrak{X}_\lambda$ is smooth.

*Remark* 3.5. If in the above proof we take the smooth locus of $q : \mathfrak{X}_\lambda \to \mathfrak{X}^\lambda$, then the same argument shows that $q$ is smooth by taking into account that $q$ is smooth in a neighborhood of $\mathfrak{X}^\lambda$ (Remark 3.3).

## Acknowledgements

The author wishes to thank Olivier Mathieu for useful discussions, Brian Conrad and Cyril Demarche for their comments, and the Fields Institute in Toronto for its hospitality.

## References

[1] Bialynicki-Birula A, Some theorems on actions of algebraic groups, *Ann. Math.* **98** (1973) 480–497
[2] Conrad B, Reductive group schemes, math.stanford.edu/~conrad/papers/luminysga3.pdf
[3] Conrad B, Gabber O, Prasad G, Pseudo-reductive groups (2010) (Cambridge University Press)
[4] Demazure M and Gabriel P, *Groupes algébriques*, Masson (1970)
[5] Drinfeld V, On algebraic spaces with an action of $\mathfrak{G}_m$, preprint (2013) arXiv:1308.2604
[6] Grothendieck A (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique: Etude locale de schémas et des morphismes de schémas*, Public. de l'I.H.É.S. no. **20** (1964), **24** (1965), **28** (1966), **32** (1967)
[7] Matsumura H, *Commutative ring theory*, second edition (Cambridge University Press)
[8] Séminaire de Géométrie algébrique de l'I.H.É.S. 1963–1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck, Lecture Notes in Math., Springer, no. 151–153, 1970–1971

COMMUNICATING EDITOR: Parameswaran Sankaran