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Pullback and pushout crossed polymodules

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Abstract. In this paper, we introduce the concept of pullback and pushout crossed polymodules and we describe the construction of pullback and pushout crossed polymodules. In particular, by using the notion of fundamental relation, we obtain a crossed module from a pullback crossed polymodule.

Keywords. Crossed module; polygroup; crossed polymodule; action; pullback; pushout; fundamental relation

Mathematics Subject Classification. 13D99, 20N20, 18D35.

1. Introduction

Crossed module was defined by Whitehad in [21] and many have worked on this subject. There are many examples of crossed module such as actor crossed module, pullback crossed module, pushout crossed module and induced crossed module, etc. Pullback crossed module was defined by Brown and Wensley in [2] and [3]. They gave many examples and applications of pullback crossed module. Pushout crossed module was defined by Korkes and Porter in [14] and they presented a good example of crossed module. The notion of pushout in a general category is well known and we refer the reader to [18].

This paper has four sections. In §2, we present some basic notions about polygroups and crossed polymodules. In §3, we introduce the concept of pullback crossed polymodule and we describe the construction of pullback crossed polymodules. In particular, by using the notion of fundamental relation, we obtain a crossed module from a pullback crossed polymodule. Finally, in §4, we introduce the concept of pushout crossed polymodule and we discuss about this concept.

2. Crossed polygroups

Polygroups were studied by Comer [5] (see also [9–11]). Comer [5] and Davvaz [9] developed the algebraic theory for polygroups. A polygroup is a completely regular, reversible multigroup in itself. We recall the following definition from [5]. A *polygroup* is a multivalued system $\mathcal{M} = \langle P, \circ, e, ^{-1} \rangle$, with $e \in P$, $^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold for all x, y, z in P: (1) $(x \circ y) \circ z = x \circ (y \circ z)$, (2) $e \circ x = x \circ e = x$, (3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$. In the above definition,

$\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and if $x \in P$ and A, B are non-empty subsets of P, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$. The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$ and $(x^{-1})^{-1} = x$. In the rest of this section, we present the facts about polygroups that underlie the subsequent material. For further discussion of polygroups, we refer to Davvaz's book [9]. Many important examples of polygroups are given in [9] such as double coset algebra, Prenowitz algebras, conjugacy class polygroups, character polygroups, extension of polygroups, and chromatic polygroups. Here, we present two of them.

Example 1. Suppose that H is a subgroup of a group G. Define a system $G /\!/ H = \langle \{HgH \mid g \in G\}, *, H, ^{-I} \rangle$, where $(HgH)^{-I} = Hg^{-1}H$ and

$$(Hg_1H) * (Hg_2H) = \{Hg_1hg_2H \mid h \in H\}.$$

The algebra of double cosets $G//H$ is a polygroup.

Example 2. Let $P = \{e, a, b, c\}$ together with the following table:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, b\}$	$\{a, b\}$	c
b	b	$\{a, b\}$	$\{e, a\}$	c
c	c	c	c	$\{e, a, b\}$

and for all $x \in P$, $x = x^{-1}$. It is easy to see that $\langle P, \circ, e, ^{-1} \rangle$ is a polygroup.

Every group is a polygroup. There are several kinds of homomorphisms between polygroups [9]. In this paper, we apply only the notion of strong homomorphisms. Let $\langle P, \circ, e, \ ^{-1} \rangle$ and $\langle P', \star, e, ^{-1} \rangle$ be two polygroups. A mapping ϕ from P into P' is said to be a *strong homomorphism* if $\phi(e) = e$ and for all $a, b \in P$, $\phi(a \circ b) = \phi(a) \star \phi(b)$, $a, b \in P$. A strong homomorphism ϕ is said to be an *isomorphism* if ϕ is one-to-one and onto. Two polygroups P and P' are said to be *isomorphic* if there is an isomorphism from P onto P'. The defining condition for a strong homomorphism is also valid for sets, i.e., if A, B are nonempty subsets of P, then it follows that $f(A \circ B) = f(A) \star f(B)$.

Let $\langle P, \circ, e, \ ^{-1} \rangle$ be a polygroup. We define the relation β^*_P as the smallest equivalence relation on P such that the quotient P/β^*_P, the set of all equivalence classes, is a group. In this case β^*_P is called the *fundamental equivalence relation* on P and P/β^*_P is called the *fundamental group*. The product $\odot$ in P/β^*_P is defined as follows: $\beta^*_P(x) \odot \beta^*_P(y) = \beta^*_P(z)$ for all $z \in \beta^*_P(x) \circ \beta^*(y)$. This relation is introduced by Koskas [15] and studied mainly by Corsini [6], Leoreanu-Fotea *et al*. [16, 17] and Freni [12, 13] concerning hypergroups, Vougiouklis [20] concerning H_v-groups, Davvaz concerning polygroups [8, 19], and many others. We consider the relation β_P as follows:

$$x \ \beta_P \ y \ \Leftrightarrow \ \text{there exist } z_1, \ldots z_n \text{ such that } \{x, y\} \subseteq \circ \prod_{i=1}^{n} z_i.$$

Freni [12] proved that for hypergroups, $\beta = \beta^*$. Since polygroups are certain subclasses of hypergroups, we have $\beta^*_P = \beta_P$. The kernel of the *canonical map* $\varphi_P : P \longrightarrow P/\beta^*_P$ is called the *core* of P and is denoted by ω_P. Here we also denote by ω_P the unit of P/β^*_P. It is easy to prove that the following statements: $\omega_P = \beta^*_P(e)$ and $\beta^*_P(x)^{-1} = \beta^*_P(x^{-1})$, for all $x \in P$.

By using the concept of generalized permutation, Davvaz in [7] defined permutation polygroups and action of a polygroup on a set. For the definition of crossed polymodule, we need the notion of polygroup action. We recall the following definition from [7]. Let $\mathcal{P} = \langle P, \circ, e, {}^{-1}\rangle$ be a polygroup and Ω be a non-empty set. A map $\alpha : P \times \Omega \to \mathcal{P}^*(\Omega)$, where $\alpha(g, \omega) := {}^g\omega$ is called a *(left) polygroup action* on Ω if the following axioms hold:

(1) ${}^e\omega = \omega$,
(2) ${}^h({}^g\omega) = {}^{h\circ g}\omega$, where ${}^gA = \bigcup_{a\in A} {}^ga$ and ${}^B\omega = \bigcup_{b\in B} {}^b\omega$, for all $A \subseteq \Omega$ and $B \subseteq P$,
(3) $\bigcup_{\omega\in\Omega} {}^g\omega = \Omega$,
(4) for all $g \in P$, $a \in^g b \Rightarrow b \in^{g^{-1}} a$.

Example 3. Suppose that $\langle P, \circ, e, {}^{-1}\rangle$ is a polygroup. Then, P acts on itself by conjugation. Indeed, if we consider the map $\alpha : P \times P \to \mathcal{P}^*(P)$ by $\alpha(g, x) = {}^gx := g\circ x\circ g^{-1}$, then

(1) ${}^ex = x$,
(2) ${}^h({}^gx) = {}^h(g \circ x \circ g^{-1}) = h \circ g \circ x \circ g^{-1} \circ h^{-1} = (h \circ g) \circ x \circ (h \circ g)^{-1} = \bigcup_{b\in h\circ g}(b \circ x \circ b^{-1}) = \bigcup^b_{b\in h\circ g} x = {}^{h\circ g}x$,
(3) $\bigcup_{x\in P} {}^gx = \bigcup_{x\in P} g \circ x \circ g^{-1} = P$,
(4) if $a \in^g b = g \circ b \circ g^{-1}$, then $g \in a \circ g \circ b^{-1}$ and hence $b^{-1} \in g^{-1} \circ a^{-1} \circ g$. This implies that $b \in g^{-1} \circ a \circ g$.

Example 4. Consider the polygroup defined in Example 2. Then, the following table shows the action of P by conjugation.

α	e	a	b	c
e	e	a	b	c
a	$\{e, b\}$	$\{a, b\}$	$\{e, a, b\}$	c
b	$\{e, a\}$	$\{e, a, b\}$	$\{a, b\}$	c
c	$\{e, a, b\}$	$\{e, a, b\}$	$\{e, a, b\}$	c

Note that the above definition is a generalization of the group action. Let G be a group and Ω be a non-empty set. A *(left) group action* is a binary operator from $G \times \Omega$ to Ω that satisfies the following two axioms: ${}^{gh}\omega = {}^g({}^h\omega)$ and ${}^e\omega = \omega$, for all $g, h \in G$ and $\omega \in \Omega$.

Now, we give the notion of crossed polymodule.

DEFINITION 2.1

A *crossed polymodule* $\mathcal{X} = (C, P, \partial, \alpha)$ consists of polygroups $\langle C, \star, e, ^{-1} \rangle$ and $\langle P, \circ, e, ^{-1} \rangle$ together with a strong homomorphism $\partial : C \to P$ and a (left) action $\alpha : P \times C \to \mathcal{P}^*(C)$ on C, satisfying the conditions:

(1) $\partial({}^{p}c) = p \circ \partial(c) \circ p^{-1}$, for all $c \in C$ and $p \in P$,
(2) ${}^{\partial(c)}c' = c \star c' \star c^{-1}$, for all $c, c' \in C$.

Example 5. A conjugation crossed polymodule is an inclusion of a normal subpolygroup N of P, with action given by conjugation. In particular, for any polygroup P the identity map $\mathrm{Id}_P : P \to P$ is a crossed polymodule with the action of P on itself, by conjugation. Indeed, there are two canonical ways in which a polygroup P may be regarded as a crossed polymodule: via the identity map or via the inclusion of the trivial subpolygroup.

Example 6. If C is a P-polymodule, then there is a well defined action α of P on C. This together with the zero homomorphism yields a crossed polymodule $(C, P, 0, \alpha)$.

Example 7. The direct product of $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed polymodules has source $C_1 \times C_2$, range $P_1 \times P_2$ and boundary homomorphism $\partial_1 \times \partial_2$ with $P_1 \times P_2$ acting trivially on $C_1 \times C_2$.

Note that the above definition is a generalization of the notion of crossed module. We recall that a *crossed module* $X = (M, G, \partial, \tau)$ consists of groups M and G together with a homomorphism $\partial : M \to G$ and a (left) action $\tau : G \times M \to M$ on M, satisfying the conditions: $\partial({}^{g}m) = g\partial(m)g^{-1}$, for all $m \in M$, $g \in G$ and ${}^{\partial(m)}m' = mm'm^{-1}$, for all $m, m' \in M$.

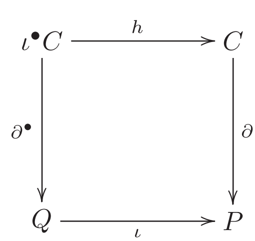
3. Pullback crossed polymodules

In this section we define pullback crossed polymodule.

DEFINITION 3.1

Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule and $\iota : Q \to P$ be a morphism of polygroups. Then $\iota^{\bullet}\mathcal{X} = (\iota^{\bullet}C, Q, \partial^{\bullet}, \alpha^{\bullet})$ is the pullback of $\mathcal{X}$ by ι, where $\iota^{\bullet}C = \{(q, c) \in Q \times C \mid \iota(q) = \partial(c)\}$ and $\partial^{\bullet}(q, c) = q$. The polygroup action of Q on $\iota^{\bullet}C$ is given by

$${}^{q}(q_1, c) = \{(x, y) \mid x \in q \cdot q_1 \cdot q^{-1},\ y \in {}^{\iota(q)} c\},$$



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Note that the above definition is a generalization of pullbacks of crossed modules [1].

Theorem 3.2. *Every pullback crossed module is a pullback crossed polymodule.*

Theorem 3.3. $\iota^{\bullet}\mathcal{X} = (\iota^{\bullet}C, Q, \partial^{\bullet}, \alpha^{\bullet})$ *is a crossed polymodule.*

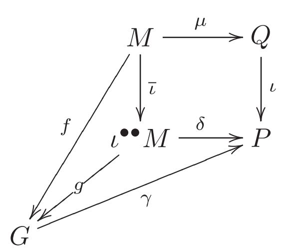
Proof. The verification of the crossed polymodule axioms is given as follows:

$$\begin{aligned} \partial^{\bullet}({}^{q'}(q,c)) &= \partial^{\bullet}(\{(x,y) \mid \; x \in q' \cdot q \cdot {q'}^{-1}, \;\; y \in {}^{\iota(q')}c\}) \\ &= \{x \mid x \in q' \cdot q \cdot {q'}^{-1}\} \\ &= q' \cdot \partial^{\bullet}(q,c) \cdot {q'}^{-1}, \end{aligned} \tag{1}$$

$$\begin{aligned} {}^{\partial^{\bullet}(q',c')}(q,c) &= \{(x,y) \mid (x,y) \in {}^{\partial^{\bullet}(q',c')}(q,c)\} \\ &= \{(x,y) \mid (x,y) \in {}^{q'}(q,c)\}, \;\; \text{by definition of } \partial^{*} \\ &= \{(x,y) \mid (x,y) \in (q' \cdot q \cdot {q'}^{-1}, \; {}^{\iota q'}c)\} \text{ by action definition} \\ &= \{(x,y) \mid x \in q' \cdot q \cdot {q'}^{-1}, y \in {}^{\iota q'}c\} \\ &= \{(x,y) \mid x \in q' \cdot q \cdot {q'}^{-1}, y \in {}^{\partial c'}c\} \text{ since } \iota(q) = \partial(c) \\ &= \{(x,y) \mid x \in q' \cdot q \cdot {q'}^{-1}, y \in c' \star c \star {c'}^{-1}\} \\ &= (q',c') * (q,c) * (q',c')^{-1}, \end{aligned} \tag{2}$$

where $(q,s), (q',s') \in \iota^{*}C$. □

The universal property of induced crossed polymodules is the following [2]. Let $\mathcal{X} = (\mu : M \to Q)$ be a crossed polymodule and let $\iota^{\bullet\bullet}\mathcal{X} = (\delta : \iota^{\bullet\bullet}M \to P)$ be induced by the homomorphism $\iota : Q \to P$. In the diagram



the pair $(\bar{\iota}, \iota)$ is a morphism of crossed polymodules such that for any crossed R-module $\mathcal{Y} = (\gamma : G \to P)$ and any morphism of crossed polymodules $(f, \iota) : \mathcal{X} \to \mathcal{Y}$, there is a unique morphism $(g, 1) : \iota^{\bullet\bullet}\mathcal{X} \to \mathcal{Y}$ of crossed polymodules such that $g\bar{\iota} = f$.

PROPOSITION 3.4

Let $\langle C, \star, e, {}^{-1}\rangle$ *and* $\langle P, \circ, e, {}^{-1}\rangle$ *be two polygroups and let* $\partial : C \to P$ *be a strong homomorphism. Then,* ∂ *induces a group homomorphism* $\mathcal{D} : C/\beta^{*}_{C} \to P/\beta^{*}_{P}$ *by setting*

$$\mathcal{D}(\beta^{*}_{C}(c)) = \beta^{*}_{P}(\partial(c)), \quad \textit{for all } c \in C.$$

We say the action of P on C is *productive*, if for all $c \in C$ and $p \in P$ there exist $c_1, \ldots, c_n$ in C such that $c^{p} = c_1 \star \cdots \star c_n$.

Example 8. The action defined in Example 3 is productive.

Let $\langle C, \star, e, ^{-1} \rangle$ and $\langle P, \circ, e, ^{-1} \rangle$ be two polygroups and let $\alpha : P \times C \to \mathcal{P}^*(C)$ be a productive action on C. We define the map $\psi : P/\beta^*_P \times P/\beta^*_C \to \mathcal{P}^*(P/\beta^*_C)$ in the usual manner:

$$\psi(\beta^*_P(p), \beta^*_C(c)) = \left\{ \beta^*_C(x) \mid x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ z \in \beta^*_P(p)}} {}^z y \right\}.$$

By definition of β^*_C, since the action of P on C is productive, we conclude that $\psi(\beta^*_P(p), \beta^*_C(c)$ is singleton, i.e., we have

$$\psi : P/\beta^*_P \times P/\beta^*_C \to P/\beta^*_C,$$
$$\psi(\beta^*_P(p), \beta^*_C(c)) = \beta^*_C(x), \text{ for all } x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ z \in \beta^*_P(p)}} {}^z y.$$

We denote $\psi(\beta^*_P(p), \beta^*_C(c)) = {}^{[\beta^*_P(p)]}\big[\beta^*_C(c)\big]$.

PROPOSITION 3.5

Let $\langle C, \star, e, ^{-1} \rangle$ and $\langle P, \circ, e, ^{-1} \rangle$ be two polygroups and let $\alpha : P \times C \to \mathcal{P}^*(C)$ be a productive action on C. Then, ψ is an action of the group P/β^*_P on the group P/β^*_C.

Theorem 3.6. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule such that the action of P on C is productive. Then, $\mathcal{X}_{\beta^*} = (C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$ is a crossed module.*

COROLLARY 3.7

Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule such that the action of P on C is productive and $\iota : Q \to P$ be a morphism of polygroups. Then, $(\iota^*)^\bullet = ((\iota^*)^\bullet(C/\beta^*_C), Q/\beta^*_Q, \mathcal{D}^\bullet, \psi^*)$ is the pullback of $\mathcal{X}_{\beta^*} = (C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$ by ι^*, where

$$\iota^* : Q/\beta^*_Q \to P/\beta^*_P, \quad \iota^*_Q(\beta^*(q)) = \beta^*_P(\iota(q)),$$
$$(\iota^*)^\bullet(C/\beta^*_C) = \{(\beta^*_Q(q), \beta^*_C(c)) \mid \iota^*(\beta^*(q)) = \mathcal{D}(\beta^*_C(c))\},$$
$$\mathcal{D}^\bullet(\beta^*_Q(q), \beta^*_C(c)) = \beta^*_Q(q).$$

We can conclude the following theorem from §3 in [1]. For the sake of completeness, we give the proof by our notations.

Theorem 3.8. $(\iota^*)^\bullet = ((\iota^*)^\bullet(C/\beta^*_C),\ Q/\beta^*_Q, \mathcal{D}^\bullet,\ \psi^*)$ *is a crossed module.*

Proof. For the first axiom of crossed module, we have

$$\begin{aligned}
&\mathcal{D}^\bullet({}^{[\beta^*_Q(q)]}[(\beta^*_Q(q), \beta^*_C(c))]) \\
&\quad = \mathcal{D}^\bullet(\beta^*_Q(q') \oslash \beta^*_Q(q) \oslash \beta^*_Q(q')^{-1},\ {}^{[\iota^*(\beta^*_Q(q))]}[\beta^*_C(c)]) \\
&\quad = \beta^*_Q(q') \oslash \beta^*_Q(q) \oslash \beta^*_Q(q')^{-1} \\
&\quad = \beta^*_Q(q') \oslash \mathcal{D}^\bullet(\beta^*_Q(q), \beta^*_C(c)) \oslash \beta^*_Q(q')^{-1}.
\end{aligned}$$

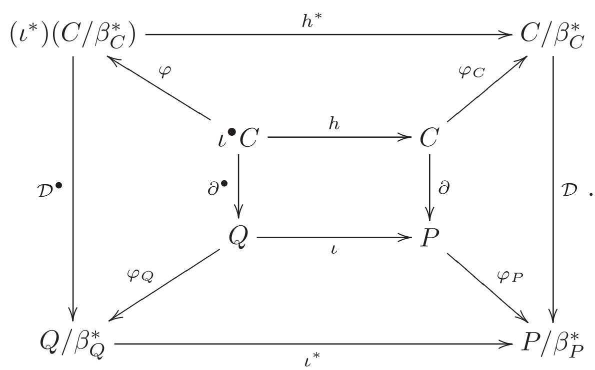
For the second axiom of crossed module, we have

$$
\begin{aligned}
&(\beta_Q^*(q'), \beta_C^*(c'))^{-1} \circledast (\beta_Q^*(q), \beta_C^*(c)) \circledast (\beta_Q^*(q'), \beta_C^*(c')) \\
&\quad = (\beta_Q^*(q')^{-1}, \beta_C^*(c')^{-1}) \circledast (\beta_Q^*(q), \beta_C^*(c)) \circledast (\beta_Q^*(q'), \beta_C^*(c')) \\
&\quad = (\beta_Q^*(q')^{-1} \oslash \beta_Q^*(q) \oslash \beta_Q^*(q'),\ \beta_C^*(c')^{-1} \otimes \beta_C^*(c) \otimes \beta_C^*(c')) \\
&\quad = ({}^{[\beta_Q^*(q')]}[\beta_Q^*(q)],\ {}^{[\mathcal{D}\beta_C^*(c')]}[\beta_C^*(c)]) \\
&\quad = {}^{[\beta_Q^*(q')]}[\beta_Q^*(q), \beta_C^*(c)] \\
&\quad = {}^{[\mathcal{D}^\bullet(\beta_Q^*(q), \beta_C^*(c))]}[(\beta_Q^*(q), \beta_C^*(c))].
\end{aligned}
$$

□

COROLLARY 3.9

The following diagram is commutative:



4. Pushouts of crossed polymodules

Let X be a nonempty subset of a polygroup $\langle P, \circ, e, ^{-1} \rangle$. Let $\{A_i \mid i \in J\}$ be the family of all subpolygroups of P which contain X. Then, $\bigcap_{i \in J} A_i$ is called the subpolygroup generated by X. This subpolygroup is denoted by $\langle X \rangle$ and we have $\langle X \rangle = \cup\{x_1^{\varepsilon_1} \circ \cdots \circ x_k^{\varepsilon_k} \mid x_i \in X, k \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}\}$.

Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule and let $\langle H, \cdot, e, ^{-1} \rangle$ be a polygroup. We know that $\mathcal{P}^*(H)$ is a monoid. Suppose that H' is the group of units of $\mathcal{P}^*(H)$. So, we can consider H' as a polygroup too. Suppose that H' be the polygroup of invertible and reversible elements of $\mathcal{P}^*(H)$.

DEFINITION 4.1

Let $\phi : P \to H'$ be a continuous strong homomorphism of polygroups. Consider polygroup $\iota_\bullet(C)$ generated by $C \times H'$ with relations

(1) $(c_1, h) * (c_2, h) = \{(c, h) \mid c \in c_1 \star c_2\}$,

(2) $({}^{p}c, h) = (c, h \cdot \phi(p))$, that is,

$$\{(c', h) \mid c' \in {}^{p}c\} = \{(c, h') \mid h' \in h \cdot \phi(p)\},$$

(3) $(c_1, h_1) * (c_2, h_2) * (c_1, h_1)^{-1} = \{(c_2, h') \mid h' \in h_1 \cdot (\phi\partial(c_1)) \cdot h_1^{-1} \cdot h_2)\}$, for all $h, h_1, h_2 \in H'$, $c, c_1, c_2 \in C$ and $p \in P$.

Define a continuous homomorphism $\partial_\bullet : \iota_\bullet(C) \to H'$ by extending

$$\partial_\bullet(c, h) = h \cdot (\phi\partial(c)) \cdot h^{-1}$$

to the whole of $\iota_\bullet(C)$ and define a continuous H'-polygroup action on the left of $\iota_\bullet(C)$ by ${}^{h}(c, h_1) = \{(c, h') \mid h' \in h \cdot h_1\}$ for $h, h_1 \in H'$, $c \in C$ and a continuous strong homomorphism $\psi : C \to \iota_\bullet(C)$ by $\psi(c) = (c, e)$.

$$\begin{array}{ccc} C & \xrightarrow{\psi} & \iota_\bullet(C) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial_\bullet} \\ P & \xrightarrow[\phi]{} & H' \end{array} .$$

Now, pushout crossed polymodule induces by $\iota_\bullet(C)$, $\partial_\bullet$ and the action.

PROPOSITION 4.2

With the notation above, $\partial_\bullet : \iota_\bullet(C) \to H'$ *is a crossed polymodule over* H'.

Proof. The statement about continuity are fairly trivial. We check the axioms of crossed polymodule as follows:

$$\begin{aligned} \partial_\bullet({}^{h}(c, h_1)) &= \partial_\bullet(\{(c, h') \mid h' \in h \cdot h_1\}) \\ &= \{\partial_\bullet(c, h') \mid h' \in h \cdot h_1\} \\ &= \{h' \cdot (\phi\partial(c)) \cdot h'^{-1} \mid h' \in h \cdot h_1\} \\ &= h \cdot h_1 \cdot (\partial_\bullet(c, h_1)) \cdot h_1^{-1} \cdot h^{-1} \\ &= h \cdot (\partial_\bullet(c, h_1)) \cdot h^{-1}, \end{aligned} \tag{3}$$

$$\begin{aligned} {}^{\partial_\bullet(c,h)}(c_1, h_1) &= {}^{h\cdot(\phi\partial(c))\cdot h^{-1}}(c_1, h_1) \\ &= \{(c_1, h') \mid h' \in h \cdot (\phi\partial(c)) \cdot h^{-1} \cdot h_1\} \\ &= (c, h) * (c_1, h_1) * (c, h)^{-1}. \end{aligned} \tag{4}$$

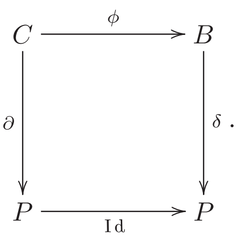
□

PROPOSITION 4.3

Let $\langle P, \circ, e, ^{-1}\rangle$, $\langle C, \star, e, ^{-1}\rangle$ and $\langle B, \cdot, e, ^{-1}\rangle$ *be polygroups. Let* $\partial : C \to P$ *and* $\delta : B \to P$ *be two crossed polymodules and let* $(\phi, \mathrm{Id}) : (\partial : C \to P) \to (\delta : B \to P)$

be a morphism of crossed polymodules. Then, defining a continuous B-action on C by ${}^{b}c = {}^{\delta(b)}c$, we have $\phi : C \to B$ is a crossed polymodule.

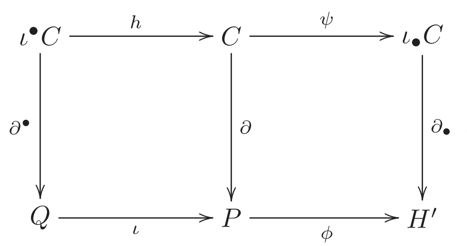
Proof. We can show two crossed polymodules as follows:



where $\partial = \delta\phi$ and $\phi({}^{p}c) = {}^{p}\phi(c)$. We can verify the axioms of crossed polymodule as follows:

(1) $\phi({}^{b}c) = \phi({}^{\delta(b)}c) = {}^{\delta(b)}(\phi(c)) = b \cdot \phi(c) \cdot b^{-1}$.
(2) ${}^{\phi(c_2)}c_1 = {}^{\delta(\phi(c_2))}c_1 = {}^{\delta\phi(c_2)}(c_1) = {}^{\partial(c_2)}c_1 = c_2 \star c_1 \star c_2^{-1}$. □

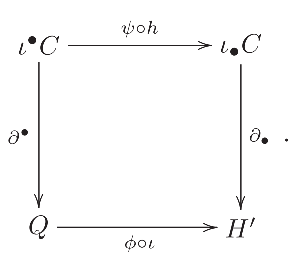
If we consider the pullback and pushout diagrams together, we get the following diagram:



The diagram is commutative, since $\phi\iota\partial^{\bullet} = \partial_{\bullet}\psi h$. Therefore

$$\begin{aligned} \phi\iota\partial^{\bullet}(q, c) &= \phi\iota(q) \\ &= \phi\partial(h(q, c)) \\ &= \partial_{\bullet}(h(q, c), e) \\ &= \partial_{\bullet}\psi(h(q, c)) \\ &= \partial_{\bullet}\psi h(q, c). \end{aligned}$$

Then, we can have the following commutative diagram:



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