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Strongly minimal triangulations of $(S^3 \times S^1)^{\#3}$ and $(S^3 \times\!\!\!\!\_\; S^1)^{\#3}$

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Abstract. A triangulated d-manifold K, satisfies the inequality $\binom{f_0(K)-d-1}{2} \geq \binom{d+2}{2}\beta_1(K;\mathbb{Z}_2)$ for $d \geq 3$. The triangulated d-manifolds that meet the bound with equality are called *tight neighbourly*. In this paper, we present tight neighbourly triangulations of 4-manifolds on 15 vertices with $\mathbb{Z}_3$ as an automorphism group. One such example was constructed by Bagchi and Datta (*Discrete Math.* **311** (2011) 986–995). We show that there are exactly 12 such triangulations up to isomorphism, 10 of which are orientable.

Keywords. Stacked sphere; tight neighbourly triangulation; minimal triangulation.

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1. Introduction

Minimal triangulations play an important role in combinatorial topology. In this regard, lower bounds on the number of vertices needed in a triangulation of a topological space, in terms of its topological invariants, are particulary important. For compact surfaces, Heawood's inequality is one such lower bound, which states that a surface with Euler characteristic χ, requires at least $\lceil \frac{1}{2}(7+\sqrt{49-24\chi})\rceil$ vertices for its triangulation. An analogue for higher dimensions was proved by Novik and Swartz in [10]. They prove that a triangulation of any manifold K, of dimension $d \geq 3$, satisfies

$$\binom{f_0(K)-d-1}{2} \geq \binom{d+2}{2}\beta_1(K;\mathbb{Z}_2). \tag{1}$$

The triangulations which satisfy (1) with equality were called *tight neighbourly* by Lutz *et al.* in [9]. Equation (1) implies that tight neighbourly triangulations, when they exist, are vertex minimal. In fact, for $d \geq 4$, they are *strongly minimal* in the sense that they have component-wise minimum face vector among all triangulations of a given manifold. This follows from Theorem 3.12 in [2]. A trivial example of a tight neighbourly triangulation is the $(d+2)$-vertex sphere S^d_{d+2} for $d \geq 3$. In 1986, Kühnel [8] constructed $2d+3$ vertex triangulations of $S^{d-1}\times S^1$ for even d, and of $S^{d-1} \times\!\!\!\!\_\; S^1$ for odd d. Kühnel's triangulations are tight neighbourly with $\beta_1 = 1$. Until recently, very few examples of tight neighbourly triangulations apart from Kühnel's series were known. The triangulation of a 4-manifold constructed by Bagchi and Datta in [1], was a first sporadic example of a tight neighbourly triangulation. Very recently, Datta and Singh [5] have given another infinite family of tight neighbourly triangulations, exploiting some unique combinatorial properties of such triangulations. These techniques were further extended in [11] to show the non-existence of tight neighbourly triangulations for $(S^{d-1} \times S^1)^{\#2}$ and $(S^{d-1} \dot{\times} S^1)^{\#2}$. In this paper, our aim is to employ the recently developed combinatorial criteria in [5] and [11] to enumerate all possible tight neighbourly triangulations of $(S^3 \dot{\times} S^1)^{\#3}$ and $(S^3 \times S^1)^{\#3}$, which have a non-trival $\mathbb{Z}_3$ action as in the Bagchi–Datta example. As a result, we obtain one more tight neighbourly triangulation of $(S^3 \dot{\times} S^1)^{\#3}$ and 10 new tight neighbourly triangulations of $(S^3 \times S^1)^{\#3}$.

2. Preliminaries

2.1 *Triangulations*

All simplicial complexes considered here are finite and abstract. Members of a simplicial complex are called its *faces*. The empty set is a face of every simplicial complex. A simplicial complex is called *pure* if all its maximal faces (called *facets*) have the same dimension. A pure d-dimensional simplicial complex is called a *weak pseudomanifold* without boundary (resp., with boundary), if all its $d-1$ dimensional faces occur in exactly two (resp., at most two) facets. With a weak pseudomanifold X, we associate the graph $\Lambda(X)$, whose vertices are the facets of X, and two facets are adjacent in $\Lambda(X)$ if they intersect in a face of co-dimension one. A weak pseudomanifold X of dimension d is called a d-pseudomanifold, if $\Lambda(X)$ is a connected graph. Triangulations of connected manifolds are naturally pseudomanifolds.

If X is a d-dimensional simplicial complex then, for $0 \leq j \leq d$, the number of its j-faces are denoted by $f_j = f_j(X)$. The vector $(f_0, \ldots, f_d)$ is called the *face vector* of X. For simplicial complexes X and Y, we define the *join* of X and Y, denoted by $X * Y$ as

$$X * Y := \{\alpha \sqcup \beta : \alpha \in X, \beta \in Y\}.$$

(Here $\sqcup$ denotes the disjoint union.) If X and Y are pseudomanifolds of dimension m and n respectively then, $X * Y$ is a pseudomanifold of dimension $m + n + 1$. When X consists of a single vertex x, we denote the join as $x * Y$, and call it the *cone* over Y.

For a vertex $x \in X$, we define the subcomplex $\mathrm{lk}_X(x)$, called the *link* of x in X as

$$\mathrm{lk}_X(x) := \{\alpha \in X : \alpha \cup \{x\} \in X, x \notin \alpha\}.$$

The cone $x * \mathrm{lk}_X(x)$ is called the *star* of x in X and is denoted by $\mathrm{st}_X(x)$. By the k-skeleton of a simplicial complex X, denoted by $\mathrm{skel}_k(X)$, we shall mean the subcomplex of X consisting of faces of dimension at most k. The 1-skeleton of a simplicial complex is said to be its *edge graph*. When the edge graph is a complete graph on the vertex set, we will call the complex to be *neighbourly*.

2.2 *Walkup's class $\mathcal{K}(d)$*

A very natural class of triangulations of a topological ball is the class of *stacked* balls. The *standard d-ball* is the pure d-dimensional simplicial complex with one facet. A simplicial complex X is called a *stacked d-ball* if it is obtained from the standard d-ball by successively pasting d-simplices along $(d-1)$-faces (cf. section 2 of [5]). A *stacked d*-sphere is defined as the boundary of a stacked $(d+1)$-ball. Clearly a stacked d-ball and a stacked d-sphere triangulate the d-ball and the d-sphere, respectively. The following result from [5], gives a combinatorial characterization of a stacked ball.

PROPOSITION 2.1

Let X be a pure d-dimensional simplicial complex.

(*a*) *If $\Lambda(X)$ is a tree then $f_0(X) \leq f_d(X) + d$.*
(*b*) *$\Lambda(X)$ is a tree and $f_0(X) = f_d(X) + d$ if and only if X is a stacked d-ball.*

In [12], Walkup introduced the class $\mathcal{K}(d)$ of simplicial complexes, all whose vertex-links are *stacked spheres*. Clearly members of $\mathcal{K}(d)$ are triangulated d-manifolds. The following result of Novik and Swartz shows that for dimensions four and above, tight neighbourly triangulations lie within the class $\mathcal{K}(d)$.

PROPOSITION 2.2 [10]

For $d \geq 4$, if M is tight neighbourly, then M is a neighbourly member of $\mathcal{K}(d)$.

Further, the following result by Kalai specifies the topological space determined by members of $\mathcal{K}(d)$.

PROPOSITION 2.3 [7]

Let $X \in \mathcal{K}(d)$ and let $\beta_1 = \beta_1(X; \mathbb{Z}_2)$. If $d \geq 4$, then X triangulates $(S^{d-1} \times S^1)^{\#\beta_1}$ if it is orientable, and it triangulates $(S^{d-1} \underline{\times} S^1)^{\#\beta_1}$ if it is non-orientable.

Analogous to the class $\mathcal{K}(d)$, we define the class $\bar{\mathcal{K}}(d)$ of triangulated manifolds, where the link of each vertex is a stacked $(d-1)$-ball. We will use the notation $\mathcal{K}^*(d)$ and $\bar{\mathcal{K}}^*(d)$ to denote neighbourly members of class $\mathcal{K}(d)$ and $\bar{\mathcal{K}}(d)$ respectively. From Remark 2.21 in [3], we have the following correspondence.

PROPOSITION 2.4 [3]

If $d \geq 4$, then $M \mapsto \partial M$ is a bijection from $\bar{\mathcal{K}}(d+1)$ to $\mathcal{K}(d)$.

Proposition 2.4 immediately suggests that one can look for members of $\mathcal{K}(d)$ as boundaries of members of $\bar{\mathcal{K}}(d+1)$. In particular, we can obtain members of $\mathcal{K}(4)$ as boundaries of members of $\bar{\mathcal{K}}(5)$. This is exactly the approach we take in this paper. Another easy consequence of the above correspondence is the following:

COROLLARY 2.5

For $d \geq 4$, let $M, N \in \bar{\mathcal{K}}(d+1)$. Then $\varphi : V(M) \to V(N)$ is an isomorphism from M to N, if and only if it is an isomorphism from ∂M to ∂N.

In particular, for $M = N$, we get $\mathrm{Aut}(M) = \mathrm{Aut}(\partial M)$ for $M \in \bar{\mathcal{K}}(d)$.

3. Results

Example 3.1. Let $V = \bigcup_{i=1}^{5}\{a_i, b_i, c_i\}$ be a 15-element set and let $\varphi : V \to V$ be the permutation defined by $\varphi := (a_1, b_1, c_1)(a_2, b_2, c_2)(a_3, b_3, c_3)(a_4, b_4, c_4)(a_5, b_5, c_5)$. For $1 \le i \le 12$, let N_i denote the simplicial complex with vertex set V and the set of facets

$$\{z_i\} \cup \{u_{i,1}, \ldots, u_{i,8}\} \cup \{v_{i,1}, \ldots, v_{i,8}\} \cup \{w_{i,1}, \ldots, w_{i,8}\}, \tag{2}$$

where $z_i = a_1b_1c_1a_2b_2c_2$, $v_{i,k} = \varphi(u_{i,k})$ and $w_{i,k} = \varphi^2(u_{i,k})$ for $1 \le i \le 12$, $1 \le k \le 8$. The facets $u_{i,k}$, $1 \le i \le 12$, $1 \le k \le 8$ are given in Table 1.

The following are the main results of this paper:

Theorem 3.2. *Let $N \in \bar{\mathcal{K}}^*(5)$ with $f_0(N) = 15$ and $\mathrm{Aut}(N) \supseteq \mathbb{Z}_3$. Then $N \cong N_i$ for some $i \in \{1, 2, \ldots, 12\}$.*

Table 1. Facets of N_i, $1 \le i \le 12$ modulo automorphism φ.

$u_{1,1} = a_1b_1c_1a_2b_2a_3$, $u_{1,2} = a_1b_1a_2b_2a_3a_4$, $u_{1,3} = a_1a_2b_2a_3a_4a_5$, $u_{1,4} = a_1a_2a_3a_4b_4a_5$,
$u_{1,5} = a_1a_3a_4b_4a_5b_5$, $u_{1,6} = a_3b_3a_4b_4a_5b_5$, $u_{1,7} = c_2a_3b_3b_4a_5b_5$, $u_{1,8} = b_1c_2b_3b_4a_5b_5$;

$u_{2,1} = a_1b_1c_1a_2b_2a_3$, $u_{2,2} = a_1b_1a_2b_2a_3a_4$, $u_{2,3} = a_1a_2b_2a_3a_4a_5$, $u_{2,4} = a_1a_2a_3a_4b_4a_5$,
$u_{2,5} = a_1a_2a_3b_4a_5b_5$, $u_{2,6} = a_2a_3b_3b_4a_5b_5$, $u_{2,7} = a_3b_3b_4c_4a_5b_5$, $u_{2,8} = b_1b_3b_4c_4a_5b_5$;

$u_{3,1} = a_1b_1c_1a_2b_2a_3$, $u_{3,2} = a_1b_1a_2b_2a_3a_4$, $u_{3,3} = a_1b_1a_2a_3a_4a_5$, $u_{3,4} = a_1a_2a_3a_4b_4a_5$,
$u_{3,5} = a_1a_2a_3b_4a_5b_5$, $u_{3,6} = a_2a_3b_3b_4a_5b_5$, $u_{3,7} = a_3b_3b_4c_4a_5b_5$, $u_{3,8} = b_2b_3b_4c_4a_5b_5$;

$u_{4,1} = a_1b_1c_1a_2b_2a_3$, $u_{4,2} = a_1b_1a_2b_2a_3a_4$, $u_{4,3} = a_1b_1a_2a_3a_4a_5$, $u_{4,4} = b_1a_2a_3a_4a_5c_5$,
$u_{4,5} = a_2a_3b_3a_4a_5c_5$, $u_{4,6} = a_2a_3b_3a_4b_4a_5$, $u_{4,7} = a_2b_3a_4b_4a_5b_5$, $u_{4,8} = c_1b_3a_4b_4a_5b_5$;

$u_{5,1} = a_1b_1c_1a_2b_2a_3$, $u_{5,2} = a_1b_1c_1a_2a_3a_4$, $u_{5,3} = a_1b_1a_2a_3a_4a_5$, $u_{5,4} = a_1a_2a_3a_4a_5b_5$,
$u_{5,5} = a_2a_3a_4b_4a_5b_5$, $u_{5,6} = a_2a_3b_3b_4a_5b_5$, $u_{5,7} = a_2a_3b_3b_4c_4b_5$, $u_{5,8} = a_2b_3b_4c_4b_5c_5$;

$u_{6,1} = a_1b_1c_1a_2b_2a_3$, $u_{6,2} = a_1b_1c_1a_2a_3a_4$, $u_{6,3} = a_1b_1a_2a_3a_4a_5$, $u_{6,4} = a_1a_2a_3a_4a_5b_5$,
$u_{6,5} = a_2a_3a_4b_4a_5b_5$, $u_{6,6} = a_2a_3b_3a_4b_4b_5$, $u_{6,7} = a_2a_3b_3b_4b_5c_5$, $u_{6,8} = a_2b_3b_4c_4b_5c_5$;

$u_{7,1} = a_1b_1c_1a_2b_2a_3$, $u_{7,2} = a_1b_1c_1a_2a_3a_4$, $u_{7,3} = a_1b_1a_2a_3a_4a_5$, $u_{7,4} = b_1a_2a_3a_4a_5c_5$,
$u_{7,5} = a_2a_3a_4b_4a_5c_5$, $u_{7,6} = a_2a_3b_3b_4a_5c_5$, $u_{7,7} = a_2a_3b_3b_4c_4a_5$, $u_{7,8} = a_2b_3b_4c_4a_5b_5$;

$u_{8,1} = a_1b_1c_1a_2b_2a_3$, $u_{8,2} = a_1b_1c_1a_2a_3a_4$, $u_{8,3} = a_1b_1a_2a_3a_4a_5$, $u_{8,4} = b_1a_2a_3a_4a_5c_5$,
$u_{8,5} = a_2a_3a_4b_4a_5c_5$, $u_{8,6} = a_2a_3b_3a_4b_4a_5$, $u_{8,7} = a_2a_3b_3b_4a_5b_5$, $u_{8,8} = a_2b_3b_4c_4a_5b_5$;

$u_{9,1} = a_1b_1c_1a_2b_2a_3$, $u_{9,2} = a_1b_1c_1a_2a_3a_4$, $u_{9,3} = a_1b_1a_2a_3a_4a_5$, $u_{9,4} = a_1a_2a_3a_4a_5b_5$,
$u_{9,5} = a_2a_3a_4c_4a_5b_5$, $u_{9,6} = a_2a_3b_3a_4c_4b_5$, $u_{9,7} = a_2a_3b_3a_4b_5c_5$, $u_{9,8} = a_2b_3a_4b_4b_5c_5$;

$u_{10,1} = a_1b_1c_1a_2b_2a_3$, $u_{10,2} = a_1b_1c_1a_2a_3a_4$, $u_{10,3} = a_1b_1a_2a_3a_4a_5$, $u_{10,4} = b_1a_2a_3a_4a_5c_5$,
$u_{10,5} = a_2a_3a_4c_4a_5c_5$, $u_{10,6} = a_2a_3b_3a_4c_4a_5$, $u_{10,7} = a_2a_3b_3a_4a_5b_5$, $u_{10,8} = a_2b_3a_4b_4a_5b_5$;

$u_{11,1} = a_1b_1c_1a_2b_2a_3$, $u_{11,2} = a_1b_1c_1a_2a_3a_4$, $u_{11,3} = a_1b_1a_2a_3a_4a_5$, $u_{11,4} = a_1a_2a_3a_4a_5b_5$,
$u_{11,5} = a_2a_3b_3a_4a_5b_5$, $u_{11,6} = a_2b_3a_4b_4a_5b_5$, $u_{11,7} = b_2b_3a_4b_4a_5b_5$, $u_{11,8} = b_2b_3c_3a_4b_4b_5$;

$u_{12,1} = a_1b_1c_1a_2b_2a_3$, $u_{12,2} = a_1b_1c_1a_2a_3a_4$, $u_{12,3} = a_1b_1a_2a_3a_4a_5$, $u_{12,4} = b_1a_2a_3a_4a_5c_5$,
$u_{12,5} = a_2a_3b_3a_4a_5c_5$, $u_{12,6} = a_2b_3a_4b_4a_5c_5$, $u_{12,7} = b_2b_3a_4b_4a_5c_5$, $u_{12,8} = b_2b_3c_3a_4b_4a_5$;

Theorem 3.3. *Let* $M \in \mathcal{K}^*(4)$ *with* $f_0(M) = 15$ *and* $\mathrm{Aut}(M) \supseteq \mathbb{Z}_3$. *Then* $M \cong \partial N_i$ *for some* $i \in \{1, 2, \ldots, 12\}$.

3.1 *Geometric carrier*

By construction, the complexes ∂N_i, $1 \leq i \leq 12$ are neighbourly members of $\mathcal{K}(4)$ and hence by Kalai's result (Proposition 2.3), they triangulate $(S^3 \times S^1)^{\#3}$ or $(S^{d-1} \underline{\times} S^1)^{\#3}$. Using a combinatorial topology software such as `simpcomp` [6], one can check that complexes ∂N_1 and ∂N_2 are non-orientable, while the complexes ∂N_i for $i \in \{3, 4, \ldots, 12\}$ are orientable. Thus, ∂N_i triangulates $(S^{d-1} \underline{\times} S^1)^{\#3}$ for $i = 1, 2$ and triangulates $(S^3 \times S^1)^{\#3}$ for $3 \leq i \leq 12$. We also point out that the example N_1 obtained here is isomorphic to the triangulation N^5_{15} obtained by Bagchi and Datta in [1]. Consequently, the triangulation M^4_{15} of $(S^{d-1} \underline{\times} S^1)^{\#3}$ in [1] is isomorphic to ∂N_1.

4. Overview

In this section, we give a broad outline of the enumeration strategy. By Proposition 2.4 and Corollary 2.5, to obtain neighbourly members of $\mathcal{K}(4)$, with $\mathbb{Z}_3$ automorphism, we can instead look for neighbourly members of $\bar{\mathcal{K}}(5)$ with $\mathbb{Z}_3$ automorphism. This has the advantage that all vertex-links are stacked balls, and by Proposition 2.1, we have a succint combinatorial description for their dual graphs; that they are trees. Moreover, the dual graph of the link of a vertex x in a pseudomanifold X, is isomorphic to the induced subgraph $\Lambda(X)[V_x]$, where V_x is the set of facets containing x. For $M \in \bar{\mathcal{K}}(d)$, let T_x denote the subtree of $\Lambda(M)$ induced by facets containing x. Then from [11], we have the following:

PROPOSITION 4.1

For $M \in \bar{\mathcal{K}}^*(d)$, *let* T_x *for* $x \in V(M)$ *be as defined above. Then*

(a) $\Lambda(M)$ *is a two connected graph.*
(b) $\Lambda(M)$ *contains* $n(n-d)/(d+1)$ *vertices and* $n(n-d-1)/d$ *edges where* $n = f_0(M)$.
(c) T_x *contains* $n - d$ *vertices for each* $x \in V(M)$.

In [11], a set of facets S of $M \in \bar{\mathcal{K}}^*(d)$ was defined to be *critical* in M if each of the connected components of $\Lambda(M) - S$ contained fewer than $f_0(M) - d$ vertices. The following observations were also made there.

PROPOSITION 4.2

Let S *be a critical set of facets of* $M \in \bar{\mathcal{K}}^*(d)$. *Then the facets in* S *together contain all the vertices of* M.

PROPOSITION 4.3

Let $M \in \bar{\mathcal{K}}^*(d)$ *with* $f_0(M) > 2d + 1$. *Then the set of facets of degree three or more in* $\Lambda(M)$ *together contain all the vertices of* M.

Identifying critical sets of facets helps us reduce the possibilities for members of $\bar{\mathcal{K}}^*(5)$. We already know from Proposition 4.1, that the dual graph $\Lambda(M)$ for $M \in \bar{\mathcal{K}}^*(d)$ is two connected. When working with complexes with non-trivial automorphism groups, we can further narrow down the admissible dual graphs, due to the following observation.

PROPOSITION 4.4 [5]

Let $M \in \bar{\mathcal{K}}(d)$, *then* $\mathrm{Aut}(M)$ *is a subgroup of* $\mathrm{Aut}(\Lambda(M))$.

We summarize the above observations as follows:

PROPOSITION 4.5

Let $M \in \bar{\mathcal{K}}^*(5)$, *with* $f_0(M) = 15$ *and* $\mathrm{Aut}(M) \subseteq \mathbb{Z}_3$. *Then we have*

(a) $\Lambda(M)$ *is a two connected graph*,
(b) $\Lambda(M)$ *contains* 25 *vertices and* 27 *edges*,
(c) $\mathbb{Z}_3$ *is a subgroup of* $\mathrm{Aut}(\Lambda(M))$,
(d) *for each vertex* x *of* M, x *appears in* 10 *facets of* M, *which induce a tree on* $\Lambda(M)$,
(e) *at least three vertices of* $\Lambda(M)$ *are of degree* ≥ 3.

The program we carry out in the next section, leading to the classification is the following. First we classify all graphs satisfying the conclusion of Proposition 4.5. Then for each of these graphs, we consider all possible members of $\bar{\mathcal{K}}^*(5)$ which will have the graph as their dual graph.

5. Classification

Example 5.1. For $r, s \geq 1$, let $G_{r,s}$ be the graph on $3r + 3s - 2$ vertices, with vertex set $V = \{z_0\} \cup (\bigcup_{i=1}^{r+s-1} \{u_i, v_i, w_i\})$ and consisting of six edge disjoint paths, namely (see figure 1(a)),

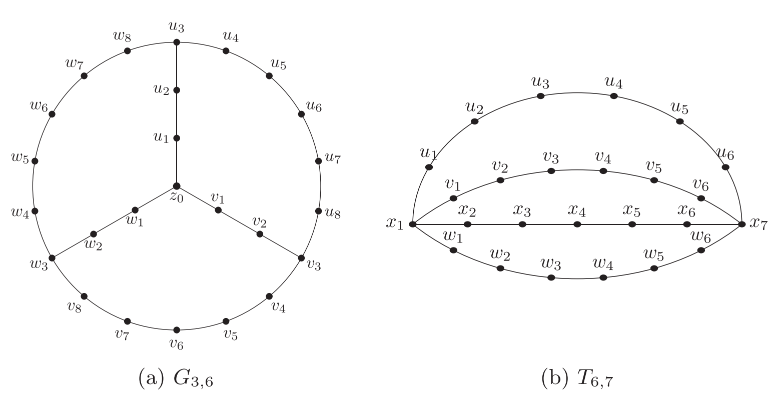


Figure 1. Graphs $G_{3,6}$ **(a)** and $T_{6,7}$ **(b)**.

$p_{zu} := z_0 u_1 \cdots u_r$, $p_{zv} := z_0 v_1 \cdots v_r$, $p_{zw} := z_0 w_1 \cdots w_r$,
$p_{uv} := u_r u_{r+1} \cdots u_{r+s-1} v_r$, $p_{vw} := v_r v_{r+1} \cdots v_{r+s-1} w_r$, $p_{wu} := w_r w_{r+1} \cdots w_{r+s-1} u_r$.

Example 5.2. For $r, s \geq 1$, let $T_{r,s}$ be the graph on $3r + s$ vertices with vertex set $V = \{x_1, \ldots, x_s\} \cup (\bigcup_{i=1}^{r} \{u_i, v_i, w_i\})$ and consisting of four edge disjoint paths, namely (see figure 1(b)),

$$p_0 := x_1 x_1 \cdots x_s, \quad p_1 := x_1 u_1 \cdots u_r x_s, \quad p_2 := x_1 v_1 \cdots v_r x_s, \quad p_3 := x_1 w_1 \cdots w_r x_s.$$

Lemma 5.3. *Let G be a* 2*-connected graph on* 25 *vertices with* 27 *edges. If* $\mathrm{Aut}(G) \supseteq \mathbb{Z}_3$, *then* $G \cong G_{r,s}$ *for some* $r, s > 0$ *with* $r + s = 9$ *or* $G \cong T_{r,s}$ *for some* $r, s > 0$ *with* $3r + s = 25$.

We will defer the proof of Lemma 5.3 to Appendix A. For $M \in \bar{\mathcal{K}}^*(5)$ with $f_0(M) = 15$, by Proposition 4.3, we see that the set of facets with degree three or more in $\Lambda(M)$ together must contain all the vertices of M. But the graphs $T_{r,s}$ contain only two vertices of degree three or more, and hence can contain at most $2 \times 6 = 12$ vertices, which cannot cover all vertices of M, as $f_0(M) = 15$. Thus, we have a further constraint on the graph $\Lambda(M)$. In particular, from Proposition 4.4 and Lemma 5.3 we have as follows:

Lemma 5.4. *Let* $M \in \bar{\mathcal{K}}^*(5)$ *with* $f_0(M) = 15$ *and* $\mathrm{Aut}(M) \supseteq \mathbb{Z}_3$. *Then* $\Lambda(M) \cong G_{r,s}$ *for some* $r, s > 0$ *with* $r + s = 9$.

The following result from [11] further restricts the structure of $\Lambda(M)$ for $M \in \bar{\mathcal{K}}^*(d)$.

PROPOSITION 5.5

Let $M \in \bar{\mathcal{K}}^*(d)$ *with* $f_0(M) > 2d + 1$. *Let* $u_0 u_1 \cdots u_r$ *be a path in* $\Lambda(M)$ *where all the internal vertices* u_i, $1 \leq i \leq r - 1$, *have degree two in* $\Lambda(M)$. *Let* x_i *be the unique element in* $u_{i-1} \backslash u_i$ *for* $1 \leq i \leq r$. *Then we have*

(a) $x_1, \ldots, x_r$ *are distinct*,
(b) $x_i \in u_0$ *for* $1 \leq i \leq r$,
(c) $r \leq d + 1$.

If $M \in \bar{\mathcal{K}}^*(5)$ and P is any path in $\Lambda(M)$ all whose internal vertices have degree two, then the length of P is at most 6. Since $G_{r,s}$ contains such paths of length at least $\max(r, s)$, we infer that $\Lambda(M) \cong G_{r,s}$ (Lemma 5.4), where $\max(r, s) \leq 6$. Together with the constraint $r + s = 9$, we have as follows:

Lemma 5.6. *Let* $M \in \bar{\mathcal{K}}^*(5)$ *with* $f_0(M) = 15$. *If* $\mathrm{Aut}(M) \supseteq \mathbb{Z}_3$, *then* $\Lambda(M) \cong G_{r,9-r}$ *for some* $r \in \{3, 4, 5, 6\}$.

Let $M \in \bar{\mathcal{K}}(d)$ and for each $x \in V(M)$, let T_x denote the subtree of $\Lambda(M)$ induced by the facets of M containing x. We note the following:

Lemma 5.7. *For* $M \in \bar{\mathcal{K}}(d)$, *let* T_x, $x \in V(M)$, *be as defined. Then*,

(a) $T_x \neq T_y$ *for* $x \neq y$,
(b) *If* $\sigma \in V(\Lambda(M))$ *is a leaf of some tree* T_x, *then* $d_{\Lambda(M)}(\sigma) < 3$.

Proof. We first prove (a). Suppose $T_x = T_y$ for some $x \neq y$. Since $T_x \neq \Lambda(M)$, and $\Lambda(M)$ is a connected graph, there exists an edge uv in $\Lambda(M)$ such that $u \in V(T_x)$ and $v \notin V(T_x)$. Now since $u \in V(T_x) = V(T_y)$, we have $\{x, y\} \subseteq u$. Since $v \neq T_x, T_y$, we have $\{x, y\} \cap v = \emptyset$. Thus $\{x, y\} \subseteq u \backslash v$, which is a contradiction as uv is an edge in $\Lambda(M)$.

To prove (b), suppose σ is a leaf of the tree T_x and $d_{\Lambda(M)}(\sigma) \geq 3$. Since at most one neighbour of σ is on T_x, we have at least two neighbours of σ, say α and β which are not on T_x. Thus $x \notin \alpha, \beta$. But then $\sigma \backslash \{x\} \subseteq \alpha, \beta$. This is a contradiction as $\sigma \backslash \{x\}$ is a face of co-dimension one which is contained in three facets, namely σ, α and β. This completes the proof. □

Let $M \in \bar{\mathcal{K}}(d)$ and let $\{T_x : x \in V(M)\}$ be the collection of trees as before. Clearly the trees T_x and T_y intersect if and only if $x \in \mathrm{st}_M(y)$. Thus the number of trees that a tree T_x intersects (counting itself), equals the number of vertices in its star. From Proposition 2.1, we have (since $\mathrm{st}_M(x)$ is a stacked d-ball)

$$f_0(\mathrm{st}_M(x)) = f_d(\mathrm{st}_M(x)) + d.$$

However, $f_d(\mathrm{st}_M(x))$ is the number of vertices in the tree T_x, and hence we have

$$\text{Number of trees intersecting } T_x = |V(T_x)| + d. \tag{3}$$

We can however, count the number of trees intersecting T_x in the following way. Designate a fixed vertex $r \in V(T_x)$ to be the *root*. Next we orient each edge uv of T_x as $\overrightarrow{uv}$, where v is the vertex nearer to r (see figure 2). To each oriented edge $\overrightarrow{uv}$ of T_x, we associate a label $l(\overrightarrow{uv}) = y$, where y is the unique element of $u \backslash v$. Now there are $d + 1$ trees that intersect T_x (including itself) at vertex r. For any tree T_y which intersects T_x, but does not contain r, there must be an oriented edge e in T_x such that $l(e) = y$. Thus the number of trees that T_x intersects is at most $(d + 1) + |V(T_x)| - 1 = |V(T_x)| + d$. For the equality to hold, which it should for $M \in \bar{\mathcal{K}}(d)$, all the edge labels must be distinct and different from the vertices in the facet r, which we have proved.

Lemma 5.8. *For $M \in \bar{\mathcal{K}}(d)$ and $x \in V(M)$, let T_x be the oriented tree with root r, as described before. Then the labels on the oriented edges are distinct, and are different from the vertices in the facet r.*

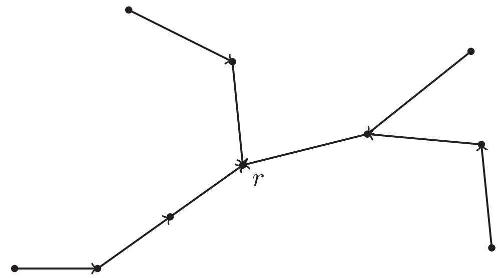


Figure 2. Oriented tree with root r.

Lemma 5.9. *Let* $M \in \bar{\mathcal{K}}(d)$ *and let* $u_0u_1\cdots u_r$ *be a path in* $\Lambda(M)$ *with* $r < d+1$. *Let* x_i *be the unique element in* $u_{i-1}\backslash u_i$, *and* y_i *be the unique element in* $u_i\backslash u_{i-1}$ *for* $1 \le i \le r$. *Then we have*

(a) $\{x_1, x_2, \ldots, x_r\} \subseteq u_0\backslash u_r$, *and*
(b) $\{y_1, y_2, \ldots, y_r\} \subseteq u_r\backslash u_0$.

Proof. Since $r < d+1$, there exists $z \in M$ such that $\{u_0, u_1, \ldots, u_r\} \subseteq V(T_z)$. Part (a) follows by orienting T_z with u_r as the root and applying Lemma 5.8, and Part (b) follows similarly by orienting T_z with u_0 as the root. □

Let $M \in \bar{\mathcal{K}}^*(5)$ with $f_0(M) = 15$ and $\mathrm{Aut}(M) \supseteq \mathbb{Z}_3$. Recall that $\Lambda(M) \cong G_{r,9-r}$ for some $3 \le r \le 6$ (see Example 5.1). We identify the facets of M with vertices of $G_{r,9-r}$. Let φ be an automorphism of M. Then φ induces an automorphism $\bar{\varphi} : u \mapsto \varphi(u)$ of $\Lambda(M)$. The following lemma has been proved in [5].

Lemma 5.10. *Let* $M \in \bar{\mathcal{K}}(d)$. *For* $\varphi \in \mathrm{Aut}(M)$, *let* $\bar{\varphi}$ *be an induced automorphism of* $\Lambda(M)$. *Then* $\varphi \mapsto \bar{\varphi}$ *is an injective homomorphism from* $\mathrm{Aut}(M)$ *to* $\mathrm{Aut}(\Lambda(M))$.

Now let φ be an order three automorphism of M. Then, it induces an order three automorphism $\bar{\varphi}$ of $\Lambda(M) \cong G_{r,9-r}$.Without loss of generality, we may assume φ such that $\bar{\varphi} = \prod_{i=1}^{8}(u_i, v_i, w_i)$. We show that φ does not have any fixed points.

Lemma 5.11. *Let* $M \in \bar{\mathcal{K}}^*(5)$ *with* $f_0(M) = 15$ *and* $\mathrm{Aut}(M) \supseteq \mathbb{Z}_3$. *If* φ *is an order three automorphism of* M, *then* φ *has no fixed points.*

Proof. By Lemma 5.6, $\Lambda(M) \cong G_{r,9-r}$ for some $3 \le r \le 6$. We identify facets of M with vertices of $G_{r,9-r}$. Without loss of generality, assume φ induces the automorphism $\bar{\varphi} = \prod_{i=1}^{8}(u_i, v_i, w_i)$ of $G_{r,9-r}$. Now the orbits of φ are either singleton or contain 3 elements. Let x be a fixed point of φ. Let V_x be the facets of M containing x. By definition of $\bar{\varphi}$, we have $\bar{\varphi}(V_x) = V_x$. Thus V_x is union of orbits of $\bar{\varphi} = \prod_{i=1}^{8}(u_i, v_i, w_i)$. Since $|V_x| = 10$ and each orbit of $\bar{\varphi}$ has cardinality three or one, V_x must contain a fixed point of $\bar{\varphi}$. Since z_0 is the only fixed point of $\bar{\varphi}$, we conclude $z_0 \in V_x$. Also, observe that we must have

$$|V_x \cap \{u_1, \ldots, u_8\}| = |V_x \cap \{v_1, \ldots, v_8\}| = |V_x \cap \{w_1, \ldots, w_8\}|.$$

Thus we must have $|V_x \cap \{u_1, \ldots, u_8\}| = 3$. Since V_x induces a connected subgraph of $G_{r,9-r}$ and $r \ge 3$, we have $V_x \cap \{u_1, \ldots, u_8\} = \{u_1, u_2, u_3\}$. Similarly, $V_x \cap \{v_1, \ldots, v_8\} = \{v_1, v_2, v_3\}$ and $V_x \cap \{w_1, \ldots, w_8\} = \{w_1, w_2, w_3\}$. Thus $V_x = \{z_0\} \cup (\cup_{i=1}^{3}\{u_i, v_i, w_i\})$. However if φ has one fixed point, it has at least three fixed points. Let $y \neq x$ be another fixed point of φ. Then we will have $V_y = \{z_0\} \cup (\cup_{i=1}^{3}\{u_i, v_i, w_i\}) = V_x$, and thus $T_x = T_y$, which contradicts Lemma 5.7. This proves the lemma. □

DEFINITION 5.12 (Class $\mathcal{C}$ of complexes)

Let $V := \bigcup_{i=1}^{5}\{a_i, b_i, c_i\}$ be a 15-vertex set. Let V be ordered as $a_1 < b_1 < c_1 < a_2 < b_2 < \cdots < b_5 < c_5$. Let $\Phi := \prod_{i=1}^{5}(a_i, b_i, c_i)$. We will denote the orbit $\{a_i, b_i, c_i\}$ of Φ as Φ_i. Let us define the class $\mathcal{C}$ of complexes by

$$\mathcal{C} := \{M \in \bar{\mathcal{K}}^*(5) : V(M) = V, \Phi \in \mathrm{Aut}(M)\}. \tag{4}$$

We notice that any $M \in \bar{\mathcal{K}}^*(5)$ with $f_0(M) = 15$ and $\mathrm{Aut}(M) \supseteq \mathbb{Z}_3$ is isomorphic to a member of the collection $\mathcal{C}$. Therefore it suffices to enumerate $\mathcal{C}$ up to isomorphism. Before we proceed with enumeration, we consider an efficient string representation of a member of the class $\mathcal{C}$.

If $M \in \mathcal{C}$, then we know that $\Lambda(M) \cong G_{r,9-r}$ for some $3 \leq r \leq 6$. We identify the facets of M with the vertices of $G_{r,9-r}$. We may assume that Φ induces the automorphism $\prod_{i=1}^{8}(u_i, v_i, w_i)$ of $G_{r,9-r}$. To each facet u of M, we associate a string $[u] = a_1a_2 \ldots a_k$ where $a_1 < a_2 < \cdots < a_k$ are the vertices of u. With the complex M, we associate the string representation, which we denote by $\mathrm{str}(M)$ as

$$\mathrm{str}(M) := [z_0] + [u_1] + [u_2] + \cdots + [u_8]. \tag{5}$$

Here $+$ denotes the concatenation of strings. Note that the above representation uniquely specifies a complex in $\mathcal{C}$ as the remaining facets may be obtained by applying the automorphism Φ. Clearly we could have used the facets $\{z_0, v_1, \ldots, v_8\}$ or $\{z_0, w_1, \ldots, w_8\}$. However, we assume that $\{z_0, u_1, \ldots, u_8\}$ is the one that yields lexicographically least representation, i.e, we assume $[u_1] < \min([v_1], [w_1])$ (in the dictionary order).

5.1 *Lexicographic enumeration*

Throughout the remainder of the paper, let $V, \mathcal{C}$ and Φ be as in Definition 5.12. We order $\mathcal{C}$ with the ordering $M_1 \leq M_2$ for $M_1, M_2 \in \mathcal{C}$ if $\mathrm{str}(M_1) \leq \mathrm{str}(M_2)$, where the ordering on strings is lexicographic. We say that $M \in \mathcal{C}$ is *minimal* if $M \leq N$ for all $N \in \mathcal{C}$ such that N is isomorphic to M. Thus each isomorphism class contains exactly one minimal element. We will look for such minimal members of $\mathcal{C}$.

Lemma 5.13. *Let $M \in \mathcal{C}$. If Γ is a permutation of V such that $\Phi\Gamma = \Gamma\Phi^i$ for some $i \in \{1, 2\}$, then $\Gamma(M) \in \mathcal{C}$.*

Proof. Clearly $\Gamma(M) \in \bar{\mathcal{K}}^*(5)$ for $M \in \bar{\mathcal{K}}^*(5)$. We need to show that Φ is an automorphism of $\Gamma(M)$. Suppose $\Phi\Gamma = \Gamma\Phi^i$, where $i \in \{1, 2\}$. Let u be a facet of $\Gamma(M)$. Then $u = \Gamma(v)$ for some facet v of M. Now $\Phi(u) = \Phi\Gamma(v) = \Gamma\Phi^i(v) = \Gamma(w)$, where $w = \Phi^i(v)$ is a facet of M, and hence $\Gamma(w)$ is a facet of $\Gamma(M)$. This Φ maps facets to facets in $\Gamma(M)$, and hence is an automorphism of $\Gamma(M)$. The lemma follows. □

Lemma 5.13 implies that for a mimimal complex $M \in \mathcal{C}$ and for Γ, a permutation of V satisfying the conditions in the lemma, we must have $M \leq \Gamma(M)$. Next we define some permutations of V, which satisfy the conditions in Lemma 5.13.

- $\pi_i := (a_i, b_i, c_i)$ for $1 \leq i \leq 5$,
- $\pi_{i,j} := (a_i, a_j)(b_i, b_j)(c_i, c_j)$ for $1 \leq i < j \leq 5$,
- $\gamma_{\alpha,\beta} := (\alpha_1, \beta_1)(\alpha_2, \beta_2)(\alpha_3, \beta_3)(\alpha_4, \beta_4)(\alpha_5, \beta_5)$, where $\{\alpha_i, \beta_i\} \subseteq \{a_i, b_i, c_i\}$ for $1 \leq i \leq 5$.

We can think of permutations π_i as shifting the elements cyclically within an orbit. The permuatations $\pi_{i,j}$ 'interchange' the orbits i and j, while the permutations $\gamma_{\alpha,\beta}$ interchange an adjacent pair in each of the orbits. We will need the above permutations in pruning our candidates for minimal element of $\mathcal{C}$.

Lemma 5.14. *Let* $M \in \mathcal{C}$ *be minimal. Then* $[z_0] = a_1 b_1 c_1 a_2 b_2 c_2$.

Proof. Since $\Phi(z_0) = z_0$, z_0 must be a union of Φ-orbits. Since $|z_0| = 6$, it contains exactly two orbits. For M to be minimal, $[z_0]$ should be minimal for M, among all complexes in its isomorphism class. Thus z_0 should be union of orbits Φ_1 and Φ_2, for otherwise using one of the permutations $\pi_{i,j}$ we can get the complex $\pi_{i,j}(M)$ with lexicographically smaller z_0. Thus $z_0 = \Phi_1 \cup \Phi_2$, or $[z_0] = a_1 b_1 c_1 a_2 b_2 c_2$. □

We introduce a succint representation of complexes in $\mathcal{C}$. Recall that with an oriented edge $\overrightarrow{uv}$ in $\Lambda(M)$, we associated a label $l(\overrightarrow{uv})$ as the unique element of $u \backslash v$. To a complex $M \in \mathcal{C}$, with $\Lambda(M) \cong G_{r,9-r}$, we associate tuples $X = (x_1, x_2, \ldots, x_9)$ and $Y = (y_1, y_2, \ldots, y_9)$, where (see figure 3)

$$x_i = \begin{cases} l(\overrightarrow{z_0 u_1}) & \text{if } i = 1, \\ l(\overrightarrow{u_{i-1} u_i}) & \text{if } 2 \le i \le 8, \\ l(\overrightarrow{u_8 v_r}) & \text{if } i = 9, \end{cases} \quad \text{and} \quad y_i = \begin{cases} l(\overrightarrow{u_1 z_0}) & \text{if } i = 1, \\ l(\overrightarrow{u_i u_{i-1}}) & \text{if } 2 \le i \le 8, \\ l(\overrightarrow{v_r u_8}) & \text{if } i = 9. \end{cases} \tag{6}$$

Clearly, the triple (z_0, X, Y) uniquely specifies a complex in $\mathcal{C}$. Further, by Lemma 5.14, for minimal complexes, z_0 is constant, and hence the pair (X, Y) uniquely specifies a minimal complex in $\mathcal{C}$. We will frequently make use of the following lemma. In the remainder of the paper, we shall always assume that for a minimal complex $M \in \mathcal{C}$, $X = (x_1, \ldots, x_9)$ and $Y = (y_1, \ldots, y_9)$ are the tuples as defined in (6).

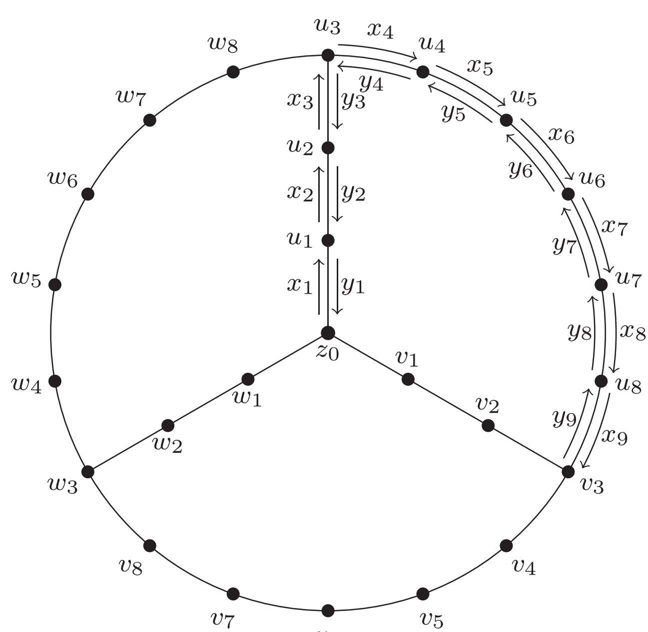


Figure 3. Succint representation of a minimal complex in $\mathcal{C}$.

Lemma 5.15. *Let $M \in \mathcal{C}$ be minimal. For $i \in \{3, 4, 5\}$, define $m_i = \min\{j : y_j \in \Phi_i\}$. For $i \in \{1, 2\}$, define $n_i = \min\{j : x_j \in \Phi_i\}$. Then we have*

(a) $m_3 < m_4 < m_5$,
(b) $n_2 < n_1$,
(c) $y_{m_i} = a_i$ *for* $i \in \{3, 4, 5\}$,
(d) $x_{n_i} = c_i$ *for* $i \in \{1, 2\}$.

Proof. To prove part (a), we prove $m_3 < m_4$ and $m_4 < m_5$. Assume that $m_3 > m_4$. Let $\pi = \pi_{3,4}$. Consider the complex $M' = \pi(M)$. Since the set of facets $\{\pi(z_0), \pi(u_1), \ldots, \pi(u_8)\}$ is one of the candidate sets for string representation of M', we have

$$\mathrm{str}(M') \leq [\pi(z_0)] + [\pi(u_1)] + \cdots + [\pi(u_8)].$$

Note that for $i < m_4 < m_3$, we have $\pi(u_i) = u_i$ as none of the u_i's contain elements from Φ_3 or Φ_4. However, for $i = m_4$, we have $u_i = S \cup \{y\}$ where $S \cap \Phi_3 = S \cap \Phi_4 = \emptyset$ and $y \in \Phi_4$. Thus $\pi(u_i) = \pi(S) \cup \{\pi(y)\} = S \cup \{\pi(y)\}$. However, $\pi(y) \in \Phi_3$ as $y \in \Phi_4$, and hence $\pi(y) < y$, or $[\pi(u_i)] < [u_i]$. Thus $\pi(M) < M$, a contradiction to minimality of M. This proves $m_3 < m_4$. Similarly we can show that $m_4 < m_5$.

To prove part (b), assume that $n_1 < n_2$. Let $\pi = \pi_{1,2}$. For $i < n_1 < n_2$, $\Phi_1 \subseteq u_i$ and $\Phi_2 \subseteq u_i$, and hence $\pi(u_i) = u_i$. For $i = n_1$, we have $u_i = S_1 \cup S_2 \cup S_3$, where $S_1 = \Phi_1 \backslash \{x_i\}$, $S_2 = \Phi_2$ and $S_3 \subseteq \Phi_3 \cup \Phi_4 \cup \Phi_5$. Thus, $\pi(u_i) = \pi(S_1) \cup \pi(S_2) \cup \pi(S_3)$. But $\pi(S_2) = \Phi_1$ as $S_2 = \Phi_2$. Thus $[\pi(u_i)]$ contains first three positions from Φ_1, whereas $[u_i]$ contains first two positions from Φ_1. Hence $\pi(u_i) < u_i$ for $i = n_1$, and hence $\pi(M) < M$, a contradiction to minimality of M. This proves $n_2 < n_1$.

Parts (c) and (d) may be proved using the permutations $\pi_i = (a_i, b_i, c_i)$. Informally, in the string representation of M, when an orbit element appears for the first time, we can always permute the orbit, so that it is the least element a_i, for the i-th orbit. Similarly, when an orbit element leaves for the first time, we can always permute the elements so that it is the greatest element c_i for the i-th orbit. □

For $i \in \{1, 2, \ldots, 12\}$, let N_i be the simplicial complexes as defined in Example 3.1.

Lemma 5.16. *Let $M \in \mathcal{C}$ be minimal with $\Lambda(M) \cong G_{3,6}$. Then $M \cong N_1$.*

Proof. Let $X = (x_1, x_2, \ldots, x_9)$ and $Y = (y_1, y_2, \ldots, y_9)$ be the tuple associated with M. We claim the following:

(a) $(y_1, y_2, y_3) = (a_3, a_4, a_5)$,
(b) $(x_1, x_2) = (c_2, c_1)$,
(c) $x_3 \in \{b_1, b_2\}$.

Recall that a set S of facets of M is critical in M if each component of $\Lambda(M) - S$ has less than 10 vertices. Thus the set of facets u_3, v_3, w_3 is critical. By Proposition 4.2, $V = u_3 \cup v_3 \cup w_3$. Since $v_3 = \Phi(u_3)$ and $w_3 = \Phi^2(u_3)$, we conclude that u_3 intersects each Φ-orbit. Since $z_0 = \Phi_1 \cup \Phi_2$ and $u_3 \backslash z_0 = \{y_1, y_2, y_3\}$, we conclude that y_1, y_2 and y_3 each come from distinct orbits among Φ_3, Φ_4 and Φ_5. By parts (a) and (c) of Lemma 5.15, we must have $(y_1, y_2, y_3) = (a_3, a_4, a_5)$. This proves (a).

By Lemma 5.9, $\{x_1, x_2, x_3\} \subseteq z_0 \subseteq \Phi_1 \cup \Phi_2$. Thus, by parts (b) and (d) of Lemma 5.15, we have $x_1 \in \Phi_2$ and further that $x_1 = c_2$. We claim that $x_2 = l(\overrightarrow{u_1u_2}) \in \Phi_1$. If possible, let $l(\overrightarrow{u_1u_2}) \in \Phi_2$. Then $x_3 = l(\overrightarrow{u_2u_3}) \in \Phi_1$, otherwise u_3 will not intersect Φ_2. Let us count the vertices in T_{x_3}. Clearly T_{x_3} contains z_0, u_1, u_2. Further notice that none of the edges oriented away from z_0 on the paths $z_0 \cdots v_3$ and $z_0 \cdots w_3$ have the label x_3. Thus, $\{v_1, v_2, v_3, w_1, w_2, w_3\} \subseteq V(T_{x_3})$. We have already accounted for $6 + 3 = 9$ vertices of T_{x_3}. By part (b) of Lemma 5.7, v_3, w_3 cannot be leaves of T_{x_3}, therefore we must have at least two more vertices in T_{x_3}. Thus, T_{x_3} contains at least 11 vertices, a contradiction. Therefore, $x_2 \in \Phi_1$, and hence by part (d) of Lemma 5.15, $x_2 = c_1$. This proves (b).

We first show that if $x_3 \in \Phi_1$, then $x_3 = b_1$ and if $x_3 \in \Phi_2$, then $x_3 = b_2$. Assume that $x_3 \in \Phi_1$ and $x_3 \neq b_1$. Then $x_3 = a_1$. Let $\pi := (a_1, c_1)(a_2, c_2)(a_3, b_3)(a_4, b_4)(a_5, b_5)$. We will show that $\pi(M) < M$. Let $M' = \pi(M)$. For clarity, we will denote the facet of M corresponding to vertex u of the dual graph $G_{3,6}$ as $M(u)$ and similarly facet of M' corresponding to vertex u of $G_{3,6}$ as $M'(u)$. We notice that $M'(z_0) = M(z_0) = a_1b_1c_1a_2b_2c_2$. Since $M'(u_1)$ is the lexicographically least neighbour of $M'(z_0)$, we see that $M'(u_1) = \pi(M(v_1))$ (lexicographically least neighbour is the one along which c_2 leaves). It follows that $M'(u_i) = \pi(M(v_i))$ for $1 \leq i \leq 3$. From parts (a) and (b), we have

$$\begin{aligned}
\mathrm{str}(M) &= a_1b_1c_1a_2b_2c_2 + a_1b_1c_1a_2b_2a_3 + a_1b_1a_2b_2a_3a_4 + b_1a_2b_2a_3a_4a_5 + \cdots \\
\mathrm{str}(M') &= [M'(z_0)] + [M'(u_1)] + [M'(u_2)] + [M'(u_3)] + \cdots \\
&= [\pi(M(z_0))] + [\pi(M(v_1))] + [\pi(M(v_2))] + [\pi(M(v_3))] + \cdots \\
&= a_1b_1c_1a_2b_2c_2 + a_1b_1c_1a_2b_2a_3 + a_1b_1a_2b_2a_3a_4 + a_1a_2b_2a_3a_4a_5 + \cdots \\
&< \mathrm{str}(M).
\end{aligned}$$

This contradicts the minimality of M. Hence $x_3 \in \Phi_1$ implies $x_3 = b_1$. Similarly it can be shown that $x_3 \in \Phi_2$ implies $x_3 = b_2$. This proves (c).

Let us deduce the arrangement of the trees T_x for $x \in V$. Because of the automorphism Φ, we only need to know the trees of a_1, a_2, a_3, a_4, a_5. We note that for $x \in \{a_3, a_4, a_5\}$, $x \notin z_0 \cup v_3 \cup w_3$, and hence $T_x \subseteq \Lambda(M) - \{z_0, v_3, w_3\}$. Thus the trees $T_{a_3}, T_{a_4}, T_{a_5}$ are contained in the union of the paths $w_4w_5 \cdots u_3 \cdots u_8$ and $u_3 \cdots u_1$. Since $y_3 = a_5$ leaves along the edge $\overrightarrow{u_3u_2}$, we conclude that T_{a_5} is contained in the arc $w_4 \cdots u_3 \cdots u_8$. As T_{a_5} has 10 vertices, we have following cases:

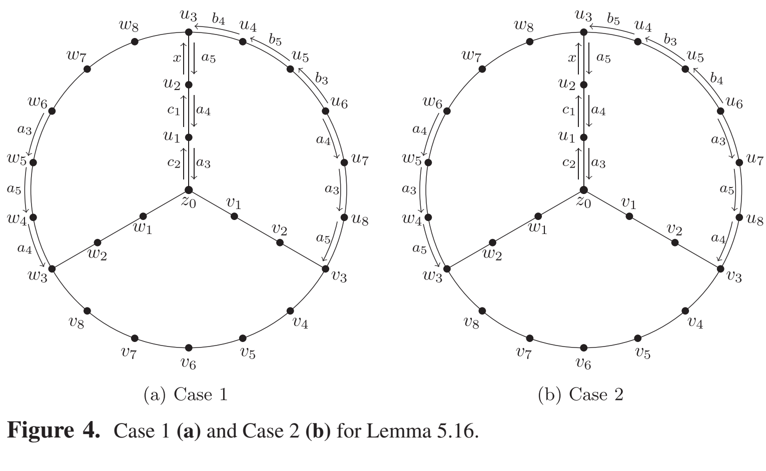


Figure 4. Case 1 **(a)** and Case 2 **(b)** for Lemma 5.16.

Case 1. $T_{a_5} = w_5 \cdots u_3 \cdots u_8$. We immediately have $x_9 = l(\overrightarrow{u_8 v_3}) = a_5$ and $l(\overrightarrow{w_5 w_4}) = a_5$, and hence $y_5 = l(\overrightarrow{u_5 u_4}) = \Phi(l(\overrightarrow{w_5 w_4})) = b_5$ (see figure 4(a)). Now T_{a_4} contains the vertex u_2 from the path $u_3 \cdots u_1$, hence it must induce an arc of 9 vertices on the outer cycle. By Lemma 5.7, it cannot share its end points with T_{a_5}. We see that the only possibility for T_{a_4} is $u_2 u_3 \cup w_4 \cdots u_3 \cdots u_6$. Similarly we have T_{a_3} as $u_1 \cdots u_3 \cup w_6 \cdots u_3 \cdots u_7$. From these, we conclude as before that $x_8 = a_3$, $x_7 = a_4$, $y_4 = b_4$ and $y_6 = b_3$. From Proposition 5.5, we must have $u_3 = \{x_4, \ldots, x_9\}$ and $v_3 = \{y_4, \ldots, y_9\}$. Hence we have $\{x_4, x_5, x_6\} = u_3 \backslash \{a_3, a_4, a_5\}$ and $\{y_7, y_8, y_9\} = v_3 \backslash \{b_3, b_4, b_5\}$. Putting $x_3 = x \in \{b_1, b_2\}$, we have the following constraints:

$$(x_1, x_2, x_3, x_7, x_8, x_9) = (c_2, c_1, x, a_4, a_3, a_5),$$
$$\{x_4, x_5, x_6\} = u_3 \backslash \{a_3, a_4, a_5\}, \tag{7a}$$
$$(y_1, y_2, y_3, y_4, y_5, y_6) = (a_3, a_4, a_5, b_4, b_5, b_3),$$
$$\{y_7, y_8, y_9\} = v_3 \backslash \{b_3, b_4, b_5\}. \tag{7b}$$

The above equation gives 2 choices for x, $3! = 6$ each for (x_4, x_5, x_6) and (y_7, y_8, y_9). We examine the $2 \times 6 \times 6 = 72$ cases on a computer using `simpcomp` [6]. We get the following solution:

$$X_1 = (c_2, c_1, b_1, b_2, a_2, a_1, a_4, a_3, a_5),$$
$$Y_1 = (a_3, a_4, a_5, b_4, b_5, b_3, c_2, b_1, b_2).$$

The pair (X_1, Y_1) yields the complex N_1.

Case 2. $T_{a_5} = w_4 w_5 \cdots u_3 \cdots u_7$. In this case, we have $T_{a_4} = u_2 u_3 \cup w_6 \cdots u_3 \cdots u_8$ and $T_{a_3} = u_1 \cdots u_3 \cup w_5 \cdots u_3 \cdots u_6$ (see figure 4(b)). Analyzing as in Case 1, we have the following constraints:

$$(x_1, x_2, x_3, x_7, x_8, x_9) = (c_2, c_1, x, a_3, a_5, a_4),$$
$$\{x_4, x_5, x_6\} = u_3 \backslash \{a_3, a_4, a_5\}, \tag{8a}$$
$$(y_1, y_2, y_3, y_4, y_5, y_6) = (a_3, a_4, a_5, b_5, b_3, b_4),$$
$$\{y_7, y_8, y_9\} = v_3 \backslash \{b_3, b_4, b_5\}, \tag{8b}$$

where $x_3 = x \in \{b_1, b_2\}$. Examining the 72 possible cases using `simpcomp` [6], we find no member of $\bar{\mathcal{K}}^*(5)$. Thus N_1 is the only minimal element of $\mathcal{C}$ with $\Lambda(M) \cong G_{3,6}$. This proves the lemma. □

Lemma 5.17. *Let $M \in \mathcal{C}$ be minimal with $\Lambda(M) \cong G_{4,5}$. Then $M \cong N_2, N_3$ or N_4.*

Proof. Let $X = (x_1, x_2, \ldots, x_9)$ and $Y = (y_1, y_2, \ldots, y_9)$ be the tuple associated with M. We claim the following:

(a) $(x_1, x_2) = (c_2, c_1)$,
(b) $(y_1, y_2, y_3) = (a_3, a_4, a_5)$,
(c) $(x_3, x_4) \in \{(b_1, a_2), (b_1, b_2), (b_2, a_1), (b_2, b_1)\}$,
(d) $y_4 \in \{b_4, c_4, b_5, c_5\}$.

We start by proving that $u_4 \cup v_4 \cup w_4 = V$, which would imply that u_4 intersects all Φ-orbits. By Proposition 4.3, $z_0 \cup u_4 \cup v_4 \cup w_4 = V$. We show that $z_0 \subseteq u_4 \cup v_4 \cup w_4$.

Suppose not, and let $x \in z_0$ be such that $x \notin u_4 \cup v_4 \cup w_4$. Thus $T_x \subseteq \Lambda(M) - \{u_4, v_4, w_4\}$. Since $|V(T_x)| = 10$, we see that $V(T_x) = \{z_0\} \cup (\bigcup_{i=1}^{3}\{u_i, v_i, w_i\})$. But then there is at most one such $x \in V$. Thus $|u_4 \cup v_4 \cup w_4| \geq 14$. Noticing that $u_4 \cup v_4 \cup w_4$ is union of Φ-orbits, we conclude that $|u_4 \cup v_4 \cup w_4| = 15$, and hence $V = u_4 \cup v_4 \cup w_4$. Thus u_4 intersects all Φ-orbits.

We now prove (a). The claim $x_1 = c_2$ is proved exactly as in Lemma 5.16. We prove $x_2 \in \Phi_1$, and then by Lemma 5.15, part (d) claims that $x_2 = c_1$. Assume that $x_2 \notin \Phi_1$. Then $x_2 \in \Phi_2$. By Lemma 5.9, $\{x_1, x_2, x_3, x_4\} \cap u_4 = \emptyset$. Since u_4 intersects Φ_2, we must have $\Phi_2 \not\subseteq \{x_1, x_2, x_3, x_4\}$. Since $\{x_1, x_2\} \subseteq \Phi_2$, we conclude that $\{x_3, x_4\} \subseteq \Phi_1$. We will show that in this case $|V(T_{x_3})| > 10$. For the purpose of estimating $|V(T_{x_3})|$, assume $(x_3, x_4) = (a_1, b_1)$. Then looking outwards from z_0, a_1 leaves along $\overrightarrow{u_2u_3}$ and $\overrightarrow{w_3w_4}$, and does not leave along the path $z_0 \cdots v_4$. Thus $\{z_0, u_1, u_2, v_1, v_2, v_3, v_4, w_1, w_2, w_3\}$ are vertices in T_{a_1}. Since v_4 cannot be a leaf, there must be at least one more vertex in T_{a_1}, and thus number of vertices exceeds 10, a contradiction. Note that we would arrive at the same estimate for any other choice of (x_3, x_4). Thus $x_2 \in \Phi_1$ and hence $x_2 = c_1$. This proves (a).

To prove (b), we claim that y_3, y_4, y_5 belong to distinct Φ-orbits among Φ_3, Φ_4, Φ_5. Since distance between the pairs (u_3, v_2) and (u_3, w_2) in $\Lambda(M)$ is 5, there exist $x, y \in V$ such that $\{u_3, u_2, u_1, z_0, v_1, v_2\} \subseteq V(T_x)$ and $\{u_3, u_2, u_1, z_0, w_1, w_2\} \subseteq V(T_y)$. We orient the two trees with z_0 as root (edges oriented towards the root). Now if y_i and y_j belong to same Φ-orbit for some $1 \leq i < j \leq 3$, we see that y_j will be repeated as a label leaving v_i or w_i towards z_0. Thus Lemma 5.8 is violated in T_x or T_y. Hence y_3, y_4, y_5 are from distinct orbits, and by Lemma 5.15, we have $(y_3, y_4, y_5) = (a_3, a_4, a_5)$. This proves (b).

To prove (c), we prove that $x_3 = b_1$ if $x_3 \in \Phi_1$ and $x_3 = b_2$ if $x_3 \in \Phi_2$. The proof is exactly the same as of Claim (c) in Lemma 5.16. Together with part (a), and the observation that $\Phi_i \not\subseteq \{x_1, x_2, x_3, x_4\}$ for $i = 1, 2$, we have $(x_3, x_4) \in \{(b_1, a_2), (b_1, b_2), (b_2, a_1), (b_2, b_1)\}$. This proves (c).

To prove (d), it is sufficient to show that $y_4 \notin \Phi_3$. Suppose $y_4 \in \Phi_3$, say $y_4 = b_3$. Then observe that $l(\overrightarrow{w_4w_3}) = a_3$. Since w_4 and u_1 are at a distance 5 in $\Lambda(M)$, there exists $z \in V$ such that $\{w_4, w_3, w_2, w_1, z_0, u_1\} \subseteq V(T_z)$. Orient T_z with z_0 as the root (edges oriented towards the root). Then $l(\overrightarrow{u_1z_0}) = l(\overrightarrow{w_4w_3}) = a_3$, which contradicts Lemma 5.8. A similar contradiction also follows if we assume $y_4 = c_3$. Thus $y_4 \notin \Phi_3$, which proves (d).

Under conditions (a), (b), (c) and (d), we attempt to deduce the arrangement of trees T_x. From (d), we have the following cases:

Case 1. $y_4 \in \{b_4, c_4\}$. Observe that in this case $u_4 \cap \Phi_3 = \{a_3\}$ and $u_4 \cap \Phi_5 = \{a_5\}$. Consequently, $a_3 \notin v_4 \cup w_4$ and $a_5 \notin v_4 \cup w_4$. Thus $T_{a_3}, T_{a_5} \subseteq \Lambda(M) - \{z_0, v_4, w_4\}$. Therefore the trees T_{a_3} and T_{a_5} are confined to the union of the paths $u_1 \cdots u_4$ and the arc $w_5w_6 \cdots u_4 \cdots u_8$. Since a_3 leaves along the edge $\overrightarrow{u_1z_0}$, it contains vertices u_1, u_2, u_3. Hence, T_{a_3} induces an arc with 7 vertices on the outer cycle. Similarly T_{a_5} induces an arc with 9 vertices on the outer cycle. It can be seen that the only possibilities for T_{a_3} and T_{a_5} are (see figure 5),

$$
\begin{aligned}
T_{a_3} &= u_1u_2u_3u_4 \cup w_6w_7w_8u_4u_5u_6u_7, \\
T_{a_5} &= u_3u_4 \cup w_5w_6w_7w_8u_4u_5u_6u_7u_8.
\end{aligned} \tag{9}
$$

From (9), we conclude that $x_8 = a_3$, $x_9 = a_5$, $y_5 = b_5$, $y_6 = b_3$. By Lemma, 5.9, we have $\{x_5, \ldots, x_9\} = u_4 \backslash v_4$ and $\{y_5, \ldots, y_9\} = v_4 \backslash u_4$. Thus $\{x_5, x_6, x_7\} = (u_4 \backslash v_4)$

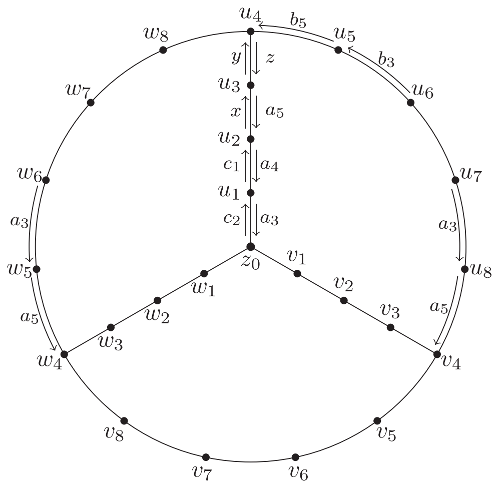


Figure 5. Illustration for Lemma 5.17, Case 1.

$\backslash\{a_3, a_5\}$, $\{y_7, y_8, y_9\} = (v_4 \backslash u_4)\backslash\{b_3, b_5\}$. Putting $(x_3, x_4) = (x, y)$ and $y_4 = z$, we have the following constraints:

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_8, x_9) &= (c_2, c_1, x, y, a_3, a_5),\\ \{x_5, x_6, x_7\} &= (u_4\backslash v_4)\backslash\{a_3, a_5\},\end{aligned} \tag{10a}$$

$$\begin{aligned}(y_1, y_2, y_3, y_4, y_5, y_6) &= (a_3, a_4, a_5, z, b_3, b_5),\\ \{y_7, y_8, y_9\} &= (v_4\backslash u_4)\backslash\{b_3, b_5\}.\end{aligned} \tag{10b}$$

Examining the cases using `simpcomp` [6], we find the following solutions:

$$\begin{aligned}X_1 &= (c_2, c_1, b_1, b_2, a_4, a_1, a_2, a_3, a_5),\\ Y_1 &= (a_3, a_4, a_5, b_4, b_5, b_3, c_4, b_1, b_2),\end{aligned}$$

$$\begin{aligned}X_2 &= (c_2, c_1, b_2, b_1, a_4, a_1, a_2, a_3, a_5),\\ Y_2 &= (a_3, a_4, a_5, b_4, b_5, b_3, c_4, b_2, b_1).\end{aligned}$$

The tuples (X_1, Y_1) and (X_2, Y_2) yield the complexes N_2 and N_3 respectively.

Case 2. $y_4 \in \{b_5, c_5\}$. In this case we see that the trees T_{a_3} and T_{a_4} are contained in the union of arc $\mathcal{A} = w_5 w_6 \cdots u_4 \cdots u_8$ and the path $\mathcal{P} = u_1 u_2 \cdots u_4$. As T_{a_4} contains 3 vertices on $\mathcal{P}$, it must induce a path containing 8 vertices (including u_4) on $\mathcal{A}$. Similarly T_{a_3}, which contains 4 vertices on $\mathcal{P}$ must induce a path containing 7 vertices on $\mathcal{A}$. It can be seen that we have the following solutions for T_{a_3} and T_{a_4}:

$$\begin{aligned}T_{a_3} &= u_1u_2u_3u_4 \cup w_5w_6w_7w_8u_4u_5u_6,\\ T_{a_4} &= u_2u_3u_4 \cup w_6w_7w_8u_4u_5u_6u_7u_8,\\ T_{a_3} &= u_1u_2u_3u_4 \cup w_7w_8u_4u_5u_6u_7u_8,\\ T_{a_4} &= u_2u_3u_4 \cup w_5w_6w_7w_8u_4u_5u_6u_7.\end{aligned} \tag{11}$$

For the first solution in (11) as illustrated in figure 6(a), we have $x_7 = a_3$, $x_9 = a_4$, $y_5 = b_3$ and $y_6 = b_4$. Letting $(x_3, x_4) = (x, y)$ and $y_4 = z$, we have the following constraints:

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_7, x_9) &= (c_2, c_1, x, y, a_3, a_4),\\ \{x_5, x_6, x_8\} &= (u_4\backslash v_4)\backslash\{a_3, a_4\},\end{aligned} \tag{12a}$$

$$
\begin{aligned}
(y_1, y_2, y_3, y_4, y_5, y_6) &= (a_3, a_4, a_5, z, b_3, b_4), \\
\{y_7, y_8, y_9\} &= (v_4 \backslash u_4) \backslash \{b_3, b_4\}.
\end{aligned} \tag{12b}
$$

Examining the above cases using `simpcomp` [6], we get the following solution:

$$
\begin{aligned}
X_3 &= (c_2, c_1, b_2, a_1, b_1, c_5, a_3, a_2, a_4), \\
Y_3 &= (a_3, a_4, a_5, c_5, b_3, b_4, b_5, c_1, b_2).
\end{aligned}
$$

The tuple (X_3, Y_3) yields the complex N_4.

For the second solution in (11) as illustrated in figure 6(b), we have $x_8 = a_4$, $x_9 = a_3$, $y_5 = b_4$ and $y_7 = b_3$. Putting $(x_3, x_4) = (x, y)$ and $y_4 = z$, we have the following constraints:

$$
\begin{aligned}
&(x_1, x_2, x_3, x_4, x_8, x_9) = (c_2, c_1, x, y, a_4, a_3), \\
&\{x_5, x_6, x_7\} = (u_4 \backslash v_4) \backslash \{a_3, a_4\},
\end{aligned} \tag{13a}
$$

$$
\begin{aligned}
&(y_1, y_2, y_3, y_4, y_5, y_7) = (a_3, a_4, a_5, z, b_4, b_3), \\
&\{y_6, y_8, y_9\} = (v_4 \backslash u_4) \backslash \{b_3, b_4\}.
\end{aligned} \tag{13b}
$$

We do not get any solutions for minimal member of $\mathcal{C}$ with the above constraints. Thus N_2, N_3, N_4 are the only minimal elements of $\mathcal{C}$ with $G_{4,5}$ as the dual graph. $\square$

Lemma 5.18. *Let* $M \in \mathcal{C}$ *be minimal with* $\Lambda(M) \cong G_{5,4}$. *Then* $M \cong N_5, N_6, \ldots, N_{11}$ *or* N_{12}.

Proof. We begin by proving the following claims:

(a) $(x_1, x_2, x_3) = (c_2, b_2, c_1)$, $(x_4, x_5) \in \{(a_1, b_1), (b_1, a_1)\}$.
(b) $(y_1, y_2, y_3) = (a_3, a_4, a_5)$, $y_4 \in \{b_5, c_5\}$.
(c) $y_5 \in \{b_3, c_3, b_4, c_4\}$.
(d) $y_4 = b_5 \Rightarrow (x_4, x_5) = (b_1, a_1)$, $y_4 = c_5 \Rightarrow (x_4, x_5) = (a_1, b_1)$.

By Lemma 5.9, we know that $\{x_1, x_2, \ldots, x_5\} = z_0 \backslash u_5$. Since $z_0 = \Phi_1 \cup \Phi_2$, we have $\Phi_1 \subseteq \{x_1, x_2, \ldots, x_5\}$ or $\Phi_2 \subseteq \{x_1, x_2, \ldots, x_5\}$. By Lemma 5.15, we have $x_1 = c_2$. Let $1 \le p < q < r \le 5$ be such that $\{x_p, x_q, x_r\}$ is one of the orbits Φ_1 or Φ_2. Then it can be seen that for $x \in \{x_p, x_q, x_r\}$, T_x contains $p + q + r - 2$ vertices. Thus we must have $p + q + r - 2 = 10$, from which it follows that $(p, q, r) = (3, 4, 5)$. Since $x_1 = c_2$,

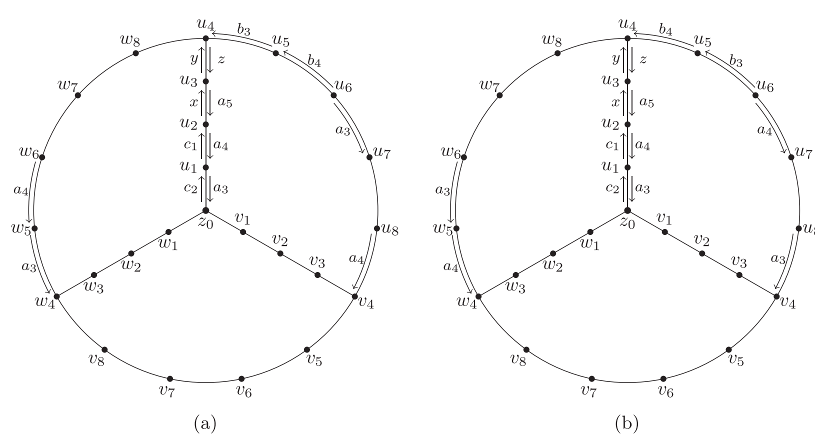


Figure 6(a, b). Illustration for Lemma 5.17, Case 2.

$\{x_p, x_q, x_r\}$ must be the orbit $\Phi_1 = \{a_1, b_1, c_1\}$. Hence $x_2 \in \Phi_2$. As in Claim (c) in Lemma 5.16, we conclude that $x_2 = b_2$. From Lemma 5.15, we further have $x_3 = c_1$. Therefore $\{x_4, x_5\} = \{a_1, b_1\}$. This proves (a).

For part (b), $(y_1, y_2, y_3) = (a_3, a_4, a_5)$ follows exactly as in the proof of Claim (b) in Lemma 5.17. We now show that $y_4 \in \Phi_5$. Suppose $y_4 \notin \Phi_5$. We show that in this case the trees T_{c_1} and T_{a_5} do not intersect. Since $\Phi_1 \subseteq \{x_1, x_2, \ldots, x_5\}$, we see that T_{c_1} is contained in the union of the three paths $z_0u_1 \cdots u_4$, $z_0v_1 \cdots v_4$ and $z_0w_1 \cdots w_4$. Thus T_{a_5} and T_{c_1} must intersect along one of the above three paths. Clearly T_{c_1} and T_{a_5} do not intersect along the path $z_0u_1 \cdots u_4$. However, if $y_4 \notin \Phi_5$, we see that T_{a_5} does not contain any vertex on the paths $z_0v_1 \cdots v_4$ and $z_0w_1 \cdots w_4$ and hence T_{a_5} cannot intersect T_{c_1}, a contradiction to neighbourliness of M. Thus $y_4 \in \Phi_5$.

For part (c), it is enough to show that $y_5 \notin \Phi_5$. If $y_5 \in \Phi_5$, then we see that $\Phi_5 \subseteq u_5$. Hence $\Phi_5 \subseteq v_5$ and $\Phi_5 \subseteq w_5$. But then $|u_5 \backslash v_5| \leq 3$. But this contradicts Lemma 5.9 along the path $u_5u_6 \cdots v_5$, as the four oriented edges $\overrightarrow{u_5u_6}, \overrightarrow{u_6u_7}, \overrightarrow{u_7u_8}, \overrightarrow{u_8v_5}$ cannot all have distinct labels. This proves (c).

For part (d), we again use the neighbourliness of M. Suppose $y_4 = b_5$. As before, the trees T_{a_1} and T_{b_5} must intersect along one of the paths $z_0u_1 \cdots u_4$, $z_0v_1 \cdots v_4$ and $z_0w_1 \cdots w_4$. We see that if $y_4 = b_4$, T_{b_5} does not contain any vertex from the path $z_0w_1 \cdots w_4$ (as both z_0 and w_5 do not contain b_5). On the path $z_0v_1 \cdots v_4$, T_{b_5} contains v_4, v_3, whereas T_{a_1} contains z_0, v_1, v_2. Thus T_{a_1} and T_{b_5} must intersect along $z_0u_1 \cdots u_4$. Now $y_4 = b_5$, implies only vertex on $z_0u_1 \cdots u_4$ on T_{b_5} is u_4. Hence T_{a_1} must also contain u_4, and $x_4 \neq a_1$. Thus $(x_4, x_5) = (b_1, a_1)$. Similarly if $y_4 = c_5$, we can show that $(x_4, x_5) = (a_1, b_1)$.

Case 1. $y_5 \in \{b_3, c_3\}$, $y_4 = b_5$. From Claim (d), we must have $(x_4, x_5) = (b_1, a_1)$ (see figure 7). Consider the tree T_{a_4}. Notice that $u_5 \cap \Phi_4 = \{a_4\}$, and hence $a_4 \notin v_5, w_5$.

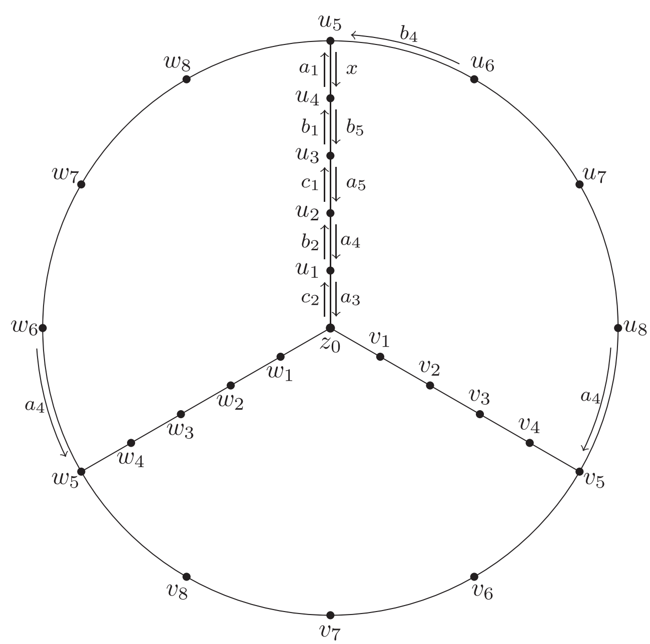


Figure 7. Illustration for Lemma 5.18, Case 1.

Thus $T_{a_4} \subseteq \Lambda(M) - \{z_0, v_5, w_5\}$. In other words, T_{a_4} is contained in the union of the path $\mathcal{P} = z_0 u_1 \cdots u_5$ and the arc $\mathcal{A} = w_6 w_7 \cdots u_5 \cdots u_8$. Since T_{a_4} contains 4 vertices on $\mathcal{P}$, it must induce a path containing 7 vertices (including u_5) on $\mathcal{A}$. The only possible solution is $T_{a_4} = u_2 u_3 u_4 u_5 \cup w_6 w_7 w_8 u_5 u_6 u_7 u_8$. This gives us $x_9 = a_4$, $y_6 = b_4$. Putting $y_5 = x \in \{b_3, c_3\}$, we get the following constraints:

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5, x_9) &= (c_2, b_2, c_1, b_1, a_1, a_4), \\ \{x_6, x_7, x_8\} &= (u_5 \backslash v_5) \backslash \{a_4\}, \end{aligned} \tag{14a}$$

$$\begin{aligned} (y_1, y_2, y_3, y_4, y_5, y_6) &= (a_3, a_4, a_5, b_5, x, b_4), \\ \{y_7, y_8, y_9\} &= (v_5 \backslash u_5) \backslash \{b_4\}. \end{aligned} \tag{14b}$$

This case yields the following solution when examined using `simpcomp` [6]:

$$\begin{aligned} X_1 &= (c_2, b_2, c_1, b_1, a_1, a_3, a_2, a_5, a_4), \\ Y_1 &= (a_3, a_4, a_5, b_5, b_3, b_4, b_2, c_3, c_5). \end{aligned}$$

The pair (X_1, Y_1) yields the complex N_{11}.

Case 2. $y_5 \in \{b_3, c_3\}$, $y_4 = c_5$. This is similar to Case 1, except that $y_4 = c_5$ and hence from Claim (d), $(x_4, x_5) = (a_1, b_1)$. Letting $y_5 = x \in \{b_3, c_3\}$, we get the constraints:

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5, x_9) &= (c_2, b_2, c_1, a_1, b_1, a_4), \\ \{x_6, x_7, x_8\} &= (u_5 \backslash v_5) \backslash \{a_4\}, \end{aligned} \tag{15a}$$

$$\begin{aligned} (y_1, y_2, y_3, y_4, y_5, y_6) &= (a_3, a_4, a_5, c_5, b_3, b_4), \\ \{y_7, y_8, y_9\} &= (v_5 \backslash u_5) \backslash \{b_4\}. \end{aligned} \tag{15b}$$

Using `simpcomp` [6] we get the following solution for this case:

$$\begin{aligned} X_2 &= (c_2, b_2, c_1, a_1, b_1, a_3, a_2, c_5, a_4), \\ Y_2 &= (a_3, a_4, a_5, c_5, b_3, b_4, b_2, c_3, b_5). \end{aligned}$$

The pair (X_2, Y_2) gives the complex N_{12}.

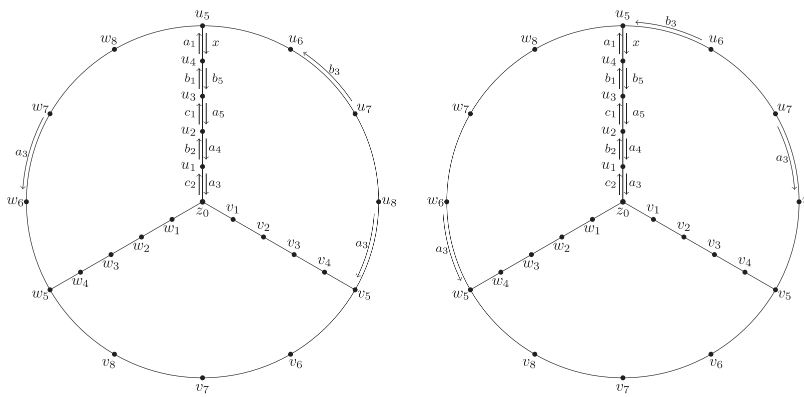
Case 3. $y_5 \in \{b_4, c_4\}$, $y_4 = b_5$. From Claim (d), we have $(x_4, x_5) = (b_1, a_1)$. We notice that $u_5 \cap \Phi_3 = \{a_3\}$, and hence $a_3 \notin z_0 \cup v_3 \cup w_3$. Thus T_{a_3} is contained in the union of the arc $\mathcal{A} = w_6 w_7 \cdots u_5 \cdots u_8$ and the path $\mathcal{P} = u_1 u_2 \cdots u_5$. Since T_{a_3} contains 5 vertices on the path $\mathcal{P}$, it must induce a 6 vertex path on $\mathcal{A}$. We have the following possibilities for T_{a_3}:

$$\begin{aligned} T_{a_3} &= u_1 u_2 u_3 u_4 u_5 \cup w_7 w_8 u_5 u_6 u_7 u_8, \\ T_{a_3} &= u_1 u_2 u_3 u_4 u_5 \cup w_6 w_7 w_8 u_5 u_6 u_7. \end{aligned} \tag{16}$$

For the first solution in (16), we get $x_9 = a_3$ and $y_7 = b_3$ (see figure 8(a)). Setting $y_5 = x \in \{b_4, c_4\}$, we have the following constraints:

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5, x_9)7 &= (c_2, b_2, c_1, b_1, a_1, a_3), \\ \{x_6, x_7, x_8\} &= (u_5 \backslash v_5) \backslash \{a_3\}, \end{aligned} \tag{17a}$$

$$\begin{aligned} (y_1, y_2, y_3, y_4, y_5, y_7) &= (a_3, a_4, a_5, b_5, x, b_3), \\ \{y_6, y_8, y_9\} &= (v_5 \backslash u_5) \backslash \{b_3\}. \end{aligned} \tag{17b}$$



(a) (b)

Figure 8. Illustration for Lemma 5.18, Case 3.

However, using simpcomp [6] we observe that the above constraints do not yield any member of $\mathcal{C}$.

For the second solution in (16), we get $x_8 = a_3$ and $y_6 = b_3$ (see figure 8(b)). Setting $y_5 = x \in \{b_4, c_4\}$, we have the following constraints:

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_8) &= (c_2, b_2, c_1, b_1, a_1, a_3),\\ \{x_6, x_7, x_9\} &= (u_5 \backslash v_5) \backslash \{a_3\}, \end{aligned} \tag{18a}$$

$$\begin{aligned}(y_1, y_2, y_3, y_4, y_5, y_6) &= (a_3, a_4, a_5, b_5, x, b_3),\\ \{y_7, y_8, y_9\} &= (v_5 \backslash u_5) \backslash \{b_3\}. \end{aligned} \tag{18b}$$

Using simpcomp [6] we get the following solutions:

$$\begin{aligned}
X_3 &= (c_2, b_2, c_1, b_1, a_1, a_4, a_5, a_3, a_2),\\
Y_3 &= (a_3, a_4, a_5, b_5, b_4, b_3, c_4, c_5, b_2),\\
X_4 &= (c_2, b_2, c_1, b_1, a_1, a_5, a_4, a_3, a_2),\\
Y_4 &= (a_3, a_4, a_5, b_5, b_4, b_3, c_5, c_4, b_2),\\
X_5 &= (c_2, b_2, c_1, b_1, a_1, a_5, c_4, a_3, a_2),\\
Y_5 &= (a_3, a_4, a_5, b_5, c_4, b_3, c_5, b_4, b_2).
\end{aligned}$$

The pairs (X_3, Y_3), (X_4, Y_4) and (X_5, Y_5) yield N_5, N_6 and N_9 respectively.

Case 4. $y_5 \in \{b_4, c_4\}$, $y_4 = c_5$. From Claim (d), we must have $(x_4, x_5) = (a_1, b_1)$. The rest of the analysis is the same as Case 3, where the possiblities for T_{a_3} are given by (16). For the first solution in (16), we have $x_9 = a_3$ and $y_7 = b_3$. Setting $y_5 = x \in \{b_4, c_4\}$, we have the constraints

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_9) &= (c_2, b_2, c_1, a_1, b_1, a_3),\\ \{x_6, x_7, x_8\} &= (u_5 \backslash v_5) \backslash \{a_3\}, \end{aligned} \tag{19a}$$

$$\begin{aligned}(y_1, y_2, y_3, y_4, y_5, y_7) &= (a_3, a_4, a_5, c_5, x, b_3),\\ \{y_6, y_8, y_9\} &= (v_5 \backslash u_5) \backslash \{b_3\}. \end{aligned} \tag{19b}$$

We obtain no solutions for members of $\mathcal{C}$ meeting the above constraints.

For the second solution for T_{a_3} in (16), we have $x_8 = a_3$ and $y_6 = b_3$. Setting $y_5 = x$, we have the constraints

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_8) &= (c_2, b_2, c_1, a_1, b_1, a_3),\\ \{x_6, x_7, x_9\} &= (u_5 \backslash v_5)\backslash\{a_3\}, \qquad (20a)\\ (y_1, y_2, y_3, y_4, y_5, y_6) &= (a_3, a_4, a_5, c_5, x, b_3),\\ \{y_7, y_8, y_9\} &= (v_5 \backslash u_5)\backslash\{b_3\}. \qquad (20b)\end{aligned}$$

Using simpcomp [6] we obtain the following solutions for members in $\mathcal{C}$:

$$\begin{aligned}X_6 &= (c_2, b_2, c_1, a_1, b_1, a_4, c_5, a_3, a_2),\\ Y_6 &= (a_3, a_4, a_5, c_5, b_4, b_3, c_4, b_5, b_2),\\ X_7 &= (c_2, b_2, c_1, a_1, b_1, c_5, a_4, a_3, a_2),\\ Y_7 &= (a_3, a_4, a_5, c_5, b_4, b_3, b_5, c_4, b_2),\\ X_8 &= (c_2, b_2, c_1, a_1, b_1, c_5, c_4, a_3, a_2),\\ Y_8 &= (a_3, a_4, a_5, c_5, c_4, b_3, b_5, b_4, b_2).\end{aligned}$$

The pairs (X_6, Y_6), (X_7, Y_7) and (X_8, Y_8) yield the complexes N_7, N_8 and N_{10}, respectively. The above cases complete the proof of the lemma. □

Lemma 5.19. *If* $M \in \mathcal{C}$ *is minimal, then* $\Lambda(M) \ncong G_{6,3}$.

Proof. Assume that M is a minimal member of $\mathcal{C}$ with $\Lambda(M) \cong G_{6,3}$. Let (X, Y) be the pair associated with M. Now by Lemma 5.8, we have $\{x_1, x_2, x_3, x_4, x_5, x_6\} = z_0 = \Phi_1 \cup \Phi_2$. Let p, q, r be such that $\{x_p, x_q, x_r\} = \Phi_1$. Then it can be seen that T_{a_1} contains $p + q + r - 2$ vertices. Similarly, let i, j, k be such that $\{x_i, x_j, x_k\} = \Phi_2$. Now T_{a_2} contains $i + j + k - 2$ vertices. But since T_{a_1}, T_{a_2} each contain 10 vertices, we must have $p + q + r + i + j + k = 24$. However, p, q, r, i, j, k is a permutation of $1, 2, \dots, 6$, and hence we must have $p + q + r + i + j + k = 21$, a contradiction. This proves the lemma. □

Proof of Theorem 3.2. The proof follows from Lemmas 5.6, 5.16, 5.17, 5.18 and 5.19. □

Proof of Theorem 3.3. The proof follows from Theorem 3.2 and Proposition 2.4. □

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Appendix A

We give a proof for the graph theoretic Lemma 5.3. For standard terminology on graphs, see [4]. For a vertex v in a graph G, $d_G(v)$ will denote the degree of the vertex v in G. For vertices u, v in G, $d_G(u, v)$ will denote the length of a shortest path between u and v in G. For a vertex $a \in V(G)$ and a subset B of $V(G)$, a path $v_0 v_1 \cdots v_k$ such that $v_0 = a$ and

$\{v_0, v_1, \ldots, v_k\} \cap B = \{v_k\}$ is called an a-B path. The following is an easy consequence of the *Fan lemma* (cf. Chapter 9 of [4]).

Lemma A.1. Let G be a two connected graph and let $B \subseteq V(G)$. If $a \notin B$ and $|B| \geq 2$ then, there exist two $a - B$ paths in G, which intersect only in a.

We prove the following generalization of Lemma 5.3.

Theorem A.2. *Let graphs $G_{r,s}$ and $T_{r,s}$ be as defined in Examples 5.1 and 5.2, respectively. Let G be a two connected graph on n vertices with $n+2$ edges. If $\mathrm{Aut}(G) \supseteq \mathbb{Z}_3$, then $G \cong G_{r,s}$ for some $r, s > 0$ with $3(r+s) = n+2$ or $G \cong T_{r,s}$ for some $r, s > 0$ with $3r + s = n$.*

Proof. Let φ be an order three automorphism of G. Let $\mathrm{Fix}(\varphi) = \{v \in V(G) : \varphi(v) = v\}$ denote the set of vertices fixed by the automorphism φ. Let T be the set of vertices with degree three or more in G. Since G is two connected, we have $d_G(v) \geq 2$ for all $v \in V(G)$. Then from the identity

$$\sum_{v \in V(G)} d_G(v) = 2(n+2) = 2n+4, \tag{A1}$$

it follows that $|T| \leq 4$. We have the following cases:

Case 1. $T \nsubseteq \mathrm{Fix}(\varphi)$. Let $u \in T$ be such that $\varphi(u) \neq u$. Let $v = \varphi(u)$ and $w = \varphi^2(u)$. As φ is an automorphism of G, we have $d_G(u) = d_G(v) = d_G(w)$. Let k be the degree of u, v and w in G. Clearly $k \geq 3$. Now from (A1), it follows that $k = 3$, and there exists $z \notin \{u, v, w\}$ with $d_G(z) = 3$. Since φ orbits are either singleton or three element subsets and $|T| \leq 4$, we have $\varphi(z) = z$, or $z \in \mathrm{Fix}(\varphi)$. Thus we have $d_G(z, u) = d_G(z, v) = d_G(z, w) = r$ for some $r \geq 1$. Let $p_{zu} := z u_1 \cdots u_r (= u)$ be a shortest z-u path in G. Then $p_{zv} := z v_1 \cdots v_r (= v)$ and $p_{zw} := z w_1 \cdots w_r (= w)$ are z-v and z-w paths respectively, where $v_i = \varphi(u_i)$ and $w_i = \varphi^2(u_i)$ for $1 \leq i \leq r$. As G is two connected, $G' := G - z$ is a connected graph. Note that φ is an automorphism of G'. Let $p_{uv} := u_r u_{r+1} \cdots u_{r+s-1} v_r$ be a shortest u-v path in G', where $s = d_{G'}(u, v) = d_{G'}(v, w) = d_{G'}(w, u)$. Let $p_{vw} := v_r v_{r+1} \cdots v_{r+s-1} w_r$ and $p_{wu} := w_r w_{r+1} \cdots w_{r+s-1} u_r$ where $v_i = \varphi(u_i)$ and $w_i = \varphi^2(u_i)$ for $r \leq i \leq r+s-1$. We claim the following:

(a) $p_{zu} \cap p_{zv} = p_{zv} \cap p_{zw} = p_{zw} \cap p_{zu} = \{z\}$.
(b) $p_{uv} \cap p_{vw} = \{v\}$, $p_{vw} \cap p_{wu} = \{w\}$ and $p_{wu} \cap p_{uv} = \{u\}$.
(c) $p_{zu} \cap p_{uv} = p_{zu} \cap p_{wu} = \{u\}$, $p_{zv} \cap p_{uv} = p_{zv} \cap p_{vw} = \{v\}$ and $p_{zw} \cap p_{vw} = p_{zw} \cap p_{wu} = \{w\}$.
(d) $p_{zu} \cap p_{vw} = p_{zv} \cap p_{wu} = p_{zw} \cap p_{uv} = \emptyset$.

We first prove (a). Let $i > 0$ be maximum such that $u_i \in p_{zu} \cap p_{zv}$. Then $u_i = v_j$ for some $1 \leq j \leq r$. Since $i = d_G(z, u_i) = d_G(z, v_j) = j$, we have $i = j$. Because $u_r = u \neq v = v_r$, we have $i < r$. Further, as $d_G(z, w) = r > i$, we have $u_i \neq w$. Thus $u_i \notin \{z, u, v, w\}$, and hence $d_G(u_i) = 2$. However by maximality of i, we have $\{v_{i-1}, v_{i+1}, u_{i+1}\}$ as three distinct neighbours of u_i, a contradiction. Therefore $p_{zu} \cap p_{zv} = \{z\}$, and similar argument works for other pairs. This proves (a). Claim (b) can be proved in a manner similar to Claim (a).

To prove Claim (c), we first show that $p_{zu} \cap p_{uv} = \{u\}$. Clearly $z \notin p_{uv}$ as $p_{uv} \subseteq G - z$. Let $0 < i < r$ be maximum such that u_i is a vertex on p_{uv}. By Claim (a), $u_i \notin \{z, u, v, w\}$, and hence $d_G(u_i) = 2$. Again maximality of i implies that u_{i+1} is distinct from the two neighbours of u_i on p_{uv}, a contradiction. Therefore $p_{zu} \cap p_{uv} = \{u\}$. Similarly, we can show for other pairs. This proves (c). Claim (d) can be proved in a manner similar to Claim (c).

Define the subgraph

$$H := p_{zu} \cup p_{zv} \cup p_{zw} \cup p_{uv} \cup p_{vw} \cup p_{wu}.$$

Observe that $d_H(v) = d_G(v)$ for all $v \in V(H)$. Since G is connected, this implies $G = H$. It can be seen that $G \cong G_{r,s}$ and $3(r + s) = n + 2$.

Case 2. $T \subseteq \mathrm{Fix}(\varphi)$. Let $u_1u_2 \cdots u_r$ be a maximal path in $G - \mathrm{Fix}(\varphi)$. Let x, y be neighbours of u_1 and u_r in G respectively, which are not on the path (such neighbours exist as $d_G(v) \geq 2$ for all $v \in V(G)$). By maximality of the path, we conclude that $x, y \in \mathrm{Fix}(\varphi)$. Let $p_u := xu_1 \cdots u_r y$. Then, observe that $p_v := xv_1 \cdots v_r y$ and $p_w := xw_1 \cdots w_r y$ are also x-y paths in G, where $v_i = \varphi(u_i)$ and $w_i = \varphi^2(u_i)$ for $1 \leq i \leq r$. We note that $x \neq y$, for otherwise we would have $d_G(x) = 6$ and it can be seen that $G - x$ cannot be connected in this case ($G - x$ would be a graph on $n - 1$ vertices with $n + 2 - 6 = n - 4$ edges). Let $H := p_u \cup p_v \cup p_w$. We claim the following:

(a) Paths p_u, p_v and p_w are vertex-independent.
(b) If $V(H) = V(G)$ then $G = H + xy$. Further $G \cong T_{r,2}$.
(c) If $V(H) \neq V(G)$, then $G = H \cup p$ where p_u, p_v, p_w and p are vertex-independent $x - y$ paths. Further $G \cong T_{r,s}$, where s is the number of vertices in the path p.

We first prove (a). Assume that p_u and p_v intersect in a vertex other than x or y. Let i be maximum such that $u_i \in p_v$. Then $u_i = v_j$ for some $1 \leq j \leq r$. Since $u_r \neq v_r$, we have $\min(i, j) < r$. Without loss, assume $i < r$. Since $T \subseteq \mathrm{Fix}(\varphi)$ and $u_i \notin \mathrm{Fix}(\varphi)$, we have $d_G(u_i) = 2$. But we see that $v_{j-1}, v_{j+1}, u_{i+1}$ are three distinct neighbours of u_i, a contradiction. Therefore p_u and p_v are vertex-independent. Similarly we can prove for other pairs.

To prove (b), assume that $V(H) = V(G)$. From Claim (a), we see that $d_H(v) = d_G(v) = 2$ for $v \in V(G)$, $v \notin \{x, y\}$. It can be seen that to satisfy (A1), we must have $G = H + xy$. In this case, it is readily seen that $G \cong T_{r,2}$ and $3r + 2 = n$. This proves (b).

To prove (c), assume that $V(H) \neq V(G)$. Let $a \in V(G)\backslash V(H)$. By Lemma 5.20, there exist two a-$V(H)$ paths p_1 and p_2 in G such that $p_1 \cap p_2 = \{a\}$. Since $d_G(v) = 2$ for all $v \in V(H)\backslash\{x, y\}$, we conclude that the paths p_1 and p_2 meet H in vertices x and y. Without loss, let p_1 be an a-x path and p_2 be an a-y path. Then $p = p_1 \cup p_2$ is an x-y path. Clearly p is vertex-independent to p_u, p_v and p_w. Let $H_1 = H \cup p$. Then we observe that $d_{H_1}(v) = 2$ for $v \in V(H_1)\backslash\{x, y\}$ and $d_{H_1}(x) = d_{H_1}(y) = 4$. From (A1), it follows that $d_{H_1}(v) = d_G(v)$ for all $v \in V(H_1)$. Since G is connected, we have $G = H_1$. It can be seen that $G \cong T_{r,s}$ in this case, where s is the number of vertices in p. Further, we have $3r + s = n$. This completes the proof of the theorem. □

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