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# Co-Roman domination in graphs

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**Abstract.** Let $G = (V, E)$ be a graph and let $f : V \to \{0, 1, 2\}$ be a function. A vertex $u$ is said to be protected with respect to $f$ if $f(u) > 0$ or $f(u) = 0$ and $u$ is adjacent to a vertex with positive weight. The function $f$ is a co-Roman dominating function (CRDF) if: (i) every vertex in $V$ is protected, and (ii) each $v \in V$ with $f(v) > 0$ has a neighbor $u \in V$ with $f(u) = 0$ such that the function $f_{vu} : V \to \{0, 1, 2\}$, defined by $f_{vu}(u) = 1$, $f_{vu}(v) = f(v) - 1$ and $f_{vu}(x) = f(x)$ for $x \in V \setminus \{u, v\}$ has no unprotected vertex. The weight of $f$ is $w(f) = \sum_{v \in V} f(v)$. The co-Roman domination number of a graph $G$, denoted by $\gamma_{cr}(G)$, is the minimum weight of a co-Roman dominating function on $G$. In this paper we initiate a study of this parameter, present several basic results, as well as some applications and directions for further research. We also show that the decision problem for the co-Roman domination number is NP-complete, even when restricted to bipartite, chordal and planar graphs.

**Keywords.** Domination; Roman domination; weak Roman domination; co-Roman domination.

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## 1. Introduction

Throughout this paper, $G = (V, E)$ represents a simple graph (with neither loops nor multiple edges) with vertex set $V$ and edge set $E$. The order of $G$ is $n = |V|$. For $v \in V$, its open neighbourhood is $N(v) = \{u \in V : uv \in E\}$, and its closed neighbourhood is $N[v] = N(v) \cup \{v\}$. Given a set $X \subseteq V$ and $v \in X$, the set of external private neighbors of $v$ with respect to $X$ (in $G$) is defined by $EPN(v, X) = \{u \in V \setminus X : N(u) \cap X = \{v\}\}$.

Suppose that one or more guards are stationed at some of the vertices of a simple $n$-vertex graph $G = (V, E)$, and a guard at vertex $v$ can deal with a problem at any vertex in its closed neighborhood. The situation can be represented as a function $f : V \to \{0, 1, 2, \ldots, k\}$ where $f(v)$ is the number of guards at $v$ and $V_i(f) = \{v \in V : f(v) = i\}$. Alternatively, we can identify $f$ with the partition of $V$ induced by $f$ and write $f = (V_0(f), V_1(f), V_2(f), \ldots, V_k(f))$, or simply $f = (V_0, V_1, V_2, \ldots, V_k)$, when the function is clear from the context. The weight of $f$, $w(f) = \sum_{v\in V} f(v) = \sum_{i=1}^{k} i|V_i(f)|$ represents the total number of guards used by $f$. Burger *et al.* [5] introduced the concept of safe function. A function $f$ is a safe function (more intuitively, $G$ is safe under $f$) if there is at least one guard available to handle a problem at any vertex. Formally, $f = (V_0, V_1, V_2, \ldots, V_k)$ is safe if each $v \in V_0$ is adjacent to at least one vertex of $V \backslash V_0$ (equivalently, if $V \backslash V_0$ is a dominating set of $G$). If a vertex $u$ is in $V \backslash V_0$ or is adjacent to a vertex $v$ in $V \backslash V_0$, we say that $u$ is protected (with respect to $f$).

Suppose that guards are stationed according to the function $f$. When a guard at $v \in V \backslash V_0$ moves along an edge to deal with a problem at $u \in V_0$, the new number of guards at each vertex are given by the function $f_{vu}$:

$$f_{vu}(x) = \begin{cases} f(x) - 1, & \text{if } x = v, \\ 1, & \text{if } x = u, \\ f(x), & \text{otherwise.} \end{cases}$$

Given a safe function $f$, $u \in V_0(f)$ and $v \in N(u) \cap (V \setminus V_0(f))$, we say that $u$ is defended by $v$ (or $v$ defends $u$) if $f_{vu}$ is safe. Alternatively, we say that $u$ replaces $v$ (or $v$ is replaced by $u$) if $f_{vu}$ is safe. Observe that $u \in V_0(f)$ is defended by every $v \in N(u) \cap V_i(f)$ for every $i > 1$.

The study of safe functions of the form $f = (V_0, V_1)$ or $f = (V_0, V_1, V_2)$ in which every vertex in $V_0$ is defended by a vertex in $V \setminus V_0$, has resulted in several new concepts such as Roman domination, weak Roman domination and secure domination, which have been investigated by several authors [5, 6, 8–11, 14, 15]. We describe six such classes of safe functions.

(*i*) *Domination*

A dominating function (DF) of $G$ is a safe function $f = (V_0, V_1)$, that is, a safe function with at most one guard per vertex. In this case, guards are not supposed to move. Clearly $f$ is a dominating function if and only if $V_1$ is a dominating set of $G$, and the minimum number of guards is $\gamma(G)$, the domination number of $G$. A DF of minimum weight is a $\gamma(G)$-function. This method of protection has, of course, been studied extensively. For an exhaustive treatment of the fundamentals of domination and surveys of several advanced topics we refer to the books by Haynes *et al.* [12, 13].

(*ii*) *Roman domination*

A Roman dominating function (RDF) of $G$ is a function $f = (V_0, V_1, V_2)$ such that each $u \in V_0$ is adjacent to at least one vertex $v \in V_2$. In this case, if a guard at $v$ moves to $u$, there is still one guard remaining at $v$. Hence for every $u \in V_0$ and $v \in N(u) \cap V_2$, $f_{vu}$ is safe. The minimum number of guards used in an RDF is called the Roman domination number of $G$ and is denoted by $\gamma_R(G)$. An RDF of minimum weight is a $\gamma_R(G)$-function. This method of protection has historical motivation, and was introduced formally in [8].

(*iii*) *Weak Roman domination*

A weak Roman dominating function (WRDF) of $G$ is a safe function $f = (V_0, V_1, V_2)$ such that for each $u \in V_0$, there exists $v \in N(u) \cap (V \setminus V_0)$ such that $f_{vu}$ is safe. The minimum number of guards in a WRDF is called the weak Roman domination number of $G$ and is denoted by $\gamma_r(G)$. A WRDF of minimum weight is a $\gamma_r(G)$-function. This concept of protection was introduced in [14] and studied in [5, 6]. We observe that in an RDF, every vertex $u$ in $V_0$ is defended by a vertex in $V_2$ and in a WRDF, $u$ is defended by a vertex in $V_1 \cup V_2$.

(*iv*) *Secure domination*

A secure dominating function (SDF) of $G$ is a safe function $f = (V_0, V_1)$ such that for each $u \in V_0$, there exists $v \in N(u) \cap V_1$ such that $f_{vu}$ is safe. The idea is the same as in the previous case, with the restriction that at most one guard is allowed at any given vertex, instead of two. The minimum weight of an SDF is the secure domination number of $G$, denoted by $\gamma_s(G)$. An SDF of minimum weight is a $\gamma_s(G)$-function. This concept of protection was introduced in [11] and studied in [7, 9, 10, 15].

(*v*) *Foolproof weak Roman domination*

A foolproof weak Roman dominating function (FWRDF) is a safe function $f = (V_0, V_1, V_2)$ such that for each $u \in V_0$ and each $v \in N(u) \cap (V \setminus V_0)$, the function $f_{vu}$ is safe. Here, given a vertex $u$ without a guard, every guard in an adjacent vertex $v$ can move to $u$, and the new assignment of guards is still safe. The minimum weight of an FWRDF is the foolproof weak Roman domination number of $G$, denoted by $\gamma_r^*(G)$. An FWRDF of minimum weight is a $\gamma_r^*(G)$-function. This concept was introduced in [5].

(*vi*) *Foolproof secure domination*

A foolproof secure dominating function (FSDF) is a safe function $f = (V_0, V_1)$ such that for each $u \in V_0$ and each $v \in N(u) \cap V_1$, the function $f_{vu}$ is safe. The idea is the same as in the previous case, with the restriction that at most one guard is allowed at any given vertex, instead of two. The minimum weight of an FSDF is the foolproof secure domination number of $G$, denoted by $\gamma_s^*(G)$. An FSDF of minimum weight is a $\gamma_s^*(G)$-function. This concept was introduced in [5].

In all the above six strategies of graph protection, the main focus is on defending an unguarded vertex. However in several practical applications, apart from defending the network, one has to consider the problem of replacing a guard by another. To address this problem, we introduced the following concept of co-secure domination in [2].

DEFINITION 1.1 [2]

A co-secure dominating function (CSDF) is a safe function $f = (V_0, V_1)$ such that for each $v \in V_1$, there exists $u \in N(v) \cap V_0$ such that $f_{vu}$ is safe. The minimum weight of a CSDF is the co-secure domination number of $G$, denoted by $\gamma_{cs}(G)$. A CSDF of minimum weight is a $\gamma_{cs}(G)$-function.

In this paper, we consider the problem of replacement in the context of a safe function $f = (V_0, V_1, V_2)$, in which up to two guards are allowed at each vertex, and we introduce the concept of co-Roman domination. We present several basic results on the corresponding parameter $\gamma_{cr}(G)$.

We need the following definition and results.

DEFINITION 1.2

The *corona* of two disjoint graphs $G_1$ and $G_2$ is defined to be the graph $G = G_1 \circ G_2$, formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i$-th vertex of $G_1$ is adjacent to every vertex in the $i$-th copy of $G_2$.

**Theorem 1.3 [5].** *For any graph $G$,*

$$\gamma \leq \gamma_r \leq \begin{cases} \gamma_r^* \leq \gamma_R \leq 2\gamma \\ \gamma_s \leq \gamma_s^* \end{cases} .$$

PROPOSITION 1.4 [8]

*Let $f = (V_0, V_1, V_2)$ be a $\gamma_R$-function of an isolate-free graph $G$, such that $|V_1|$ is minimum. Then*

(i) *$V_1$ is independent, and $V_0 \cup V_2$ is a vertex cover.*
(ii) *$V_0$ dominates $V_1$.*
(iii) *Each vertex of $V_0$ is adjacent to at most one vertex of $V_1$.*

PROPOSITION 1.5 [2]

*Let $G$ be a graph of order $n \geq 2$. Then $\gamma_{cs}(G) = n - 1$ if and only if $G$ is a star.*

**Theorem 1.6 [2].** *Let $T$ be a tree. Then $\gamma_{cs}(T) = n - 2$ if and only if $T$ is*: (*i*) *a double star or* (*ii*) *a double star with at least one vertex of degree* $\geq 3$, *whose middle edge is subdivided once.*

## 2. Co-Roman domination – Basic results

DEFINITION 2.1

A co-Roman dominating function (CRDF) of $G$ is a safe function $f = (V_0, V_1, V_2)$ such that for each $v \in V \backslash V_0$, there exists $u \in N(v) \cap V_0$, such that $f_{vu}$ is safe. In this case we say that $u$ replaces $v$. The minimum weight of a CRDF is called the co-Roman domination number of $G$ and is denoted by $\gamma_{cr}(G)$. A CRDF of minimum weight is a $\gamma_{cr}(G)$-function.

Notice that in a graph with isolated vertices, neither a CSDF nor a CRDF exist. Moreover, if $G$ is a graph with connected components $G_1, \ldots, G_k$, and $f$ is a CSDF or a CRDF of $G$, then $w(f) = \sum_{i=1}^{k} w(f_i)$, where $f_i$ is the restriction of $f$ to $G_i$. Therefore, from now on all graphs considered will be connected and non-trivial.

We observe that a safe function $f = (V_0, V_1, V_2)$ is a weak Roman dominating function if every vertex in $V_0$ is defended, and it is a co-Roman dominating function if every vertex in $V_1 \cup V_2$ can be replaced by a vertex in $V_0$. A Roman (or weak Roman) dominating function is not necessarily a CRDF. For example, for the path $P_5 = (v_1, v_2, v_3, v_4, v_5)$, the function $f_1 = (V_0 = \{v_1, v_3\}, V_1 = \{v_4, v_5\}, V_2 = \{v_2\})$ is a $\gamma_R(G)$-function, and $f_2 = (V_0 = \{v_1, v_5\}, V_1 = \{v_2, v_3, v_4\}, V_2 = \emptyset)$ is a $\gamma_r(G)$-function, but none of them is a CRDF. Similarly, $f_3 = (V_0 = \{v_1, v_3, v_5\}, V_1 = \{v_2, v_4\}, V_2 = \emptyset)$ is a $\gamma_{cr}(G)$-function which is neither an RDF nor a WRDF.

We now proceed to determine $\gamma_{cr}$ for standard graphs.

We observe that if $G$ has a vertex of degree $n-1$, then $\gamma_{cr}(G) \leq 2$. Also, $\gamma_{cr}(G) = 1$ if and only if $G$ has at least two vertices of degree $n-1$. Hence $\gamma_{cr}(K_n) = 1$ for all $n \geq 2$. Furthermore, $\gamma_{cr}(W_n) = 2$ for all $n \geq 5$, where $W_n = C_{n-1} + \{v\}$ is the wheel on $n$ vertices.

*Lemma* 2.2. *If $P_m$ is any path of order $m \geq 2$, then $\gamma_{cr}(P_{m+5}) = \gamma_{cr}(P_m) + 2$.*

*Proof.* Let $P_m = (u_1, u_2, \ldots, u_{m-1}, u)$, $P = (v_1, v_2, \ldots, v_5)$ and $P_{m+5} = (u_1, u_2, \ldots, u_{m-1}, u, v_1, v_2, \ldots, v_5)$. For any $\gamma_{cr}(P_m)$-function $f'$, the function $f : V \to \{0, 1, 2\}$ defined by $f(v_2) = f(v_4) = 1,\ f(v_1) = f(v_3) = f(v_5) = 0,\ f(x) = f'(x)$ for every $x \in V(P_m))$, is a CRDF of $P_{m+5}$. Hence, $\gamma_{cr}(P_{m+5}) \leq \gamma_{cr}(P_m) + 2$.

For the other inequality, first we will show that there exists a minimum CRDF $f$ of $P_{m+5}$ such that $f(V(P)) = 2$. Let $f'$ be any minimum CRDF of $P_{m+5}$. It is clear that $f'(V(P)) \geq 2$. Suppose $f'(V(P)) \geq 3$. If $f'(u) > 0$ then $f : V \to \{0, 1, 2\}$ defined by $f(v_2) = f(v_4) = 1,\ f(v_1) = f(v_3) = f(v_5) = 0,\ f(x) = f'(x)$ for every $x \in V(P_m)$, is a CRDF of $P_{m+5}$, and $w(f) \leq w(f') - 1$, which is a contradiction. Therefore, $f'(u) = 0$.

Suppose $f'(u_{m-1}) = 0$. Then $f : V \to \{0,1,2\}$, defined by $f(v_2) = f(v_4) = f(u) = 1$, $f(v_1) = f(v_3) = f(v_5) = 0,\ f(x) = f'(x)$ for every $x \in V(P_m)\backslash\{u\}$, is a CRDF of $P_{m+5}$, and $w(f) \leq w(f')$. Now suppose $f'(u_{m-1}) > 0$. If $EPN(u_{m-1}, V_1(f') \cup V_2(f')) \subseteq \{u\}$, then $f : V \to \{0, 1, 2\}$, defined by $f(v_2) = f(v_4) = 1,\ f(v_1) = f(v_3) = f(v_5) = 0,\ f(x) = f'(x)$ for every $x \in V(P_m)$, is a CRDF of $P_{m+5}$, and $w(f) \leq w(f')-1$, which is a contradiction. It follows that $u_{m-2} \in EPN(u_{m-1}, V_1(f') \cup V_2(f'))$, and $f : V \to \{0, 1, 2\}$, defined by $f(v_2) = f(v_4) = f(u) = f(u_{m-2}) = 1,\ f(v_1) = f(v_3) = f(v_5) = f(u_{m-1}) = 0,\ f(x) = f'(x)$ for every $x \in V(P_m)\backslash \{u_{m-2}, u_{m-1}, u\}$, is a CRDF of $P_{m+5}$ such that $w(f) \leq w(f')$. Therefore, in any case there exists a minimum CRDF $f$ of $P_{m+5}$ such that $f(V(P)) = 2$.

Now let $f$ be a minimum CRDF of $P_{m+5}$ such that $f(V(P)) = 2$. Suppose $f(v_1) > 0$. Since $V_1(f) \cup V_2(f)$ is a dominating set and $f(V(P)) = 2$, $v_4$ must have positive weight. Hence $\{v_1, v_4\} \subseteq V_1(f)$. However, in this case neither $v_3$ nor $v_5$ can replace $v_4$. It follows that $f(v_1) = 0$, so $\{v_2, v_4\} \subseteq V_1(f)$. Therefore, the restriction of $f$ to $V(P_m)$ is a CRDF of $P_m$, since no vertex of $P_m$ is adjacent to a vertex of $P$ with positive weight. Hence, $\gamma_{cr}(P_m) \leq \gamma_{cr}(P_{m+5}) - 2$. □

PROPOSITION 2.3

*For the path $P_n$, $\gamma_{cr}(P_n) = \left\lceil \frac{2n}{5} \right\rceil$.*

*Proof.* We proceed by induction on $n$. It is straightforward to verify the result for $2 \leq n \leq 6$. Assume that the result holds for all paths of order less than $n$, where $n \geq 7$. Let $P = (v_1, v_2, \ldots, v_n)$ be a path of order $n$, and let $P'$ be the path $(v_6, \ldots, v_n)$. From the inductive hypothesis, $\gamma_{cr}(P') = \left\lceil \frac{2(n-5)}{5} \right\rceil$. By Lemma 2.2, $\gamma_{cr}(P) = \gamma_{cr}(P') + 2 = \left\lceil \frac{2(n-5)}{5} \right\rceil + 2 = \left\lceil \frac{2n}{5} \right\rceil$. □

Next we consider the co-Roman domination number of a cycle. For this purpose, we shall need the following observation:

*Observation* 2.4. If $H$ is a spanning subgraph of a graph $G$, then $\gamma_{cr}(G) \leq \gamma_{cr}(H)$, since any CRDF of $H$ is also a CRDF of $G$.

PROPOSITION 2.5

*For the cycle $C_n$ with $n \geq 4$, $\gamma_{cr}(C_n) = \gamma_{cr}(P_n) = \lceil \frac{2n}{5} \rceil$.*

*Proof.* It is straightforward to verify the result for $4 \leq n \leq 9$. Suppose that $n \geq 10$. By Proposition 2.3 and Observation 2.4, $\gamma_{cs}(C_n) \leq \gamma_{cs}(P_n) = \lceil \frac{2n}{5} \rceil$. Now, let $f = (V_0, V_1, V_2)$ be a $\gamma_{cr}$-function of $C_n = (v_1, \ldots, v_n)$. Since $n \geq 10$, at least two adjacent vertices of $C_n$ must be in $V_0$, for otherwise $w(f) \geq \lceil \frac{n}{2} \rceil > \lceil \frac{2n}{5} \rceil$. Without loss of generality, let $\{v_1, v_n\} \subseteq V_0$. Since the only vertex dominating $v_1$ is $v_2$, and the only vertex dominating $v_n$ is $v_{n-1}$, we have that $f$ is a CRDF of $C_n - v_n v_1 \cong P_n$. This implies that $\gamma_{cr}(C_n) \geq \gamma_{cr}(P_n) = \lceil \frac{2n}{5} \rceil$. Consequently, $\gamma_{cr}(C_n) = \lceil \frac{2n}{5} \rceil$. □

PROPOSITION 2.6

*For the complete bipartite graph $K_{p,q}$ with $p \leq q$,*

$$\gamma_{cr}(K_{p,q}) = \begin{cases} 1, & \textit{if } p = q = 1 \\ 2, & \textit{if } p = 1, 2, q > 1, \\ 3, & \textit{if } p = 3, \\ 4, & \textit{if } p \geq 4. \end{cases}$$

*Proof.* Let $P = \{u_1, \ldots, u_p\}$ and $Q = \{v_1, \ldots, v_q\}$ be the bipartition of $G = K_{p,q}$. The result is obvious for $p = q = 1$. If $p = 1$ and $q > 1$, then the only vertex dominating all other vertices is $u_1$ and hence $\gamma_{cr}(G) \geq 2$. On the other hand, $f = (Q, \emptyset, P)$ is a CRDF of $G$, so $\gamma_{cr}(G) = 2$. If $p = 2$, then no single vertex dominates all other vertices, so $\gamma_{cr}(G) \geq 2$. Also $f = (Q, P, \emptyset)$ is a CRDF of $G$, so $\gamma_{cr}(G) = 2$. If $p = 3$, any function $f = (V_0, V_1, V_2)$ with $V_1 \cap P = \{u_i\}$ or $V_1 \cap Q = \{v_j\}$ is not a CRDF, since neither $u_i$ nor $v_j$ can be replaced by any vertex. Hence $\gamma_{cr}(G) > 2$. Also, $f = (Q, P, \emptyset)$ is a CRDF of $G$, so $\gamma_{cr}(G) = 3$. Now, let $p \geq 4$ and let $f = (V_0, V_1, V_2)$ be any CRDF of $G$. If $w(f) \leq 3$, then either $V_1 \cap P = \{u_i\}$ or $V_1 \cap Q = \{v_j\}$. Therefore, in this case neither $u_i$ nor $v_j$ can be replaced by any vertex. Thus $\gamma_{cr}(G) \geq 4$. Also, $f = ((Q \cup P) \backslash \{u_1, v_1\}, \emptyset, \{u_1, v_1\})$ is a CRDF of $G$, so $\gamma_{cr}(G) = 4$. □

PROPOSITION 2.7

*For the graph $G = K_{p_1, p_2, \ldots, p_t}$ where $p_1 \leq p_2 \leq \cdots \leq p_t$ and $t \geq 3$,*

$$\gamma_{cr}(G) = \begin{cases} 1, & \textit{if } p_1 = p_2 = 1, \\ 2, & \textit{otherwise.} \end{cases}$$

*Proof.* Denote the partite sets by $S_i = \{u_{i,j} : j = 1, 2, \ldots, p_i\}, i = 1, 2, \ldots, t$. The result is obvious for $p_1 = p_2 = 1$. In all other cases, $f = (V(G) \setminus \{u_{11}\}, \emptyset, \{u_{11}\})$ is a CRDF of $G$ and hence $\gamma_{cr}(G) = 2$. □

## 3. $\gamma_{cr}$ and other parameters

In this section we discuss the relation of $\gamma_{cr}(G)$ with the parameters $\gamma_R(G)$, $\gamma_r(G)$, $\gamma_r^*(G)$, $\gamma_s^*(G)$ and $\gamma_{cs}(G)$. Wherever possible we will use abbreviations: $\gamma$ for $\gamma(G)$, $\gamma_R$ for $\gamma_R(G)$, etc.

*Observation* 3.1. Let $G$ be a graph and let $f = (V_0, V_1, V_2)$ be a safe function of minimum weight in $G$. Then $N(v) \cap V_0 \neq \emptyset$ for all $v \in V_2$. Also if $f = (V_0, V_1, V_2)$ is a safe function, then $f_{vu}$ is safe for all $u \in V_0$ and $v \in N(u) \cap V_2$.

**Theorem 3.2.** *For every graph $G$, we have $\gamma_{cr} \leq \gamma_R$.*

*Proof.* Let $f = (V_0, V_1, V_2)$ be a $\gamma_R$-function of $G$ such that $|V_1|$ is minimum. We claim that $f$ is a CRDF of $G$. Let $v \in V_1 \cup V_2$. If $v \in V_2$, it follows from Observation 3.1 that $f_{vu}$ is safe for every $u \in N(v) \cap V_0$. If $v \in V_1$, from Proposition 1.4 it follows that $f_{vu}$ is safe for every $u \in N(v) \cap V_0$, since $v$ is adjacent to $u$ and every vertex in $V_0(f_{vu})\backslash\{v\} = V_0(f)\backslash\{u\}$ is adjacent to a vertex in $V_2(f_{vu}) = V_2(f)$. Hence $f$ is a CRDF of $G$, so $\gamma_{cr} \leq w(f) = \gamma_R$. □

**Theorem 3.3.** *For every graph $G$, we have $\gamma_{cr} \leq \gamma_r$.*

*Proof.* We first prove that there exists a $\gamma_r$-function $g = (V_0, V_1, V_2)$ of $G$ such that $N(v) \cap V_0 \neq \emptyset$ for all $v \in V_1 \cup V_2$. Let $f = (V_0, V_1, V_2)$ be any $\gamma_r$-function of $G$. Suppose there exists $v \in V_1 \cup V_2$ such that $N(v) \cap V_0 = \emptyset$. It follows from Observation 3.1 that $v \in V_1$. Now, if there exists $w \in N(v) \cap V_2$, then $f_1 = (V_0 \cup \{v\}, V_1\backslash\{v\}, V_2)$ is a WRDF of $G$ and $w(f_1) = w(f) - 1$, a contradiction. Hence $N(v) \cap V_2 = \emptyset$. Since $N(v) \cap V_0 = N(v) \cap V_2 = \emptyset$, it follows that $N(v) \subseteq V_1$. Now, let $u \in N(v)$. Then $f' = (V_0 \cup \{v\}, V_1\backslash\{u, v\}, V_2 \cup \{u\})$ is a WRDF of $G$ and $w(f') = w(f)$. Thus for any $v \in V_1 \cup V_2$ with $N(v) \cap V_0 = \emptyset$, we have constructed another WRDF $f'$ with $w(f') = w(f)$ such that $v \in V_0$. By repeating this process, we obtain a WRDF $g = (V_0, V_1, V_2)$ with $w(g) = w(f) = \gamma_r$ such that $N(v) \cap V_0 \neq \emptyset$ for all $v \in V_1 \cup V_2$. We now claim that $g$ is a CRDF. Let $v \in V_1 \cup V_2$. If $v \in V_2$, then any vertex $u \in N(v) \cap V_0$ replaces $v$. Suppose $v \in V_1$. Since $g$ is a WRDF of $G$, we have $|EPN(v, V_1 \cup V_2)| \leq 1$. If $EPN(v, V_1 \cup V_2) = \emptyset$, then any vertex $u \in N(v) \cap V_0$ replaces $v$. If $EPN(v, V_1 \cup V_2) = \{z\}$, then $z$ replaces $v$. Thus $g$ is a CRDF of $G$ and $\gamma_{cr} \leq w(g) = \gamma_r$. □

*Observation* 3.4. Let $G$ be a graph. Then any CSDF of $G$ is a CRDF of $G$ and hence $\gamma_{cr} \leq \gamma_{cs}$.

COROLLARY 3.5

*For every graph $G$, the following inequality chains hold*:

(i) $\gamma \leq \gamma_{cr} \leq \gamma_r \leq \gamma_r^* \leq \gamma_R \leq 2\gamma$.
(ii) $\gamma \leq \gamma_{cr} \leq \gamma_r \leq \gamma_s \leq \gamma_s^*$.
(iii) $\gamma \leq \gamma_{cr} \leq \gamma_{cs}$.

*Proof.* The result follows from Theorem 1.3, Theorem 3.3 and Observation 3.4. □

The problem of determining graphs for which equality holds for any two of the parameters in the above inequality chains is an interesting direction for further research.

## 4. Co-Roman domination in trees

In this section we present a few basic results on the co-Roman domination number of a tree.

PROPOSITION 4.1

*Let $G$ be a graph. Then $\gamma_{cr}(G) = n - 1$ if and only if $G \cong K_2$ or $K_{1,2}$.*

*Proof.* Suppose $\gamma_{cr} = n - 1$. It follows from (iii) of Corollary 3.5 that $\gamma_{cs}(G) = n - 1$. Hence from Proposition 1.5, $G$ is a star. Now, if $n \geq 4$, then $\gamma_{cr}(G) = 2 \neq n - 1$. Hence $n \leq 3$ and $G \cong K_2$ or $K_{1,2}$. The converse is obvious. □

**Theorem 4.2.** *Let $T$ be a tree. Then $\gamma_{cr}(T) = n - 2$ if and only if $T = K_{1,3}$ or $T$ is a double star with at most two pendant vertices adjacent to each of its support vertices.*

*Proof.* Suppose $\gamma_{cr}(T) = n - 2$. It follows from (iii) of Corollary 3.5 that $\gamma_{cs}(T) = n - 1$ or $n - 2$. If $\gamma_{cs}(T) = n - 1$, then it follows from Proposition 1.5 that $T = K_{1,3}$. Now, suppose $\gamma_{cs}(T) = n - 2$. Then by Theorem 1.6, $T$ is a double star or $T$ is obtained from a double star with at least one vertex of degree $\geq 3$, by subdividing its middle edge exactly once. If $T$ is obtained from a double star by subdividing its middle edge exactly once, then $\gamma_{cr}(T) \leq n - 3$. Hence $T$ is a double star, and $\gamma_{cr}(T) = 2, 3$ or $4$. Therefore every support vertex has at most two adjacent leaves. The converse is obvious. □

In the following two theorems we give sharp upper bounds for the co-Roman domination number of a tree.

**Theorem 4.3.** *If $T$ is a tree, then $\gamma_{cr} \leq \frac{2n}{3}$ and the bound is sharp.*

*Proof.* The proof is by induction on $n$. If $n = 2$, then $\gamma_{cr} = 1$ and if $T$ is a star of order $n \geq 3$, then $\gamma_{cr} = 2$ and in both cases $\gamma_{cr} \leq \frac{2n}{3}$. Hence we may assume that $\mathrm{diam}(T) \geq 3$, which implies $n \geq 4$.

Let $P$ be a longest path in $T$, and let $v$ be a support vertex of $P$. Let $T'$ be the subtree of $T$ obtained by deleting $v$ and its leaf neighbors. Since $T$ is not a star, we have $|V(T')| = n' \geq 2$. From the induction hypothesis, there exists a CRDF $f'$ of $T'$ with weight at most $\frac{2n'}{3}$.

We now define $f : V(T) \to \{0, 1, 2\}$ by

$$f(u) = \begin{cases} 0, & \text{if } u \text{ is a leaf adjacent to } v,\\ 1, & \text{if } u = v \text{ and } \deg\, v = 2,\\ 2, & \text{if } u = v \text{ and } \deg\, v \geq 3,\\ f'(u), & \text{otherwise.} \end{cases}$$

Clearly $f$ is a CRDF of $T$ with $w(f) \leq \frac{2n}{3}$. Hence $\gamma_{cr}(T) \leq \frac{2n}{3}$. For the tree $T = P_k \circ \bar{K}_2$, we have $n = 3k$ and $\gamma_{cr}(T) = 2k = \frac{2n}{3}$. Hence the bound is sharp. □

PROPOSITION 4.4

*Let $T$ be a tree such that no support vertex of $T$ has exactly two leaves adjacent to it. Then $\gamma_{cr}(T) \leq \lceil \frac{n}{2} \rceil$ and the bound is sharp.*

*Proof.* The proof of the inequality is similar to the proof of Theorem 4.3. It follows from Proposition 2.3 that the bound is attained for paths with $n \leq 4$. □

## 5. Complexity

The basic complexity question of the decision problem for the co-Roman domination number takes the following form:

*Co-Roman Domination* (*CRDM*)

*Instance.* A nontrivial connected graph $G = (V, E)$ and a positive integer $k$.

*Question.* Does $G$ have a co-Roman dominating function $f$ with $w(f) \leq k$?

We proceed to prove that CRDM is *NP*-complete even when restricted to bipartite, chordal or planar graphs. The proof is by reduction from the DOMINATING SET problem which is known to be *NP*-complete even when restricted to bipartite [14], chordal [3, 4] or planar graphs (mentioned in [1]).

*Dominating set (DM)*

*Instance.* A graph $G = (V, E)$ without isolated vertices and a positive integer $k$.

*Question.* Does $G$ have a dominating set $S$ with $|S| \leq k$?

**Theorem 5.1.** *CRDM is NP-complete, even when restricted to bipartite, planar or chordal graphs.*

*Proof.* It is clear that CRDM is NP. Let $G, k$ be an instance of DM and let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $H$ be the graph obtained from $G$ by attaching a path $(v_i, v_{i_1}, v_{i_2})$ to every vertex $v_i$ of $G$ and let $k_1 = k + |V(G)|$. We claim that $\gamma_{cr}(H) = \gamma(G) + |V(G)|$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{cr}$-function of $H$. Then for each $i$ exactly one of $v_{i_1}, v_{i_2}$ is in $V_1 \cup V_2$ and we may assume that $v_{i_1} \in V_1 \cup V_2$. Clearly $v_{i_2}$ replaces $v_{i_1}$ and hence $S = (V_1 \cup V_2) \cap V(G)$ is a dominating set of $G$. Thus $\gamma_{cr}(H) = w(f) \geq \gamma(G) + |V(G)|$. Now let $S$ be any minimum dominating set of $G$. Then $f' = ((V(G) - S) \cup \{v_{i_2} : 1 \leq i \leq n\}, S \cup \{v_{i_1} : 1 \leq i \leq n\}, \emptyset)$ is a CRDF of $H$, and so $\gamma_{cr}(H) \leq w(f') = \gamma(G) + |V(G)|$. Thus $\gamma_{cr}(H) = \gamma(G) + |V(G)|$. Hence $G$ has a dominating set of cardinality $k$ or less if and only if $H$ has a CRDF of cardinality $k_1 = k + |V(G)|$ or less. □

## 6. Conclusion and scope

In this paper we have introduced the concept of co-Roman domination and have initiated a study of the corresponding parameter. The following are some interesting problems for further investigation.

*Problem* 6.1. Characterize graphs in which equality holds for any two of the parameters in the inequality chains given in Corollary 3.5.

*Problem* 6.2. Characterize graphs such that $\gamma_{cr} = n - 2$.

*Problem* 6.3. Characterize trees such that $\gamma_r = \frac{2n}{3}$.

*Problem* 6.4. Characterize trees such that $\gamma_{cr} = \lceil \frac{3n}{5} \rceil$.


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