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A sharp Rogers–Shephard type inequality for Orlicz-difference body of planar convex bodies

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Abstract. In this paper, we prove a sharp Rogers–Shephard type inequality for the Orlicz-difference body of planar convex bodies, which extend the works of Bianchini and Colesanti (*Proc. Amer. Math. Soc.* **138(7)** (2008) 2575–2582).

Keywords. Rogers–Shephard type inequality; difference body.

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1. Introduction

As usual, S^{n-1} denotes the unit sphere and o the origin in Euclidean n-space $\mathbb{R}^n$. The standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. A convex body is a non-empty convex compact subset of $\mathbb{R}^n$. Denote by $\mathscr{K}^n$ the set of convex bodies in $\mathbb{R}^n$. For $K \in \mathscr{K}^n$, let $h_K = h(K, \cdot) : S^{n-1} \to \mathbb{R}$ denote the support function of K; i.e., for $u \in S^{n-1}$, $h(K, u) = \max\{u \cdot x : x \in K\}$, where $u \cdot x$ denotes the standard inner product of u and x in $\mathbb{R}^n$. The Minkowski sum of $K, L \in \mathscr{K}^n$ is defined as

$$K + L = \{a + b : a \in K, b \in L\}.$$

Beginning in the late nineteenth century, the classical Brunn–Minkowski theory was developed by Minkowski, Blaschke, Aleksandrov, Fenchel and others. Combining two concepts, i.e. Minkowski addition and geometric inequality, it became one of the most interesting aspects of convex geometry with significant application to various other areas of mathematics. An important family of inequalities are those leading to estimate the volume of a special body associated with the addition of convex bodies. A remarkable inequality of this type is the classical Rogers–Shephard inequality for the difference body. The body $K + (-K)$ is called difference body of K and it is the Minkowski sum of K and its reflect body with respect to the origin, $-K$. Let $V_n(K)$ denote the n-dimensional volume of K.

Theorem A [18]. *If* $K \in \mathscr{K}^n$, *then*

$$V_n(K + (-K)) \leq \binom{2n}{n} V_n(K),$$

and equality holds if and only if K *is a simplex.*

During the last few decades, the core Brunn–Minkowski theory has been extended to the L_p-Brunn–Minkowski theory which blends a different way of combining sets called L_p addition, introduced by Firey [8] in the 1960's. Let us fix $K, L \in \mathscr{K}^n$ containing the origin; the p-sum of K, L, $K +_p L$, is defined by its support function in the following way:

$$h_{K+_pL}(u) = (h_K^p(u) + h_L^p(u))^{\frac{1}{p}}, \quad u \in \mathbb{R}^n.$$

Note that the extremal values $p = 1$ and $p = \infty$ correspond to the Minkowski sum and the convex hull of the union, respectively. Indeed one has $h_{K+_1L}(u) = h_{K+L}(u)$ and $h_{K+_\infty L}(u) = h_{\mathrm{conv}(K\cup L)}(u)$, $u \in \mathbb{R}^n$.

For all $K \in \mathscr{K}^n$ containing the origin, a problem was proposed in [2], which was to find the best constant $c = c_{n,p}$, depending on n and p, such that

$$V_n(K +_p (-K)) \le c_{n,p} V_n(K).$$

This problem was solved in the case $n = 2$ in [2]. Concretely, the following theorem was proved

Theorem B. *For every $p \ge 1$, there exists a constant c_p such that*

$$V_2(K +_p (-K)) \le c_p V_2(K),$$

for all $K \in \mathscr{K}^2$ containing the origin. In particular, if K is a triangle with one vertex at the origin, then equality holds.

Quite recently, Gardner *et al.* [9] constructed a general framework for the Orlicz–Brunn–Minkowski theory, which was introduced by Lutwak *et al.* (see [15, 16], also see [10, 11]), and made clear for the first time the relation to Orlicz spaces and norms. In [9], Gardner *et al.* gave the definition of Orlicz addition, and then obtained the Orlicz–Brunn–Minkowksi inequality. In the end they gave the Orlicz mixed volume of convex bodies which contain the origin in their interiors and got the Orlicz–Minkowski mixed volume inequality.

Let Φ_2 denote the set of convex functions $\varphi : [0, \infty)^2 \to [0, \infty)$ that are increasing in each variable and satisfy $\varphi(o) = 0$ and $\varphi(0, 1) = \varphi(1, 0) = 1$. The Orlicz sum $K +_\varphi L$ of $K, L \in \mathscr{K}^n$, both containing the origin, is defined by

$$h_{K+_\varphi L}(u) = \inf\left\{\lambda > 0 : \varphi\left(\frac{h_K(u)}{\lambda}, \frac{h_L(u)}{\lambda}\right) \le 1\right\},$$

for all $u \in \mathbb{R}^n$.

It is easy to check that $h_{K+_\varphi L}(u) = \lambda(u)$ iff $\varphi\left(\frac{h_K(u)}{\lambda(u)}, \frac{h_L(u)}{\lambda(u)}\right) = 1$. For $K, L \in \mathscr{K}^n$, $p \ge 1$, if $\varphi(t_1, t_2) = t_1^p + t_2^p$, then it turns out that $K +_\varphi L = K +_p L$.

In this paper, we extend Theorem A to the Orlicz version in the case $n = 2$. Concretely, the following theorem is obtained.

Theorem 1. *For every $\varphi \in \Phi_2$, there exists a constant c_φ such that*

$$V_2(K +_\varphi (-K)) \le c_\varphi V_2(K), \tag{1.1}$$

for all $K \in \mathscr{K}^2$ containing the origin. In particular, if K is a triangle with one vertex at the origin, then equality holds.

In §3, we will give an explicit expression of c_φ. We will show the Orlicz–Rogers–Shephard inequality as a consequence of a theorem about the Orlicz addition of the so-called parallel chord movements of convex bodies. Parallel chord movements are special cases of a wider class of movements of convex bodies introduced by Rogers and Shephard in [20], which have recently been applied in the proof of several inequalities in convex geometry (see [1, 3–7, 17]).

In this paper, we prove that if K_t is a parallel chord movement, then the volume of its Orlicz-difference body $V_n(K_t +_\varphi (-K_t))$ is a convex function of t, for every $\varphi \in \Phi_2$. This result, together with a technique in [3], leads to the proof of Theorem 1.

2. Shadow systems and linear parameter system

The notion of shadow system, which was introduced by Rogers and Shephard [20, 22] and was largely used by Campi and Gronchi [4–7], plays an important role in proving our theorems.

A shadow system is a family of n-dimensional convex bodies $\{K(u)\}$ obtained as the projection of a fixed convex body $\tilde{K} \subset \mathbb{R}^{n+1}$ onto the hyperplane $\{e_{n+1}^\perp\}$, which we identify with $\mathbb{R}^n$, along the direction $e_{n+1}+u$. Here u varies in $\{e_{n+1}^\perp\}$. The shadow system is said to be originated from the $(n+1)$-dimensional body $\tilde{K}$.

A linear parameter system, which is one of shadow systems in which direction u lies on a line, is a family of convex bodies $\{K_t\}$ that can be written in the form

$$K_t = \operatorname{conv}\{x_i + \lambda_i t v : i \in I\}, \quad t \in \mathscr{I}; \tag{2.1}$$

where I is an arbitrary index set, $\{x_i\}_{i\in I}$ and $\{\lambda_i\}_{i\in I}$ are bounded subsets of $\mathbb{R}^n$ and $\mathbb{R}$ respectively, $\mathscr{I}$ is an interval of $\mathbb{R}$ and $v \in \mathbb{R}^n$ is the direction of the linear parameter system.

We will need the following result stated in [2].

PROPOSITION 2.1

Let $\{K_t\}_{t\in\mathscr{I}}$ defined by (2.1) is a linear parameter system in $\mathbb{R}^n$ if and only if there exists a convex body $\tilde{K}$ in $\mathbb{R}^{n+1}$ defined by

$$\tilde{K} = \operatorname{conv}\{x_i + \lambda_i e_{n+1} : i \in I\} \tag{2.2}$$

such that for every $t \in \mathscr{I}$, K_t is the projection of $\tilde{K}$ onto the hyperplane $\{e_{n+1}^\perp\}$ along the direction $e_{n+1} - tv$.

The following formula relates the support functions of K_t and $\tilde{K}$ (see [6]):

$$h_{K_t}(u) = h_{\tilde{K}}(u + t\langle u, v\rangle e_{n+1}), u \in \mathbb{R}^n, \quad t \in \mathscr{I}. \tag{2.3}$$

We can give a cinematic interpretation of a linear parameter system viewing the numbers λ_i as the speeds of the points x_i along the direction v and t as the time parameter.

If the index set I is a convex body $K \in \mathscr{K}^n$ and the speed is a function of the point, then the linear parameter system is called continuous movement:

$$K_t = \operatorname{conv}\{x + \alpha(x) t v : x \in K\}, \quad t \in \mathscr{I},$$

where $\alpha(\cdot)$ is a bounded function on K.

Assume that the speed function is constant on each chord parallel to v, i.e. $\alpha(x) = \beta(x|v^{\perp})$ where $x|v^{\perp}$ is the projection of x onto $\{v^{\perp}\}$ and β is a function defined on the orthogonal projection of K onto $\{v^{\perp}\}$. Moreover, if β is such that convexity is preserved for any t, namely,

$$\{x + \beta(x|v^{\perp})tv : x \in K\} = \mathrm{conv}\{x + \beta(x|v^{\perp})tv : x \in K\},$$

then the continuous movement is called the parallel chord movement.

In other words, a parallel chord movement is obtained by assigning to each chord parallel to the direction v a speed vector $\beta(x|v^{\perp})v$ and considering for each fixed time t the union of these chords. Such an union has to be convex. Notice that if $\{K_t\}_{t\in\mathscr{I}}$ is a parallel chord movement, then the volume of K_t is independent of t.

The following theorem is due to Rogers and Shephard [20] which will be needed in our paper.

Theorem 2.2. *The volume $V_n(K_t)$ of a linear parameter system is a convex function of the parameter t.*

In [5], it is proved that the Minkowski sum of linear parameter systems is a linear parameter system. In [2], Bianchini and Colesanti extended this result to the p-sum. Now, we extend this result to Orlicz-sum.

Theorem 2.3. *Let $\{K_t\}_{t\in\mathscr{I}}$ and $\{L_t\}_{t\in\mathscr{I}}$ be linear parameter systems along the direction v and let $\varphi \in \Phi_2$. Then $\{K_t +_\varphi L_t\}_{t\in\mathscr{I}}$ is also a linear parameter system along the direction v.*

The proof is a straightforward consequence of Proposition 2.1 and the following lemma.

Lemma 2.4. *Let $\{K_t\}_{t\in\mathscr{I}}$ and $\{L_t\}_{t\in\mathscr{I}}$ be linear parameter systems along the same direction v and let $\tilde{K}$ and $\tilde{L}$ be the $(n+1)$-dimensional convex bodies which generate K_t and L_t respectively, defined as in (2.2). Hence for all $t \in \mathscr{I}$, $K_t +_\varphi L_t$ is the projection of $\tilde{K} +_\varphi \tilde{L}$ onto the hyperplane $\{e_{n+1}^{\perp}\}$ along the direction $e_{n+1} - tv$.*

Proof. Set $u' = u + t\langle u, v\rangle e_{n=1}$. Using (2.3), we have

$$\begin{aligned} h_{\tilde{K}+_\varphi\tilde{L}}(u') &= \inf\left\{\lambda > 0 : \varphi\left(\frac{h_{\tilde{K}(u')}}{\lambda}, \frac{h_{\tilde{L}(u')}}{\lambda}\right) \le 1\right\} \\ &= \inf\left\{\lambda > 0 : \varphi\left(\frac{h_{K_t}(u)}{\lambda}, \frac{h_{L_t}(u)}{\lambda}\right) \le 1\right\} \\ &= h_{K_t+_\varphi L_t}(u). \end{aligned}$$

This implies that $K_t +_\varphi L_t$ is the projection of a body $\tilde{K} +_\varphi \tilde{L}$ onto the hyperplane $\{e_{n+1}^{\perp}\}$ along the direction $e_{n+1} - tv$, which means, by Proposition 2.1 that $K_t +_\varphi L_t$ is a linear parameter system along v. □

3. The proof of the Orlicz–Rogers–Shephard inequality

Denote by $\mathscr{K}_0^n$ the set of convex bodies with non-empty interior and containing the origin. Consider the functional F_φ defined on $\mathscr{K}_0^n$:

$$F_\varphi(K) = \frac{V_n(K +_\varphi (-K))}{V_n(K)}.$$

It is clear that the best constant c_φ such that (1.1) holds is the supremum of F_φ in $\mathscr{K}_0^n$.

We will use linear parameter systems to find a maximum for the functional F_φ in the planar case. The starting point is the next proposition which follows from Theorems 2.2 and 2.3 and the fact that the volume is constant on each parallel chord movement.

PROPOSITION 3.1

If K_t is any parallel chord movement such that $K_t \in \mathscr{K}_0^n$ for all $t \in I$, then $F_\varphi(K_t)$ is a convex function of the parameter t.

In [3], the following fact is proved: if P is a planar convex polygon with m vertices, $m > 3$, then there exists a parallel chord movement $\{P_t\}_{t\in[t_0,t_1]}$ with $t_0 < 0 < t_1$ such that $P = P_0$ and P_{t_0} and P_{t_1} have at most $(m-1)$ vertices. By Proposition 3.1, it follows that

$$F_\varphi(P) \le \max\{F_\varphi(P_{t_0}), F_\varphi(P_{t_1})\}.$$

Using recursively this fact, we deduce that

$$\sup_{\mathscr{P}} F_\varphi = \sup_{\mathscr{T}} F_\varphi,$$

where $\mathscr{P} = \{K \in \mathscr{K}_0^2 \mid K \text{ is a polygon}\}$ and $\mathscr{T} = \{K \in \mathscr{K}_0^2 \mid K \text{ is a triangle}\}$. Moreover, by the continuity of $F_\varphi(\cdot)$ and a standard density argument, we have

$$\sup_{\mathscr{K}_0^2} F_\varphi = \sup_{\mathscr{T}} F_\varphi.$$

In particular, we are going to show that triangles with one vertex at the origin are maximizers for F_φ. In order to do this, let $T \in \mathscr{T}$ and assume that $o \in \text{int}(T)$ ('int' denotes the interior). Then there exists a parallel chord movement (whose elements are translates of T) $\{T_t\}_{t\in[t_0,t_1]}$ with $t_0 < 0 < t_1$, such that $T_0 = T$ and $o \in \text{bd}(T_{t_0})$, $o \in \text{bd}(T_{t_1})$, ('bd' denotes the boundary). Similarly, if $o \in \text{bd}(T)$, then there exists a parallel chord movement containing T, whose endpoints are triangles with one vertex at o. Using again Proposition 3.1, we have proved that

$$\sup_{\mathscr{K}_0^2} F_\varphi = \sup_{\mathscr{T}_0} F_\varphi,$$

where $\mathscr{T}_0$ denotes the set of triangles with one vertex at the origin.

By Theorem 5.2 of [9], we have F_φ is invariant under non-singular linear transformations. This implies that F_φ is constant on $\mathscr{T}_0$.

This argument proves the following result.

Theorem 3.2. *If T is a triangle in $\mathscr{K}_0^2$ with one vertex at the origin, then T is a maximizer for F_φ.*

To compute the best constant c_φ, we can choose as a maximizer the triangle with vertices at the origin at $(1, 0)$ and $(0, 1)$, denoted by A. Namely

$$c_\varphi = \frac{V_2(A +_\varphi (-A))}{V_2(A)}.$$

Then to express the value of c_φ it is necessary to know how the Orlicz-difference body $A +_\varphi (-A)$ looks. Here we use the parametrization of the boundary of a convex body in terms of its support function (see Corollary 1.7.3 of [21]).

The support function of $A +_\varphi (-A)$ is

$$h_{A+_\varphi(-A)}(\omega) = \begin{cases} \cos\theta, & \text{if } 0 \le \theta < \frac{\pi}{4}, \\ \sin\theta, & \text{if } \frac{\pi}{4} \le \theta < \frac{\pi}{2}, \\ \lambda_0, & \text{if } \frac{\pi}{2} \le \theta < \pi, \\ -\cos\theta, & \text{if } \pi \le \theta < \frac{5\pi}{4}, \\ -\sin\theta, & \text{if } \frac{5\pi}{4} \le \theta < \frac{3\pi}{2}, \\ \lambda_1, & \text{if } \frac{3\pi}{2} \le \theta < 2\pi, \end{cases}$$

where $\omega = \mathrm{e}^{i\theta} \in S^1$ and λ_0 and λ_1 satisfy $\varphi\left(\frac{\sin\theta}{\lambda_0}, \frac{-\cos\theta}{\lambda_0}\right) = 1$ and $\varphi\left(\frac{-\sin\theta}{\lambda_1}, \frac{\cos\theta}{\lambda_1}\right) = 1$. Let $\varphi'_x(x, y)$, $\varphi'_y(x, y)$ denote the partial derivative with respect to x, y respectively. Then a parametrization for the boundary of $A +_\varphi (-A)$ for $\varphi \in \Phi_2$ is $\zeta(\theta) = (x(\theta), y(\theta))$, where

$$x(\theta) = \begin{cases} 1 - \frac{2}{\pi}\theta, & \text{for } \theta \in [0, \frac{2}{\pi}], \\ x_0(\theta), & \text{for } \theta \in (\frac{\pi}{2}, \pi), \\ \frac{2}{\pi}\theta - 3, & \text{for } \theta \in [\pi, \frac{3\pi}{2}] \\ x_1(\theta), & \text{for } \theta \in (\frac{3\pi}{2}, 2\pi), \end{cases}$$

$$y(\theta) = \begin{cases} \frac{2}{\pi}\theta, & \text{for } \theta \in [0, \frac{2}{\pi}], \\ y_0(\theta), & \text{for } \theta \in (\frac{\pi}{2}, \pi), \\ 2 - \frac{2}{\pi}\theta, & \text{for } \theta \in [\pi, \frac{3\pi}{2}] \\ y_1(\theta), & \text{for } \theta \in (\frac{3\pi}{2}, 2\pi). \end{cases}$$

Here,

$$x_0(\theta) = \frac{\varphi'_y(0)\lambda_0}{\varphi'_y(0)\cos\theta - \varphi'_x(0)\sin\theta},$$

$$x_1(\theta) = \frac{\varphi'_y(1)\lambda_1}{\varphi'_y(1)\cos\theta - \varphi'_x(1)\sin\theta},$$

$$y_0(\theta) = \frac{-\varphi'_x(0)\lambda_0}{\varphi'_y(0)\cos\theta - \varphi'_x(0)\sin\theta},$$

$$y_1(\theta) = \frac{-\varphi'_x(1)\lambda_1}{\varphi'_y(1)\cos\theta - \varphi'_x(1)\sin\theta},$$

and

$$\varphi_x'(0) = \varphi_x'\left(\frac{\sin\theta}{\lambda_0}, \frac{-\cos\theta}{\lambda_0}\right),$$

$$\varphi_y'(0) = \varphi_y'\left(\frac{\sin\theta}{\lambda_0}, \frac{-\cos\theta}{\lambda_0}\right),$$

$$\varphi_x'(1) = \varphi_x'\left(\frac{-\sin\theta}{\lambda_1}, \frac{\cos\theta}{\lambda_1}\right),$$

$$\varphi_y'(1) = \varphi_y'\left(\frac{-\sin\theta}{\lambda_1}, \frac{\cos\theta}{\lambda_1}\right).$$

Using the above parametrization and Gauss–Green's formulas we can express the area of $A +_\varphi (-A)$ and then the value of the best constant c_φ as

$$c_\varphi = 2\left[1 + \int_{\frac{\pi}{2}}^{\pi} (x_0(\theta)y_0'(\theta) - x_0'(\theta)y_0(\theta))\mathrm{d}\theta + \int_{\frac{3\pi}{2}}^{2\pi} (x_1(\theta)y_1'(\theta) - x_1'(\theta)y_1(\theta))\mathrm{d}\theta\right].$$

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References

[1] Barthe F, Fradelizi M and Maurey B, A short solution to the Busemann–Petty problem, *Positive* **3** (1999) 95–100

[2] Bianchini C and Colesanti A, A sharp Rogers-Shephard type inequality for the p-difference body of planar convex bodies, *Proc. Amer. Math. Soc.* **138(7)** (2008) 2575–2582

[3] Campi S, Colesanti A and Gronchi P, A note on Sylvester's problem for random polytopes in convex body, *Rend. Ist. Mat. Univ. Trieste* **31** (1999) 79–94

[4] Campi S and Gronchi P, The L^p-Busemann–Petty centroid inequality, *Adv. Math.* **167** (2002) 128–141

[5] Campi S and Gronchi P, On the reverse L^p-Busemann–Petty centroid inequality, *Mathematika* **49** (2002) 1–11

[6] Campi S and Gronchi P, On volume product inequalities for convex sets, *Proc. Amer. Math. Soc.* **134(8)** (2006) 2393–2402

[7] Campi S and Gronchi P, Volume inequalities for L_p-zonotopes, *Mathematika* **53** (2006) 71–80

[8] Firey W J, p-means of convex bodies, *Math. Scand.* **10** (1962) 17–24

[9] Gardner R J, Hug D and Weil W, The Orlicz–Brunn–Minkowski theory: a general framework, additions, and inequalities, *J. Differental Geom.*, in Press.

[10] Haberl C, Lutwak E, Yang D and Zhang G, The even Orlicz Minkowski problem, *Adv. Math.* **224** (2000) 2485–2510

[11] Huang Q and He B, On the Orlicz Minkowski problem for polytopes, *Discrete Comput. Geom.* **48** (2012) 281–297

[12] John F, Extremum problems with inequalities as subsidiary conditions, Courant Anniversary Volume (1948) (New York: Interscience) pp. 187–204

[13] Lutwak E, The Brunn–Minkowski–Firey Theory I: Mixed Volumes and the Minkowski Problem, *J. Differental Geom.* **38(1)** (1993) 131–150
[14] Lutwak E, The Brunn–Minkowski–Firey Theory II: Affine and Geominimal Surface Areas, *Adv. Math.* **118(2)** (1996) 244–294
[15] Lutwak E, Yang D and Zhang G, Orlicz projection bodies, *Adv. Math.* **223** (2010) 220–242
[16] Lutwak E, Yang D and Zhang G, Orlicz centroid bodies, *J. Differ. Geom.* **84** (2010) 365-387
[17] Meyer M and Reisner S, Shadow systems and volumes of polar convex bodies, *Mathematika* **53** (2006) 129–148
[18] Rogers C A and Shephard G C, The difference body of a convex body, *Arch. Math.* **8** (1957) 220–233
[19] Rogers C A and Shephard G C, Convex bodies associated with a gived convex body, *J. Lond. Math. Soc.* **33** (1958) 270–281
[20] Rogers C A and Shephard G C, Some extremal problems for convex bodies, *Mathematika* **5** (1958) 93–102
[21] Schneider R, Convex bodies: the Brunn–Minkowski theory (1993) (Cambridge University Press)
[22] Shephard G C, Shadow system of convex sets, *Israel J. Math.* **2** (1964) 229–236