Proc. Indian Acad. Sci. (Math. Sci.) Vol. 125, No. 1, February 2015, pp. 71–77.

L_p-dual affine surface area forms of Busemann–Petty type problems

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MS received 27 August 2013

Abstract. Associated with the notion of L_p-intersection body which was defined by Haberl, we research L_p-dual affine surface area forms of Busemann–Petty type problems.

Keywords. L_p-dual affine surface area; L_p-intersection body; Busemann–Petty problem.

2000 Mathematics Subject Classification. 52A20, 52A40.

1. Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroid lie at the origin in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ the n-dimensional volume of body K.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, +\infty)$ is defined by (see [4])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Let S_o^n, S_c^n and S_e^n denote the set of star bodies (about the origin), the set of star bodies whose centroid lie at the origin and the set of origin-symmetric star bodies in $\mathbb{R}^n$, respectively. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

The notion of intersection bodies was introduced by Lutwak (see [10]): For $K \in \mathcal{S}_o^n$, the intersection body, IK of K is a star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^{\perp}$, the hyperplane orthogonal to u, i.e. for all $u \in S^{n-1}$,

$$\rho(IK, u) = V_{n-1}(K \cap u^{\perp}),$$

where V_{n-1} denotes the $(n-1)$-dimensional volume.

The notion of an intersection body has been shown to be fundamentally connected to the Busemann–Petty problem (see [10]), which states that if two centrally-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying

$$V_{n-1}(K \cap u^{\perp}) \leq V_{n-1}(L \cap u^{\perp})$$

for all $u \in \mathcal{S}^{n-1}$, does it follow that

$$V(K) \leq V(L)?$$

The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution appeared as a result of a sequence of papers (see [1, 3–5, 8, 9, 11, 14, 15]). In [10], Lutwak proved that the Busemann–Petty problem has a positive answer if K is an intersection body in $\mathbb{R}^n$.

Haberl and Ludwig [7] defined the L_p-intersection body as follows: Let L be a star body and nonzero $p < 1$, the L_p-intersection body of L, I_pL is the origin-symmetric star body, whose radial function is defined by

$$\rho(I_pL, u)^p = \int_L |\, u \cdot x \,|^{-p} \, \mathrm{d}x.$$

Soon afterwards, Yuan and Cheung [13] generalized the classical Busemann–Petty problem from intersection bodies to L_p-intersection bodies as follows:

Theorem 1.A. *Let K be a L_p-intersection body and L be a star body in $\mathbb{R}^n$. If $I_pK \subseteq I_pL$, then*

$$V(K) \leq V(L), \quad \textit{for } 0 < p < 1$$

and

$$V(K) \geq V(L), \quad \textit{for } p < 0.$$

In both cases equality holds if and only if $K = L$.

Theorem 1.B. *Let K be a centered star body and L be a star body in $\mathbb{R}^n$. If $I_pK = I_pL$, then*

$$V(K) \leq V(L), \quad \textit{for } 0 < p < 1$$

and

$$V(K) \geq V(L), \quad \textit{for } p < 0.$$

In both cases equality holds if and only if $K = L$.

Associated with the L_p-dual mixed volume $\tilde{V}_p(M, N)$, Wang *et al.* [12] gave the notion of L_p-dual affine surface area as follows: For $K \in \mathcal{S}_o^n$ and $0 < p < n$, the L_p-dual affine surface area $\tilde{\Omega}_p(K)$ of K is defined by

$$n^{-\frac{p}{n}} \tilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\{n\tilde{V}_p(K, Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{K}_c^n\}. \tag{1.1}$$

Here Q^* denotes the polar of Q which is defined by (see [4])

$$Q^* = \{x : x \cdot y \leq 1, y \in Q\}, \qquad x \in \mathbb{R}^n.$$

In this paper, combined with (1.1) of the L_p-dual affine surface area, we will research the Busemann–Petty type problems for the L_p-intersection bodies. For convenience, we improve (1.1) from $Q \in \mathcal{K}_c^n$ to $Q \in \mathcal{S}_c^n$:

For $K \in \mathcal{S}_o^n$ and $0 < p < n$, the L_p-dual affine surface area $\tilde{\Omega}_p(K)$ of K is defined by

$$n^{-\frac{p}{n}} \tilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\{n \tilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_c^n\}. \tag{1.2}$$

Let Z_p^n denote the polar set of L_p-intersection bodies, then $Z_p^n \subseteq S_c^n$. If $Q \in \mathcal{Z}_p^n$ in (1.2), we write $\tilde{\Omega}_p^\circ(K)$ by

$$n^{-\frac{p}{n}} \tilde{\Omega}_p^\circ(K)^{\frac{n+p}{n}} = \sup\{n \tilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{Z}_p^n\}. \tag{1.3}$$

According to equality (1.3), we first give an affirmative form of the Busemann–Petty problem for the L_p-intersection bodies.

Theorem 1.1. *For $K, L \in \mathcal{S}_o^n, 0 < p < 1$, if $I_p K \subseteq I_p L$, then*

$$\tilde{\Omega}_p^\circ(K) \le \tilde{\Omega}_p^\circ(L), \tag{1.4}$$

and equality holds if and only if $I_p K = I_p L$.

Next, we obtain a negative form of the Busemann–Petty problem for the L_p-intersection bodies.

Theorem 1.2. *For $L \in \mathcal{S}_o^n, 0 < p < 1$, if L is not an origin-symmetric star body, then there exists $K \in \mathcal{S}_e^n$, such that*

$$I_p K \subset I_p L,$$

but

$$\tilde{\Omega}_p^\circ(K) > \tilde{\Omega}_p^\circ(L). \tag{1.5}$$

The proofs of Theorems 1.1–1.2 will be completed in Section 3 of this paper.

2. Preliminaries

2.1 *L_p-dual mixed volume*

The notion of L_p-dual mixed volume was introduced as follows (see [2, 6]): For $K, L \in S_o^n$ and any real number p, the L_p-dual mixed volume $\tilde{V}_p(K, L)$ of K and L is defined by

$$\tilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-p} \rho_L(u)^p \mathrm{d}u, \tag{2.1}$$

From (2.1), we easily know that

$$\tilde{V}_p(K, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^n \mathrm{d}u.$$

2.2 L_p-dual Blaschke combination

For $K, L \in S_o^n$, $0 < p < n$ and $\lambda, \mu \geq 0$ (both not zero), the L_p-dual Blaschke combination $\lambda \otimes K \check{+}_p \mu \otimes L$ of K and L is defined by

$$\rho(\lambda \otimes K \check{+}_p \mu \otimes L, \cdot)^{n-p} = \lambda \rho(K, \cdot)^{n-p} + \mu \rho(L, \cdot)^{n-p}. \tag{2.2}$$

Take $\lambda = \mu = \frac{1}{2}$, $L = -K$ in $\lambda \otimes K \check{+}_p \mu \otimes L$, then L_p-dual Blaschke body $\bar{\nabla}_p K$ is defined by

$$\bar{\nabla}_p K = \frac{1}{2} \otimes K \check{+}_p \frac{1}{2} \otimes (-K). \tag{2.3}$$

Obviously, the L_p-dual Blaschke body $\bar{\nabla}_p K$ is origin-symmetric.

3. L_p-dual affine surface area forms of Busemann–Petty type problems

In this section, we will give L_p-dual affine surface area forms of Busemann–Petty type problems. Firstly, we prove Theorem 1.1. Here, we obtain a lemma (see [6]).

Lemma 3.1. *If $K, L \in S_o^n$, then*

$$\tilde{V}_p(K, I_p L) = \tilde{V}_p(L, I_p K). \tag{3.1}$$

Proof of Theorem 1.1. Since $I_p K \subseteq I_p L$, thus for any $Q \in S_o^n$,

$$\tilde{V}_p(Q, I_p K) \leq \tilde{V}_p(Q, I_p L). \tag{3.2}$$

From Lemma 3.1, we get

$$\tilde{V}_p(K, I_p Q) \leq \tilde{V}_p(L, I_p Q). \tag{3.3}$$

Let $M = I_p Q$, then $M \in Z_p^n$. Thus from equations (1.3) and (3.3), we obtain

$$\begin{aligned} n^{-\frac{p}{n}} \tilde{\Omega}_p^{\circ}(K) &= \sup\{n \tilde{V}_p(K, M^*) V(M)^{\frac{p}{n}} : M \in Z_p^n\} \\ &\leq \sup\{n \tilde{V}_p(L, M^*) V(M)^{\frac{p}{n}} : M \in Z_p^n\} \\ &= n^{-\frac{p}{n}} \tilde{\Omega}_p^{\circ}(L), \end{aligned}$$

i.e.

$$\tilde{\Omega}_p^{\circ}(K) \leq \tilde{\Omega}_p^{\circ}(L). \tag{3.4}$$

According to the equality condition of (3.1), we know that equality holds in (3.4) if and only if $I_p K = I_p L$. □

In order to prove the negative form for the Busemann–Petty type problem, we need the following two lemmas.

Lemma 3.2. *If $K, L \in S_o^n, \lambda, \mu \geq 0$ (not both zero) and $0 < p < n$, then*

$$\tilde{\Omega}_p^{\circ}(\lambda \otimes K \check{+}_p \mu \otimes L)^{\frac{n+p}{n}} \leq \lambda \tilde{\Omega}_p^{\circ}(K)^{\frac{n+p}{n}} + \mu \tilde{\Omega}_p^{\circ}(L)^{\frac{n+p}{n}} \tag{3.5}$$

with equality if and only if K and L are dilates.

Proof. From (1.3) and (2.2), we have

$$
\begin{aligned}
&n^{-\frac{p}{n}}\tilde{\Omega}_p^{\circ}(\lambda\otimes K\check{+}_p\mu\otimes L)^{\frac{n+p}{n}}\\
&\quad=\sup\left\{n\tilde{V}_p(\lambda\otimes K\check{+}_p\mu\otimes L,M)V(M^*)^{\frac{p}{n}}:M\in Z_p^n\right\}\\
&\quad=\sup\left\{\left[\int_{S^{n-1}}\rho(\lambda\otimes K\check{+}_p\mu\otimes L,u)^{n-p}\rho(M,u)^p\mathrm{d}u\right]V(M^*)^{\frac{p}{n}}:M\in Z_p^n\right\}\\
&\quad=\sup\left\{\left[\int_{S^{n-1}}[\lambda\rho(K,u)^{n-p}+\mu\rho(L,u)^{n-p}]\rho(M,u)^p\mathrm{d}u\right]V(M^*)^{\frac{p}{n}}:M\in Z_p^n\right\}\\
&\quad=\sup\left\{\lambda\left[\int_{S^{n-1}}\rho(K,u)^{n-p}\rho(M,u)^p\mathrm{d}u\right]V(M^*)^{\frac{p}{n}}\right.\\
&\qquad\left.+\mu\left[\int_{S^{n-1}}\rho(L,u)^{n-p}\rho(M,u)^p\mathrm{d}u\right]V(M^*)^{\frac{p}{n}}:M\in Z_p^n\right\}\\
&\quad=\sup\left\{n\lambda\tilde{V}_p(K,M)V(M^*)^{\frac{p}{n}}+n\mu\tilde{V}_p(L,M)V(M^*)^{\frac{p}{n}}:M\in Z_p^n\right\}\\
&\quad\le\sup\left\{n\lambda\tilde{V}_p(K,M)V(M^*)^{\frac{p}{n}}:M\in Z_p^n\right\}\\
&\qquad+\sup\left\{n\mu\tilde{V}_p(L,M)V(M^*)^{\frac{p}{n}}:M\in Z_p^n\right\}\\
&\quad=\lambda n^{-\frac{p}{n}}\tilde{\Omega}_p^{\circ}(K)^{\frac{n+p}{n}}+\mu n^{-\frac{p}{n}}\tilde{\Omega}_p^{\circ}(L)^{\frac{n+p}{n}}.
\end{aligned}
$$

Thus

$$\tilde{\Omega}_p^{\circ}(\lambda\otimes K\check{+}_p\mu\otimes L)^{\frac{n+p}{n}}\le\lambda\tilde{\Omega}_p^{\circ}(K)^{\frac{n+p}{n}}+\mu\tilde{\Omega}_p^{\circ}(L)^{\frac{n+p}{n}}.$$

Hence equality holds if and only if $\lambda\otimes K\check{+}_p\mu\otimes L$ are dilates with K and L, respectively. This means that equality holds in (3.5) if and only if K and L are dilates. □

COROLLARY 3.1

If $K\in S_o^n, 0<p<n$, then

$$\tilde{\Omega}_p^{\circ}(\bar{\nabla}_pK)\le\tilde{\Omega}_p^{\circ}(K), \tag{3.6}$$

with equality if and only if K is origin-symmetric.

Proof. Taking $\lambda=\mu=\frac{1}{2}$, $L=-K$ in (3.5), and combining with (2.3), we get

$$\tilde{\Omega}_p^{\circ}(\bar{\nabla}_pK)\le\frac{1}{2}\tilde{\Omega}_p^{\circ}(K)+\frac{1}{2}\tilde{\Omega}_p^{\circ}(-K). \tag{3.7}$$

Since $M\in Z_p^n$, M is origin-symmetric, i.e., $\rho_M(u)=\rho_{-M}(u)=\rho_M(-u)$. By (2.1), we get

$$
\begin{aligned}
\tilde{V}_p(-K,M)&=\frac{1}{n}\int_{S^{n-1}}\rho_{-K}(u)^{n-p}\rho_M(u)^p\mathrm{d}u\\
&=\frac{1}{n}\int_{S^{n-1}}\rho_K(-u)^{n-p}\rho_M(-u)^p\mathrm{d}u=\tilde{V}_p(K,M).
\end{aligned}\tag{3.8}
$$

Thus, associated with (1.3) and (3.8), we easily have

$$\tilde{\Omega}_p^{\circ}(-K) = \tilde{\Omega}_p^{\circ}(K). \tag{3.9}$$

Therefore, from (3.7) and (3.9), we know that

$$\tilde{\Omega}_p^{\circ}(\bar{\nabla}_p K) \le \tilde{\Omega}_p^{\circ}(K).$$

According to the equality condition of (3.5), we easily know that equality holds in (3.6) if and only if K is an origin-symmetric body. □

Lemma 3.3. *If* $K \in S_o^n$, $0 < p < 1$, *then*

$$I_p(\bar{\nabla}_p K) = I_p(K).$$

Proof. From equations (2.2) and (2.3), we get

$$\begin{aligned} \rho\left(I_p(\bar{\nabla}_p K), u\right)^p &= \frac{1}{n-p}\int_{S^{n-1}\cap u^{\perp}} \rho\left(\frac{1}{2}\otimes K \check{+}_p \frac{1}{2}\otimes(-K), v\right)^{n-p} dv \\ &= \frac{1}{n-p}\int_{S^{n-1}\cap u^{\perp}} \left[\frac{1}{2}\rho(K,v)^{n-p} + \frac{1}{2}\rho(-K,v)^{n-p}\right] dv \\ &= \frac{1}{2}\rho(I_p K, u)^p + \frac{1}{2}\rho\left(I_p(-K), u\right)^p. \end{aligned} \tag{3.10}$$

From equation (2.2) we easily know that $I_p(-K) = I_p(K)$. So combining with (3.10), for any $u \in S^{n-1}$,

$$\rho\left(I_p(\bar{\nabla}_p K), u\right)^p = \rho(I_p K, u)^p,$$

i.e.

$$I_p(\bar{\nabla}_p K) = I_p K.$$

Proof of Theorem 1.2. Since K is not an origin-symmetric body, so from Corollary 3.1, we know that

$$\tilde{\Omega}_p^{\circ}(\bar{\nabla}_p K) < \tilde{\Omega}_p^{\circ}(K).$$

Choose $\varepsilon > 0$, such that $\tilde{\Omega}_p^{\circ}\left((1+\varepsilon)\bar{\nabla}_p K\right) < \tilde{\Omega}_p^{\circ}(K)$. Therefore, let $L = (1+\varepsilon)\bar{\nabla}_p K$, then

$$\tilde{\Omega}_p^{\circ}(K) > \tilde{\Omega}_p^{\circ}(L).$$

But from Lemma 3.3 and $I_p\left((1+\varepsilon)K\right) = (1+\varepsilon)^{\frac{n-p}{p}} I_p(K)$, we get

$$I_p L = I_p\left((1+\varepsilon)\bar{\nabla}_p K\right) = (1+\varepsilon)^{\frac{n-p}{p}} I_p(\bar{\nabla}_p K) = (1+\varepsilon)^{\frac{n-p}{p}} I_p K \supset I_p K.$$

□

Acknowledgements

This research is supported in part by the Natural Science Foundation of China (Grant No. 11371224) and Excellent Foundation of Graduate Student of China Three Gorges University (Grant No. 2014PY065).

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COMMUNICATING EDITOR: Parameswaran Sankaran