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Subintegrality, invertible modules and Laurent polynomial extensions

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Abstract. Let $A \subseteq B$ be a commutative ring extension. Let $\mathcal{I}(A, B)$ be the multiplicative group of invertible A-submodules of B. In this article, we extend a result of Sadhu and Singh by finding a necessary and sufficient condition on an integral birational extension $A \subseteq B$ of integral domains with $\dim A \leq 1$, so that the natural map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ is an isomorphism. In the same situation, we show that if $\dim A \geq 2$, then the condition is necessary but not sufficient. We also discuss some properties of the cokernel of the natural map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ in the general case.

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1. Introduction

In [4], Roberts and Singh have introduced the group $\mathcal{I}(A, B)$ to generalize a result of Dayton. The relation between the group $\mathcal{I}(A, B)$ and subintegral extensions has been investigated by Reid, Roberts and Singh in a series of papers. Recently in [5], Sadhu and Singh have proved that A is subintegrally closed in B if and only if the canonical map $\mathcal{I}(A, B) \to \mathcal{I}(A[X], B[X])$ is an isomorphism. It is easy to see that the map is injective and that $\mathcal{I}(A[X], B[X]) = \mathcal{I}(A, B) \oplus N\mathcal{I}(A, B)$, where $N\mathcal{I}(A, B)$ denotes the kernel of the map $\mathcal{I}(A[X], B[X]) \stackrel{X\mapsto 0}{\to} \mathcal{I}(A, B)$. So the result of [5], just mentioned, amounts to saying that $N\mathcal{I}(A, B) = 0$ if and only if A is subintegrally closed in B.

The primary goal of this paper is to extend the result of Sadhu and Singh in [5] by finding a necessary and sufficient condition on $A \subseteq B$, so that the natural map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ is an isomorphism. It is easy to see that the map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ is always injective (see Lemma 3.8). Thus the problem reduces to the investigation of conditions for the cokernel of the above map to be zero. This cokernel will be denoted by $M\mathcal{I}(A, B)$. The secondary goal will be to investigate properties of the cokernel $M\mathcal{I}(A, B)$ in the general case.

In §2, we mainly give basic definitions and notations. In §3, we discuss conditions on $A \subseteq B$ under which the map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ is an isomorphism. We are able to prove some results in the situation when $A \subseteq B$ is an integral birational extension of domains. First, if $\dim A \leq 1$ then by using a result of Onoda-Yoshida (Theorem 1.13 of [3]), we prove the following.

Theorem 3.14. *Let $A \subseteq B$ be an integral, birational extension of domains with* $\dim A \leq 1$. *Then $\mathcal{MI}(A, B) = 0$ if and only if A is subintegrally closed in B and $A \subseteq B$ is anodal.*

For higher dimension, we show that the above conditions are necessary but not sufficient. More precisely, we prove the following

Theorem 3.17. *Let $A \subseteq B$ be an integral, birational extension of domains. Suppose $\mathcal{MI}(A, B) = 0$. Then A is subintegrally closed in B and $A \subseteq B$ is anodal.*

That the conditions are not sufficient is shown by an example of Weibel [9] (see Remark 3.18). We note that for any ring extension $A \subseteq B$, the condition $\mathcal{MI}(A, B) = 0$ implies easily that A is subintegrally closed in B (see Lemma 3.15(4)).

In §4, we examine the cokernel $\mathcal{MI}(A, B)$ in the general case. In order to do this, we first discuss the surjectivity of the natural map $\varphi(A, C, B) : \mathcal{I}(A, B) \to \mathcal{I}(C, B)$ given by $\varphi(A, C, B)(I) = IC$ for any ring extensions $A \subseteq C \subseteq B$. We show that the map $\varphi(A, C, B)$ is surjective in two cases: (1) C is subintegral over A, (2) $A \subseteq B$ is an integral extension with A being Hensel local (see Propositions 4.1 and 4.2). We show further that if C is subintegral over A, then the sequence

$$1 \to \mathcal{MI}(A, C) \to \mathcal{MI}(A, B) \to \mathcal{MI}(C, B) \to 1$$

is exact (see Proposition 4.3). Finally we prove the following

Theorem 4.7. *Let $A \subseteq B$ be a ring extension with A as Hensel local and B as seminormal. Then $\mathcal{MI}(A, B) \cong \mathcal{MI}(A, {}^{+}\!A) \oplus \mathcal{MI}({}^{+}\!A, B)$, where ${}^{+}\!A$ is the subintegral closure of A in B.*

In this section we also observe that if A is subintegrally closed in B with B a seminormal domain and A a Hensel local, then $\mathcal{MI}(A, B) = 0$ (see Proposition 4.5(4)).

2. Basic definitions and notations

All the rings we consider are commutative with 1, and all ring homomorphisms are unitary. Let X, T be indeterminates.

An *elementary subintegral* extension is an extension of the form $A \subseteq B$ with $B = A[b]$ for some $b \in B$ such that $b^2, b^3 \in A$. An extension $A \subseteq B$ is *subintegral* if it is a filtered union of elementary subintegral extensions; that is, for each $b \in B$ there is a finite sequence $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_r \subseteq B$ of ring extensions such that $b \in C_r$ and $C_{i-1} \subseteq C_i$ is elementary subintegral for each i, $1 \leq i \leq r$. We say that A is *subintegrally closed* in B if whenever $b \in B$ and $b^2, b^3 \in B$, then $b \in A$. The ring A is *seminormal* if the following condition holds: $b, c \in A$ and $b^3 = c^2$ imply that there exists $a \in A$ with $b = a^2$ and $c = a^3$. A seminormal ring is necessarily reduced and is subintegrally closed in every reduced overring. It is easily seen that if A is subintegrally closed in B with B seminormal then A is seminormal. For details, see [7, 8].

For a ring A we denote by

$U(A)$: the groups of units of A,
$H^0(A) = H^0(\mathrm{Spec}(A), \mathbb{Z})$: the group of continuous maps from $\mathrm{Spec}(A)$ to $\mathbb{Z}$,
$\mathrm{Pic}\ A$: the Picard group of A,

$KU(A)$: cokernel of the natural map $U(A) \to U(A[X])$,
$MU(A)$: cokernel of the natural map $U(A) \to U(A[X, X^{-1}])$,
$NU(A)$: kernel of the map $U(A[X]) \to U(A)$,
K Pic A: cokernel of the natural map Pic $A \to$ Pic $A[X]$,
M Pic A: cokernel of the natural map Pic $A \to$ Pic $A[X, X^{-1}]$,
N Pic A: kernel of the map Pic $A[X] \to$ Pic A,
L Pic A: cokernel of the map Pic $A[X] \times$ Pic $A[X^{-1}] \overset{\text{add}}{\to}$ Pic $A[X, X^{-1}]$.

Let $A \subseteq B$ be a ring extension. Then we denote by

$\mathcal{I}(A, B)$: the group of invertible A-submodules of B.

It is easily seen that $\mathcal{I}$ is a functor from extensions of rings to abelian groups. Some properties of $\mathcal{I}(A, B)$ can be found in [§2 of [4]].

$K\mathcal{I}(A, B)$: cokernel of the natural map $\mathcal{I}(A, B) \to \mathcal{I}(A[X], B[X])$,
$M\mathcal{I}(A, B)$: cokernel of the natural map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$,
$N\mathcal{I}(A, B)$: kernel of the map $\mathcal{I}(A[X], B[X]) \to \mathcal{I}(A, B)$ (Here the map is induced by the B-algebra homomorphism $B[X] \to B$ given by $X \mapsto 0$).

Recall from §2 of [4] that for any commutative ring extension $A \subseteq B$, we have the exact sequence

$$1 \to U(A) \to U(B) \to \mathcal{I}(A, B) \to \text{Pic } A \to \text{Pic } B.$$

Applying M, K we obtain the chain complexes:

$$1 \to MU(A) \to MU(B) \to M\mathcal{I}(A, B) \overset{\eta}{\to} M \text{ Pic } A \overset{\varphi}{\to} M \text{ Pic } B \tag{2.1}$$

and

$$1 \to KU(A) \to KU(B) \to K\mathcal{I}(A, B) \overset{\alpha}{\to} K \text{ Pic } A \overset{\beta}{\to} K \text{ Pic } B. \tag{2.2}$$

3. The map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$

In this section, we examine some conditions on $A \subseteq B$ under which the natural map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ is an isomorphism. For this, we consider the notions of quasinormal and anodal extensions (or u-closed).

Let $A \subseteq B$ be a ring extension. We say that A is *quasinormal* in B if the natural map M Pic $A \to M$ Pic B is injective. For properties of quasinormal extensions, see [3].

An inclusion $A \subseteq B$ of rings is called *anodal* or *an anodal extension*, if every $b \in B$ such that $(b^2 - b) \in A$ and $(b^3 - b^2) \in A$ belongs to A. This notion was first introduced by Asanuma and Onoda-Yoshida in [3], and they called this notion 'u-closed'. Some related details can be found in [1, 3, 9].

We first show in Proposition 3.2 below that a subintegral extension is always an anodal extension, which is perhaps a result of independent interest.

Lemma 3.1. *Let $A \subseteq C \subseteq B$ be extensions of rings. Then the following statements hold*:

(1) *If A is anodal in B, then so is A in C.*
(2) *If A is anodal in C and C is anodal in B, then so is A in B.*

Proof. Clear from the definition. □

PROPOSITION 3.2

Let $A \subseteq B$ be a ring extension. If $A \subseteq B$ is subintegral, then it is anodal.

Proof. Assume first that $A \subseteq B$ is an elementary subintegral extension, i.e., $A \subseteq B = A[b]$ such that $b^2, b^3 \in A$. Let $f \in B$ such that $f^2 - f, f^3 - f^2 \in A$. We have to show that $f \in A$. Clearly f is of the form $a + \lambda b$ where $a, \lambda \in A$. So it is enough to show that $\lambda b \in A$. Since $\lambda b(2a-1), \lambda b(3a^2-1) \in A$, $\lambda b = \lambda b.1 = \lambda b[(6a+3)(2a-1) - 4(3a^2-1)] \in A$. Hence $f \in A$.

In the general case, for $f \in B$ there exists a finite sequence $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_r \subseteq B$ of extensions such that $C_i \subseteq C_{i+1}$ is an elementary subintegral extension for each $i, 0 \leq i \leq r-1$ and $f \in C_r$. So by the above argument $C_i \subseteq C_{i+1}$ is anodal for each i. Now the result follows from Lemma 3.1(2). □

The following result is due to Sadhu and Singh (Theorem 1.5 of [5]) which we use frequently throughout this paper.

Lemma 3.3. *Let $A \subseteq B$ be a ring extension. Then A is subintegrally closed in B if and only if the canonical map $\mathcal{I}(A, B) \to \mathcal{I}(A[X], B[X])$ is an isomorphism.*

One can restate the above result in the following way: A is subintegrally closed in B $\Leftrightarrow K\mathcal{I}(A, B) = 0 \Leftrightarrow N\mathcal{I}(A, B) = 0$.

The following result is due to Weibel (Theorem 5.2 of [9]).

Lemma 3.4. *There is a natural decomposition*

$$\text{Pic } A[X, X^{-1}] \cong \text{Pic } A \oplus N \text{ Pic } A \oplus N \text{ Pic } A \oplus L \text{ Pic } A$$

for any commutative ring A.

Remark 3.5. By Swan theorem [7], N Pic $A = 0$ if and only if A_{red} is seminormal. So for a seminormal ring A, L Pic $A \cong M$ Pic A.

The next result is given in (Exercise 3.17, page 30 of [10]).

Lemma 3.6. *There is a natural decomposition*

$$U(A[X, X^{-1}]) \cong U(A) \oplus NU(A) \oplus NU(A) \oplus H^0(A)$$

for any commutative ring A.

Remark 3.7. It follows that for a reduced ring A, $H^0(A) \cong MU(A)$.

Lemma 3.8. *The natural map $\phi : \mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$, given by $I \to IA[X, X^{-1}]$, is injective. Thus, ϕ is an isomorphism if and only if $M\mathcal{I}(A, B) = 0$.*

Proof. Let $I = (b_1, b_2, \ldots, b_r)A \in \ker \phi$, where $b_i \in B$. Then $IA[X, X^{-1}] = A[X, X^{-1}]$. This implies that $b_i \in A[X, X^{-1}] \cap B = A$, for all i. So $I \subseteq A$. Similarly $I^{-1} \subseteq A$. Hence $I = A$. □

Lemma 3.9. *The sequence* (2.1) (*respectively* (2.2)) *is exact*, *except possibly at the place* $M\,\text{Pic}\,A$ (*respectively* $K\,\text{Pic}\,A$). *It is exact there too if the map* $\text{Pic}\,A \to \text{Pic}\,B$ *is surjective.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccccc}
 & & 1 & & 1 & & 1 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & U(A) & \to & U(B) & \to & \mathcal{I}(A, B) & \to & \text{Pic}\,A & \to & \text{Pic}\,B \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & U(A[X, X^{-1}]) & \to & U(B[X, X^{-1}]) & \to & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \to & \text{Pic}\,A[X, X^{-1}] & \to & \text{Pic}\,B[X, X^{-1}], \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & MU(A) & \to & MU(B) & \to & M\mathcal{I}(A, B) & \to & M\text{Pic}\,A & \to & M\text{Pic}\,B \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1 & & 0 & & 0
\end{array}$$

where the first two rows are exact and each column is exact. Now the result follows by chasing this diagram. □

Lemma 3.10. *Let* $A \subseteq B$ *be a ring extension. The map* $\text{Pic}\,A \to \text{Pic}\,B$ *is surjective if any one of the following conditions holds*:

(1) $A \subseteq B$ *is subintegral.*
(2) $A \subseteq B$ *is an integral*, *birational extension of domains with* $\dim A \leq 1$.

Proof.

(1) See Proposition 7 of [2].
(2) Let K be the quotient field of A and B. We have the commutative diagram:

$$\begin{array}{ccccc}
\mathcal{I}(A, K) & \longrightarrow & \text{Pic}\,A & \longrightarrow & 0 \\
{\scriptstyle \varphi(A,B,K)}\downarrow & & {\scriptstyle \rho}\downarrow & & , \\
\mathcal{I}(B, K) & \longrightarrow & \text{Pic}\,B & \longrightarrow & 0
\end{array}$$

where $\varphi(A, B, K)$ is surjective by Proposition 2.3 of [5]. Hence ρ is surjective. □

Lemma 3.11 (*cf. Lemma* 1.4 *of* [3]). *Let* $A \subseteq B$ *be a ring extension with* B *reduced and* A *quasinormal in* B. *Then* A *is subintegrally closed in* B.

Proof. We have not assumed B to be a domain. By Lemma 3.3, it is enough to show that $K\mathcal{I}(A, B) = 0$. We have the sequence

$$1 \to KU(A) \to KU(B) \to K\mathcal{I}(A, B) \xrightarrow{\alpha} K\ \text{Pic}\,A \xrightarrow{\beta} K\ \text{Pic}\,B$$

which is exact except possibly at the place K Pic A. Since A and B are reduced, $KU(A) = 0$ and $KU(B) = 0$. In the proof of Lemma 1.4 of [3], it is shown that the map K Pic $A \to K$ Pic B is injective, i.e., $\ker \beta = 0$. We have $\operatorname{im} \alpha \subseteq \ker \beta$. Hence $K\mathcal{I}(A, B) = 0$. □

Remark 3.12. In the above lemma we cannot drop the condition that B is reduced. For example, consider the extension $A = K \subsetneq B = K[b]$ with $b^2 = 0$, where K is any field. Since M Pic $K = 0$, clearly A is quasinormal in B. But A is not subintegrally closed in B, because $b^2 = b^3 = 0 \in K, b \notin K$.

Lemma 3.13. *Let $A \subseteq B$ be a ring extension with B a domain. Then the following statements hold*:

(1) *If A is quasinormal in B, then $M\mathcal{I}(A, B) = 0$.*
(2) *Suppose the extension $A \subseteq B$ is integral and birational with* $\dim A \leq 1$, *and $M\mathcal{I}(A, B) = 0$. Then A is quasinormal in B.*

Proof.

(1) Since A and B are domains, $MU(A) = MU(B) \cong \mathbb{Z}$. By (2.1), $\operatorname{im} \eta \subseteq \ker \varphi$. As A is quasinormal in B, $\ker \varphi = 0$. This implies that $\operatorname{im} \eta = 0$. We get $M\mathcal{I}(A, B) = 0$.
(2) By Lemma 3.10(2) and Lemma 3.9, the sequence (2.1) is exact at M Pic A also. Since $M\mathcal{I}(A, B) = 0$, we get the result. □

Theorem 3.14. *Let $A \subseteq B$ be an integral, birational extension of domains with* $\dim A \leq 1$. *Then $M\mathcal{I}(A, B) = 0$ if and only if A is subintegrally closed in B and $A \subseteq B$ is anodal.*

Proof. If $\dim A = 0$, then $A = B$ and the assertion holds trivially in this case. If $\dim A = 1$, then by Theorem 1.13 of [3], A is quasinormal in B if and only if A is subintegrally closed in B and $A \subseteq B$ is anodal. We also have that A is quasinormal in B if and only if $M\mathcal{I}(A, B) = 0$ by Lemma 3.13. Combining these two results we get the assertion. □

Next, in Theorem 3.17 and Remark 3.18, we show that in general, the condition A is subintegrally closed in B and $A \subseteq B$ is anodal are necessary but not sufficient.

Lemma 3.15.

(1) *The diagram*

$$\begin{array}{ccc} \mathcal{I}(A,B) & \xrightarrow{\psi} & \mathcal{I}(A[X],B[X]) \\ & \searrow^{\phi} & \downarrow{\scriptstyle\theta} \\ & & \mathcal{I}(A[X,X^{-1}],B[X,X^{-1}]) \end{array}$$

is commutative.
(2) *The maps ϕ, ψ and θ are injective.*
(3) *ϕ is an isomorphism if and only if ψ and θ are isomorphisms.*
(4) *If ϕ is an isomorphism, i.e., $M\mathcal{I}(A, B) = 0$, then A is subintegrally closed in B.*

Proof.

(1) Since the maps are natural, the diagram is commutative.
(2) ϕ is injective by Lemma 3.8. The injectivity of ψ and θ follows by a similar argument as in Lemma 3.8.
(3) If ψ and θ are isomorphisms then clearly ϕ is an isomorphism. Conversely, suppose ϕ is an isomorphism. Then by simple diagram chasing we get that ψ and θ are isomorphisms.
(4) If ϕ is an isomorphism, then ψ is an isomorphism. Hence by Lemma 3.3, A is subintegrally closed in B. □

Lemma 3.16. *Let $\mathfrak{a}$ be a B-ideal contained in A. Then the homomorphism $M\mathcal{I}(A, B) \to M\mathcal{I}(A/\mathfrak{a}, B/\mathfrak{a})$ is an isomorphism.*

Proof. Clearly, $\mathfrak{a}[X, X^{-1}]$ is a $B[X, X^{-1}]$-ideal contained in $A[X, X^{-1}]$. We have $\mathcal{I}(A, B) \cong \mathcal{I}(A/\mathfrak{a}, B/\mathfrak{a})$ and $\mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \cong \mathcal{I}(A/\mathfrak{a}[X, X^{-1}], B/\mathfrak{a}[X, X^{-1}])$ by Proposition 2.6 of [4]. Now by chasing a suitable diagram, we get the result. □

Theorem 3.17. *Let $A \subseteq B$ be an integral, birational extension of domains. Suppose $M\mathcal{I}(A, B) = 0$. Then A is subintegrally closed in B and $A \subseteq B$ is anodal.*

Proof. By Lemma 3.15(4), A is subintegrally closed in B. To prove $A \subseteq B$ is anodal, by Lemma 1.10 of [3], it is enough to show that for every intermediate ring C between A and B such that C is a finite A-module, the map $M \operatorname{Pic} A \to M \operatorname{Pic}(A/\mathfrak{c}) \times M \operatorname{Pic} C$ is injective, where $\mathfrak{c}$ is the conductor of C in A. We first claim that the map $\tau : M\mathcal{I}(A, C) \to M\mathcal{I}(A, B)$ is injective, where C is any intermediate ring between A and B.

We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{I}(A,C) & \longrightarrow & \mathcal{I}(A[X,X^{-1}],C[X,X^{-1}]) & \longrightarrow & M\mathcal{I}(A,C) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\tau} & & \\ 1 & \longrightarrow & \mathcal{I}(A,B) & \xrightarrow{\phi} & \mathcal{I}(A[X,X^{-1}],B[X,X^{-1}]) & \longrightarrow & M\mathcal{I}(A,B) & \longrightarrow & 1 \end{array},$$

where the first two vertical arrows are natural inclusions (because any invertible A-submodule of C is also an invertible A-submodule of B).

Let $\bar{J} \in \ker \tau$, where $J \in \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}])$. Then $J \in \operatorname{im} \phi$ and there exists $J_1 \in \mathcal{I}(A, B)$ such that $J_1 A[X, X^{-1}] = J$. Let $J_1 = (b_1, b_2, \ldots, b_r)A$ and $J = (f_1, f_2, \ldots, f_s)A[X, X^{-1}]$, where $b_i \in B$ and $f_i \in C[X, X^{-1}]$. Then clearly $b_i \in B \cap C[X, X^{-1}] = C$ for all i. So $J_1 \subseteq C$. Also $J_1^{-1} \subseteq C$. This implies that $J_1 \in \mathcal{I}(A, C)$. So $\bar{J} = 0$. This proves the claim.

Since $M\mathcal{I}(A, B) = 0$, $M\mathcal{I}(A, C) = 0$. By Lemma 3.16, $M\mathcal{I}(A/\mathfrak{c}, C/\mathfrak{c}) = 0$, where $\mathfrak{c}$ is the conductor of C in A. By (2.1), we have $MU(A) \cong MU(C)$ and $MU(A/\mathfrak{c}) \cong MU(C/\mathfrak{c})$. Now the result follows from the following exact sequence which we obtain by applying M to the unit-Pic sequence (Theorem 3.10 of [10]),

$$\begin{aligned} MU(A) &\to MU(A/\mathfrak{c}) \times MU(C) \to MU(C/\mathfrak{c}) \to M \operatorname{Pic} A \\ &\to M \operatorname{Pic}(A/\mathfrak{c}) \times M \operatorname{Pic} C \end{aligned}$$

□

Remark 3.18. The converse of the above theorem holds for $\dim A \leq 1$ as seen in Theorem 3.14. In general, the converse does not hold. This is seen by considering Example 3.5 of [9]. In that example, A is a 2-dimensional noetherian domain whose integral closure is $B = K[X, Y]$, where K is a field. So $A \subseteq B$ is an integral, birational extension. By Proposition 3.5.2 of [9], $A \subseteq B$ is anodal and A is subintegrally closed in B. Since B is a UFD, $\text{Pic } B = \text{Pic } B[T, T^{-1}] = 0$. Then we get the exact sequence

$$1 \to MU(A) \to MU(B) \to M\mathcal{I}(A, B) \to M \text{ Pic } A \to 0.$$

As A, B are domains, $MU(A) = MU(B) \cong \mathbb{Z}$. So $M\mathcal{I}(A, B) \cong M \text{ Pic } A$. By Remark 3.5, $L \text{ Pic } A \cong M \text{ Pic } A$. Hence by Proposition 3.5.2 of [9], $M\mathcal{I}(A, B) \neq 0$.

4. Some observations on $M\mathcal{I}(A, B)$

In this section we discuss some properties of the cokernel $M\mathcal{I}(A, B)$ in the general case.

Recall from §3 of [6] that for any extensions $A \subseteq C \subseteq B$ of rings, we have the exact sequence

$$1 \to \mathcal{I}(A, C) \to \mathcal{I}(A, B) \overset{\varphi(A,C,B)}{\to} \mathcal{I}(C, B)$$

where the map $\varphi(A, C, B)$ is given by $\varphi(A, C, B)(I) = IC$.

Now it is natural to ask under what conditions on $A \subseteq B$ the map $\varphi(A, C, B)$ is surjective. In [6], Singh has proved that if B is subintegral over A then the map $\varphi(A, C, B)$ is surjective. In the next Proposition, we generalize Singh's result as follows:

PROPOSITION 4.1

For all rings C between A and B such that C is subintegral over A, the map $\varphi(A, C, B)$ is surjective.

Proof. We have the commutative diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & U(A) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(A,B) & \longrightarrow & \text{Pic}\, A & \longrightarrow & \text{Pic}\, B \\
 & & \downarrow & & \downarrow = & & \downarrow \varphi(A,C,B) & & \downarrow \rho & & \downarrow =. \\
1 & \longrightarrow & U(C) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(C,B) & \longrightarrow & \text{Pic}\, C & \longrightarrow & \text{Pic}\, B
\end{array}$$

Since ρ is surjective by Lemma 3.10(1), the result follows by chasing the diagram. □

The following result gives another case where the map $\varphi(A, C, B)$ is surjective.

Recall that a local ring A is *Hensel* if every finite A-algebra B is a direct product of local rings.

PROPOSITION 4.2

Let $A \subseteq B$ be an integral extension with A Hensel local. Then for all rings C with $A \subseteq C \subseteq B$, the map $\varphi(A, C, B)$ is surjective.

Proof. By Lemma 2.2 of [5], it is enough to show that $\varphi(A, D, B)$ is surjective for every subring D of C containing A such that D is finitely generated as an A-algebra. Let such a ring D be given. Since D is integral over A, D is a finite A-algebra. As A is Hensel, D is a finite direct product of local rings. Then Pic A and Pic D are both trivial. This implies that $\mathcal{I}(A, B) = U(B)/U(A)$, $\mathcal{I}(D, B) = U(B)/U(D)$ and clearly $\varphi(A, D, B)$ is surjective. □

PROPOSITION 4.3

Let $A \subseteq C \subseteq B$ be extensions of rings with $A \subseteq C$ subintegral. Then the sequence

$$1 \to M\mathcal{I}(A, C) \to M\mathcal{I}(A, B) \to M\mathcal{I}(C, B) \to 1$$

is exact.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{I}(A, C) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, C) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, B) & \longrightarrow & 1, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{I}(C, B) & \longrightarrow & \mathcal{I}(C[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(C, B) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where the rows are clearly exact. Since $A \subseteq C$ is subintegral, so is $A[X, X^{-1}] \subseteq C[X, X^{-1}]$. Therefore by Proposition 4.1, the first two columns are exact. Hence exactness of the last column follows by chasing the diagram. □

COROLLARY 4.4

Let $A \subseteq B$ be a ring extension and let ${}^{+}\!A$ denote the subintegral closure of A in B. Then the sequence

$$1 \to M\mathcal{I}(A, {}^{+}\!A) \to M\mathcal{I}(A, B) \to M\mathcal{I}({}^{+}\!A, B) \to 1$$

is exact.

Proof. Immediate from Proposition 4.3. □

PROPOSITION 4.5

Let $A \subseteq B$ be a ring extension. Assume that A is subintegrally closed in B. Then

(1) $M\mathcal{I}(A, B) \cong M\mathcal{I}(A[T], B[T])$.
(2) *$M\mathcal{I}(A, B)$ is a torsion-free abelian group if B is a seminormal ring.*
(3) *$M\mathcal{I}(A, B)$ is a free abelian group if B is a seminormal ring and A is Hensel local.*
(4) *$M\mathcal{I}(A, B) = 0$ if B is a seminormal domain and A is Hensel local.*

Proof.

(1) Since A is subintegrally closed in B, $A[X]$ is subintegrally closed in $B[X]$ by Corollary 1.6 of [5]. Therefore $A[X, X^{-1}]$ is subintegrally closed in $B[X, X^{-1}]$. We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{I}(A,B) & \longrightarrow & \mathcal{I}(A[X,X^{-1}],B[X,X^{-1}]) & \longrightarrow & M\mathcal{I}(A,B) & \longrightarrow & 1 \\ & & \downarrow \psi & & \downarrow \xi & & \downarrow & & , \\ 1 & \longrightarrow & \mathcal{I}(A[T],B[T]) & \longrightarrow & \mathcal{I}(A[T][X,X^{-1}],B[T][X,X^{-1}]) & \longrightarrow & M\mathcal{I}(A[T],B[T]) & \longrightarrow & 1 \end{array}$$

where ψ and ξ are isomorphisms by Lemma 3.3. Hence we get the result.

(2) As A is subintegrally closed in B and B is a seminormal ring, A is seminormal. Then by Remark 3.5, L Pic $A \cong M$ Pic A. Since any seminormal ring is reduced, $MU(A) = H^0(A)$ and $MU(B) = H^0(B)$ by Remark 3.7. Now, from (2.1), we have the exact sequence

$$1 \to H^0(A) \to H^0(B) \to M\mathcal{I}(A, B) \to M \text{ Pic } A,$$

where M Pic A is a torsion-free abelian group by Corollary 2.3.1 of [9]. Let T be the cokernel of the map $H^0(A) \to H^0(B)$. Then

$$1 \to T \to M\mathcal{I}(A, B) \to M \text{ Pic } A$$

is exact and T is a free abelian group by Proposition 1.3 of [9]. Therefore $M\mathcal{I}(A, B)$ is a torsion-free abelian group.

(3) By Theorem 2.5 of [9], L Pic $A = 0$. Since A is seminormal, M Pic $A = 0$. Then we have the exact sequence

$$1 \to H^0(A) \to H^0(B) \to M\mathcal{I}(A, B) \to 1$$

and $M\mathcal{I}(A, B) = \text{Coker}[H^0(A) \to H^0(B)]$ is a free abelian group by Proposition 1.3 of [9].

(4) Since B is a domain, $H^0(A) = H^0(B) \cong \mathbb{Z}$. So $M\mathcal{I}(A, B) = 0$. □

Lemma 4.6. *Let $A \subseteq B$ be a subintegral extension. Then the map L* Pic *$A \to L$* Pic *B is surjective.*

Proof. Since $A \subseteq B$ is subintegral, so are $A[X] \subseteq B[X]$ and $A[X, X^{-1}] \subseteq B[X, X^{-1}]$. Then the maps Pic $A[X] \times$ Pic $A[X^{-1}] \to$ Pic $B[X] \times$ Pic $B[X^{-1}]$ and Pic $A[X, X^{-1}] \to$ Pic $B[X, X^{-1}]$ are surjective by Lemma 3.10(1). Hence we get the result by chasing the following commutative diagram:

$$\begin{array}{ccccccc}
\operatorname{Pic} A[X] \quad \operatorname{Pic} A[X^{-1}] & \longrightarrow & \operatorname{Pic} A[X, X^{-1}] & \longrightarrow & L\operatorname{Pic} A & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Pic} B[X] \quad \operatorname{Pic} B[X^{-1}] & \longrightarrow & \operatorname{Pic} B[X, X^{-1}] & \longrightarrow & L\operatorname{Pic} B & \longrightarrow & 1 \\
\downarrow & & \downarrow & & & & \\
1 & & 1 & & & &
\end{array}.$$

□

Theorem 4.7. *Let $A \subseteq B$ be a ring extension with A Hensel local and B seminormal. Then $\mathcal{MI}(A, B) \cong \mathcal{MI}(A, {}^{+}\!A) \oplus \mathcal{MI}({}^{+}\!A, B)$, where ${}^{+}\!A$ is the subintegral closure of A in B.*

Proof. By Lemma 4.6, L Pic $A \to L$ Pic ${}^{+}\!A$ is surjective. Since A is Hensel local, L Pic $A = 0$ by Theorem 2.5 of [9]. Therefore L Pic ${}^{+}\!A = 0$ and M Pic ${}^{+}\!A = 0$ because ${}^{+}\!A$ is seminormal. Then by the same argument as Proposition 4.5(3), $\mathcal{MI}({}^{+}\!A, B)$ is a free abelian group. Now the result follows from the following exact sequence (Corollary 4.4)

$$1 \to \mathcal{MI}(A, {}^{+}\!A) \to \mathcal{MI}(A, B) \to \mathcal{MI}({}^{+}\!A, B) \to 1.$$

□

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References

[1] Asanuma T, Pic $R[X, X^{-1}]$ for R a one-dimensional reduced noetherian ring, *J. Pure Appl. Algebra* **71** (1991) 111–128

[2] Ischebeck F, Subintegral ring extensions and some K-theoretical functors, *J. Algebra* **121** (1989) 323–338

[3] Onoda N and Yoshida K, Remarks on quasinormal rings, *J. Pure Appl. Algebra* **33** (1984) 59–67

[4] Roberts L G and Singh B, Subintegrality, invertible modules and the Picard group, *Compositio Math.* **85** (1993) 249–279

[5] Sadhu V and Singh B, Subintegrality, invertible modules and polynomial extensions, *J. Algebra* **393** (2013) 16–23

[6] Singh B, The Picard group and subintegrality in positive characteristic, *Compositio Math.* **95** (1995) 309–321

[7] Swan R G, On seminormality, *J. Algebra* **67** (1980) 210–229
[8] Vitulli M A, Weak normality and seminormality. Commutative algebra-Noetherian and non-Noetherian perspectives (2011) (New York: Springer) pp. 441–480
[9] Weibel C, Pic is a contracted functor, *Invent Math* **103** (1991) 351–377
[10] Weibel C, The K-Book: An Introduction to Algebraic K-Theory, Graduate Studies in Math. vol. 145, AMS (2013)

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