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On two functional equations originating from number theory

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Abstract. Reducing the functional equations introduced in *Proc. Indian Acad. Sci. (Math. Sci.)* **113(2)** (2003) 91–98 and in *Appl. Math. Lett.* **21** (2008) 974–977 to equations in complex variables and quaternions, we find general solutions of the equations. We also obtain the stability of the equations.

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1. Introduction

Throughout this paper we denote by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, $\mathbb{R}^n$, the set of real numbers, positive real numbers, complex numbers and the n-dimensional Euclidean space, respectively. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^4 \to \mathbb{R}$. In [4], Jung and Bae introduced the following functional equations

$$f(x_1, y_1)f(x_2, y_2) = f(x_1x_2 + y_1y_2,\ x_1y_2 - x_2y_1) \tag{1.1}$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$, and

$$\begin{aligned} &f(x_1, y_1, u_1, v_1)f(x_2, y_2, u_2, v_2)\\ &\quad = f(x_1x_2 + y_1y_2 + u_1u_2 + v_1v_2,\ x_1y_2 - y_1x_2 + u_1v_2 - v_1u_2,\\ &\qquad x_1u_2 - y_1v_2 - u_1x_2 + v_1y_2,\ x_1v_2 + y_1u_2 - u_1y_2 - v_1x_2) \end{aligned} \tag{1.2}$$

for all $x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2 \in \mathbb{R}$. As stated in [4], eqs (1.1) and (1.2) arise from a well-known theorem in number theory:

A positive integer of the form m^2n, where each divisor of n is not a squares of integer, can be represented as a sum of two squares of integer if and only if every prime factor of n is not of the form $4k + 3$.

In a proof of the above theorem, the following elementary equalities are employed:

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1x_2 + y_1y_2)^2 + (x_1y_2 - x_2y_1)^2 \tag{1.3}$$

and

$$(x_1^2 + y_1^2 + u_1^2 + v_1^2)(x_2^2 + y_2^2 + u_2^2 + v_2^2)$$

$$= (x_1x_2 + y_1y_2 + u_1u_2 + v_1v_2)^2 + (x_1y_2 - y_1x_2 + u_1v_2 - v_1u_2)^2$$
$$+(x_1u_2 - y_1v_2 - u_1x_2 + v_1y_2)^2 + (x_1v_2 + y_1u_2 - u_1y_2 - v_1x_2)^2. \quad (1.4)$$

It follows from the identities (1.3) and (1.4) that $f(x, y) = x^2 + y^2$ is a solution of (1.1) and $g(x, y, u, v) = x^2 + y^2 + u^2 + v^2$ is a solution of (1.2). More generally, one can deduce that if $m : \mathbb{R} \to \mathbb{R}$ satisfies $m(xy) = m(x)m(y)$ for all $x, y \in \mathbb{R}$, then

$$f(x, y) = m(x^2 + y^2) \tag{1.5}$$

are solutions of the equation (1.1), and

$$g(x, y, u, v) = m(x^2 + y^2 + u^2 + v^2) \tag{1.6}$$

are solutions of eq. (1.2). In this paper, reducing equations (1.1) and (1.2) to those in the complex numbers and the quaternions, we prove that all general solutions of equations (1.1) and (1.2) are of the form (1.5) and (1.6), respectively. We also consider the stability of equations (1.1) and (1.2), i.e., we investigate bounded and unbounded functions f satisfying each of the following functional inequalities:

$$|f(x_1, y_1)f(x_2, y_2) - f(x_1x_2 + y_1y_2,\ x_1y_2 - x_2y_1)| \le \phi(x_1, y_1), \tag{1.7}$$

$$|f(x_1, y_1)f(x_2, y_2) - f(x_1x_2 + y_1y_2,\ x_1y_2 - x_2y_1)| \le \phi(x_2, y_2) \tag{1.8}$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$ and for some $\phi : \mathbb{R}^2 \to \mathbb{R}^+$, and bounded and unbounded functions g satisfying each of the following functional inequalities:

$$\begin{aligned}&|g(x_1, y_1, u_1, v_1)g(x_2, y_2, u_2, v_2)\\&\quad -g(x_1x_2 + y_1y_2 + u_1u_2 + v_1v_2,\ x_1y_2 - y_1x_2 + u_1v_2 - v_1u_2,\\&\quad x_1u_2 - y_1v_2 - u_1x_2 + v_1y_2,\ x_1v_2 + y_1u_2 - u_1y_2 - v_1x_2)| \le \psi(x_1, y_1, u_1, v_1),\end{aligned} \tag{1.9}$$

$$\begin{aligned}&|g(x_1, y_1, u_1, v_1)g(x_2, y_2, u_2, v_2)\\&\quad -g(x_1x_2 + y_1y_2 + u_1u_2 + v_1v_2,\ x_1y_2 - y_1x_2 + u_1v_2 - v_1u_2,\\&\quad x_1u_2 - y_1v_2 - u_1x_2 + v_1y_2,\ x_1v_2 + y_1u_2 - u_1y_2 - v_1x_2)| \le \psi(x_2, y_2, u_2, v_2)\end{aligned} \tag{1.10}$$

for all $x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2 \in \mathbb{R}$ and for some $\psi : \mathbb{R}^4 \to \mathbb{R}^+$. We also refer the reader to [5] for another proof for finding general solutions of (1.1).

2. General solutions of equations (1.1) and (1.2)

In this section we find general solutions of the functional equations (1.1) and (1.2). Hereafter we denote by $m : \mathbb{R} \to \mathbb{R}$ a multiplicative function, that is, $m(xy) = m(x)m(y)$ for all $x, y \in \mathbb{R}$.

Theorem 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (1.1). Then f has the form*

$$f(x, y) = m\left(\sqrt{x^2 + y^2}\right) \tag{2.1}$$

for all $x, y \in \mathbb{R}$.

Proof. Define $F : \mathbb{C} \to \mathbb{R}$ by

$$F(x + yi) = f(x, y), \quad x, y \in \mathbb{R}. \tag{2.2}$$

Then the functional equation (1.1) is equivalent to

$$F(\bar{z}w) = F(z)F(w) \tag{2.3}$$

for all $z, w \in \mathbb{C}$. Letting $z = 1$ in (2.3), we have

$$F(w) = F(1)F(w)$$

for all $w \in \mathbb{C}$. If $F(1) = 0$, then we have $F \equiv 0$ and $f \equiv 0$. Otherwise, we have $F(1) = 1$. Letting $w = 1$ in (2.3), we have

$$F(\bar{z}) = F(z) \tag{2.4}$$

for all $z \in \mathbb{C}$. Replacing $\bar{z}$ and w by z in (2.3) and using (2.3) and (2.4), we have

$$\begin{aligned} F(z^2) &= F(\bar{z})F(z) = F(z)F(z) \\ &= F(\bar{z}z) = F(|z|^2) \end{aligned} \tag{2.5}$$

for all $z \in \mathbb{C}$. Thus, we have

$$F(z) = F(|z|) \tag{2.6}$$

for all $z \in \mathbb{R}$. Let $m(x) = F(x)$ for $x \in \mathbb{R}$. Then from (2.3), m is a multiplicative function. Thus, from (2.6) we get (2.1). This completes the proof. □

Now, we find general solutions of (1.2) by reducing it to an equation in the quaternion group. Let $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ be the quaternion group. Recall that $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$ and the conjugate of $q = a + bi + cj + dk \in \mathbb{H}$ is given by $q^* = a - bi - cj - dk$. We denote by $\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Theorem 2.2. *Let $g : \mathbb{R}^4 \to \mathbb{R}$ satisfy (1.2). Then g has the form*

$$g(x, y, u, v) = m(x^2 + y^2 + u^2 + v^2) \tag{2.7}$$

for all $x, y, u, v \in \mathbb{R}$.

Proof. Define $F : \mathbb{C} \to \mathbb{R}$ by

$$F(x + yi + uj + vk) = g(x, y, u, v) \quad x, y, u, v \in \mathbb{R}. \tag{2.8}$$

Let $p = x_1 + y_1 i + u_1 j + v_1 k$, $q = x_2 + y_2 i + u_2 j + v_2 k$. Then the functional equation (1.2) is equivalent to

$$F(p)F(q) = F(qp^*). \tag{2.9}$$

Letting $p = 1$ in (2.9) we have

$$F(1)F(q) = F(q)$$

for all $q \in \mathbb{H}$. If $F(1) = 0$, then we have $F \equiv 0$ and $g \equiv 0$. Otherwise, we have $F(1) = 1$. Letting $q = 1$ in (2.9) we have

$$F(p) = F(p^*) \tag{2.10}$$

for all $p \in \mathbb{H}$. Replacing p by q^* in (2.9) and using (2.10) we have

$$\begin{aligned} F(q^2) &= F(q^*)F(q) = F(q)F(q) \\ &= F(qq^*) = F(\|q\|^2) \end{aligned} \tag{2.11}$$

for all $q \in \mathbb{H}$. Since $\{q^2 \mid q \in \mathbb{H}\} = \mathbb{H}$, it follows from (2.11) that

$$F(q) = F(\|q\|) \tag{2.12}$$

for all $q \in \mathbb{H}$. Let $m(x) = F(x)$ for $x \in \mathbb{R}$. Then from (2.9), m is a multiplicative function. Thus, from (2.12) we get (2.7). This completes the proof. □

3. Stability of equations (1.1) and (1.2)

In this section we consider the stability of equations (1.1) and (1.2), i.e., the inequalities (1.7) $\sim$ (1.10). We first consider unbounded functions f satisfying (1.7) and (1.8) and unbounded functions g satisfying (1.9) and (1.10).

Theorem 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an unbounded function satisfying* (1.7)*. Then f has the form*

$$f(x, y) = m(\sqrt{x^2 + y^2}) \tag{3.1}$$

for all $x, y \in \mathbb{R}$.

Proof. Let $F(x + yi) = f(x, y)$ and $\Phi(x + yi) = \phi(x, y)$ for all $x, y \in \mathbb{R}$. Then the functional inequality (1.7) is converted to

$$|F(\bar{z}w) - F(z)F(w)| \le \Phi(z) \tag{3.2}$$

for all $z, w \in \mathbb{C}$. Choose a sequence $c_n \in \mathbb{C}$, $n = 1, 2, 3, \ldots$, such that $|F(c_n)| \to \infty$ as $n \to \infty$. Replacing w by c_n in (3.2) and dividing the result by $|F(c_n)|$, we have

$$\left| F(z) - \frac{F(\bar{z}c_n)}{F(c_n)} \right| \le \frac{\Phi(z)}{|F(c_n)|}. \tag{3.3}$$

Letting $n \to \infty$ in (3.3), we have

$$F(z) = \lim_{n\to\infty} \frac{F(\bar{z}c_n)}{F(c_n)} \tag{3.4}$$

for all $z \in \mathbb{C}$. Multiplying both sides of (3.4) by $F(w)$ and using (3.2) and (3.4) we have

$$F(w)F(z) = \lim_{n\to\infty} \frac{F(w)F(\bar{z}c_n)}{F(c_n)} = \lim_{n\to\infty} \frac{F(\bar{w}\bar{z}c_n)}{F(c_n)} = F(wz) \tag{3.5}$$

for all $z, w \in \mathbb{C}$. From (3.2) and (3.5), we have

$$|F(\bar{z}w) - F(z)F(w)| = |F(\bar{z}) - F(z)||F(w)| \le \Phi(z) \tag{3.6}$$

for all $z, w \in \mathbb{C}$. Since F is unbounded, we have

$$F(\bar{z}) = F(z) \tag{3.7}$$

for all $z \in \mathbb{C}$. From (3.5) and (3.7) we get

$$F(\bar{z}w) - F(z)F(w) = 0 \tag{3.8}$$

for all $z, w \in \mathbb{C}$. Thus, f satisfies eq. (1.1). Using Theorem 2.1, we get (3.1). This completes the proof. □

Theorem 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an unbounded function satisfying (1.8). Then f has the form*

$$f(x, y) = m\left(\sqrt{x^2 + y^2}\right) \tag{3.9}$$

for all $x, y \in \mathbb{R}$.

Proof. Let F and Φ be the functions defined in Theorem 3.1. Then the functional inequality (1.8) is converted to

$$|F(\bar{z}w) - F(z)F(w)| \le \Phi(w) \tag{3.10}$$

for all $z, w \in \mathbb{C}$. Putting $w = 1$ in (3.10), we have

$$|F(\bar{z}) - F(z)F(1)| \le \Phi(1) \tag{3.11}$$

for all $z \in \mathbb{C}$. Since F is unbounded, we have $F(1) \neq 0$. Replacing z by $z\bar{w}$ in (3.11) we have

$$|F(\bar{z}w) - F(z\bar{w})F(1)| \le \Phi(1) \tag{3.12}$$

for all $z, w \in \mathbb{C}$. Using the triangle inequality with (3.10) and (3.12), replacing (z, w) by (w, z) in the result and dividing the result by $|F(1)|^2$, we have

$$|F_0(w\bar{z}) - F_0(w)F_0(z)| \le \Phi_0(z) \tag{3.13}$$

for all $z, w \in \mathbb{C}$, where

$$F_0(z) = \frac{F(z)}{F(1)}, \quad \Phi_0(z) = \frac{\Phi(z) + \Phi(1)}{|F(1)|^2}.$$

Thus, we get the inequality of the form (3.2). By the proof of Theorem 3.1, we have

$$F_0(w\bar{z}) - F_0(w)F_0(z) = 0 \tag{3.14}$$

for all $z, w \in \mathbb{C}$. Now, by the proof of Theorem 2.1, we get

$$F_0(z) = m(|z|), \quad z \in \mathbb{C} \tag{3.15}$$

for some multiplicative function $m : \mathbb{R} \to \mathbb{R}$. Putting (3.15) in (3.10) we have

$$|F(1)m(|\bar{z}w|) - F(1)^2 m(|z|)m(|w|)| = |F(1)(1 - F(1))||m(|zw|)| \le \Phi(w) \tag{3.16}$$

for all $z, w \in \mathbb{C}$. Putting $w = 1$ in (3.16), we have

$$|F(1)(1 - F(1))||m(|z|)| \leq \Phi(1) \tag{3.17}$$

for all $z \in \mathbb{C}$. Since m is unbounded, we have $F(1) = 1$. Thus, we have $F(z) = m(|z|)$ for all $z \in \mathbb{C}$ and (3.10) follows. This completes the proof. □

Theorem 3.3. *Let $g : \mathbb{R}^4 \to \mathbb{R}$ be an unbounded function satisfying (1.9). Then g has the form*

$$g(x, y, u, v) = m\left(\sqrt{x^2 + y^2 + u^2 + v^2}\right) \tag{3.18}$$

for all $x, y, u, v \in \mathbb{R}$.

Proof. Let $F(x + yi + uj + vk) = g(x, y, u, v)$ and $\Psi(x + yi + uj + vk) = \psi(x, y, u, v)$ for all $x, y, u, v \in \mathbb{R}$. Then the functional inequality (1.9) is reduced to

$$|F(q)F(p) - F(qp^*)| \leq \Psi(p) \tag{3.19}$$

for all $p, q \in \mathbb{H}$. Choose a sequence $r_n \in \mathbb{H}$, $n = 1, 2, 3, \ldots,$ such that $|F(r_n)| \to \infty$ as $n \to \infty$. Replacing q by r_n in (3.19) and dividing the result by $|F(r_n)|$ we have

$$\left|F(p) - \frac{F(r_n p^*)}{F(r_n)}\right| \leq \frac{\Psi(p)}{|F(r_n)|}. \tag{3.20}$$

Letting $n \to \infty$ in (3.20) we have

$$F(p) = \lim_{n\to\infty} \frac{F(r_n p^*)}{F(r_n)} \tag{3.21}$$

for all $p \in \mathbb{H}$. Multiplying both sides of (3.21) by $F(q)$ and using (3.19) and (3.21) we have

$$\begin{aligned} F(p)F(q) &= \lim_{n\to\infty} \frac{F(r_n p^*)F(q)}{F(r_n)} = \lim_{n\to\infty} \frac{F(r_n p^* q^*)}{F(r_n)} \\ &= \lim_{n\to\infty} \frac{F(r_n (qp)^*)}{F(r_n)} = F(qp) \end{aligned} \tag{3.22}$$

for all $p, q \in \mathbb{H}$. From (3.19) and (3.22) we have

$$|F(q)F(p) - F(qp^*)| = |F(q)||F(p) - F(p^*)| \leq \Psi(p) \tag{3.23}$$

for all $p, q \in \mathbb{H}$. Since F is unbounded, it follows from (3.23) that

$$F(p) = F(p^*) \tag{3.24}$$

for all $p \in \mathbb{H}$. Now, from (3.22) and (3.24) we obtain

$$F(q)F(p) = F(q)F(p^*) = F(qp^*) \tag{3.25}$$

for all $p, q \in \mathbb{H}$. Thus, by Theorem 2.2, g has the form (3.18). This completes the proof. □

Theorem 3.4. *Let $g : \mathbb{R}^4 \to \mathbb{R}$ be an unbounded function satisfying (1.10). Then g has the form*

$$g(x, y, u, v) = m\left(\sqrt{x^2 + y^2 + u^2 + v^2}\right) \tag{3.26}$$

for all $x, y, u, v \in \mathbb{R}$.

Proof. Let F and Ψ be the functions defined in Theorem 3.3. Then the functional inequality (1.10) is reduced to

$$|F(p)F(q) - F(qp^*)| \leq \Psi(q)$$

for all $p, q \in \mathbb{H}$, which is

$$|F(q)F(p) - F(pq^*)| \leq \Psi(p) \tag{3.27}$$

for all $p, q \in \mathbb{H}$. Letting $p = 1$ in (3.27) we have

$$|F(q)F(1) - F(q^*)| \leq \Psi(1) \tag{3.28}$$

for all $q \in \mathbb{H}$. Since F is unbounded, we have $F(1) \neq 0$. Replacing q by qp^* in (3.28) we have

$$|F(qp^*)F(1) - F(pq^*)| \leq \Psi(1) \tag{3.29}$$

for all $p, q \in \mathbb{H}$. Using the triangle inequality with (3.27) and (3.29) we have

$$|F(q)F(p) - F(qp^*)F(1)| \leq \Psi(p) + \Psi(1) \tag{3.30}$$

for all $p, q \in \mathbb{H}$. Dividing (3.30) by $|F(1)|^2$ we have

$$|F_0(q)F_0(p) - F_0(qp^*)| \leq \Psi_0(p) \tag{3.31}$$

for all $p, q \in \mathbb{H}$, where

$$F_0(p) = \frac{F(p)}{F(1)}, \ \ \Psi_0(q) = \frac{\Psi(q) + \Psi(1)}{|F(1)|^2}.$$

Thus, we get the inequality of the form (3.19). By the proof of Theorem 3.3, we have

$$F_0(q)F_0(p) = F_0(qp^*) \tag{3.32}$$

for all $p, q \in \mathbb{H}$. Thus, by Theorem 2.2, we have $F_0(q) = m(q)$ for all $q \in \mathbb{H}$ and hence

$$F(q) = F(1)m(\|q\|), \ \ q \in \mathbb{H} \tag{3.33}$$

for some multiplicative function $m : \mathbb{R} \to \mathbb{R}$. Putting (3.33) in (3.27) and letting $p = 1$ in the result we have

$$|F(1)(1 - F(1))||m(\|q\|)| \leq \Psi(1) \tag{3.34}$$

for all $q \in \mathbb{H}$. Now, since m is unbounded, from (3.34) we have $F(1) = 1$. Thus, we have $F(q) = m(\|q\|)$ and (3.26) follows. This complete the proof. □

Finally, we investigate bounded functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying inequalities (1.7), (1.8) and bounded functions $g : \mathbb{R}^4 \to \mathbb{R}$ satisfying (1.9) and (1.10). We use the following lemma [3].

Lemma 3.5. *Let G be a group and $\sigma : G \times G \to G$ be given. Let $h : G \to \mathbb{R}$ be a bounded function satisfying*

$$|h \circ \sigma(z, w) - h(z)h(w)| \le \Phi(z) \tag{3.35}$$

for all $z, w \in G$ and for some $\Phi : G \to \mathbb{R}^+$. Then h satisfies

$$|h(z)| \le \frac{1}{2}(1 + \sqrt{1 + 4\Phi(z)}) \tag{3.36}$$

for all $z \in G$. Furthermore, let Γ be the set of all $z \in G$ such that $\sigma(z, \cdot) : G \to G$ is surjective and let $K := \{z \in \Gamma : \Phi(z) < \frac{1}{4}\}$. Then h satisfies

$$\frac{1}{2}(1 + \sqrt{1 - 4\Phi(z)}) \le |h(z)| \le \frac{1}{2}(1 + \sqrt{1 + 4\Phi(z)}) \tag{3.37}$$

for all $z \in K$, or else

$$|h(z)| \le \frac{1}{2}(1 - \sqrt{1 - 4\Phi(z)}) \tag{3.38}$$

for all $z \in K$.

Hereafter we denote by $\mathbb{R}^2_0 = \mathbb{R}^2 \setminus \{(0, 0)\}$, $\mathbb{R}^4_0 = \mathbb{R}^4 \setminus \{(0, 0, 0, 0)\}$. Using Lemma 3.5, we have the following:

Theorem 3.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function satisfying (1.7) or (1.8). Then f satisfies*

$$|f(x, y)| \le \frac{1}{2}(1 + \sqrt{1 + 4\phi(x, y)}) \tag{3.39}$$

for all $x, y \in \mathbb{R}$. Furthermore, let $K := \{(x, y) \in \mathbb{R}^2_0 \mid \phi(x, y) < \frac{1}{4}\}$. Then f satisfies

$$\frac{1}{2}(1 + \sqrt{1 - 4\phi(x, y)}) \le |f(x, y)| \le \frac{1}{2}(1 + \sqrt{1 + 4\phi(x, y)}) \tag{3.40}$$

for all $(x, y) \in K$, or else

$$|f(x, y)| \le \frac{1}{2}(1 - \sqrt{1 - 4\phi(x, y)}) \tag{3.41}$$

for all $(x, y) \in K$.

Proof. Let $G = \mathbb{C}$ and let $\sigma(z, w) = \bar{z}w$ or $\sigma(z, w) = z\bar{w}$ in Lemma 3.5. Then we have $\Gamma = \mathbb{C} \setminus \{0\}$. Applying Lemma 3.5 to the inequality (3.2) with $\sigma(z, w) = \bar{z}w$, we get the result for (1.7). Also, applying Lemma 3.5 to the inequality (3.10) with $\sigma(z, w) = z\bar{w}$, we get the result for (1.8). □

Theorem 3.7. *Let $g : \mathbb{R}^4 \to \mathbb{R}$ be a bounded function satisfying* (1.9) *or* (1.10). *Then g satisfies*

$$|g(x, y, u, v)| \le \frac{1}{2}(1 + \sqrt{1 + 4\psi(x, y, u, v)}) \tag{3.42}$$

for all $x, y, u, v \in \mathbb{R}$. Furthermore, let $K := \{(x, y, u, v) \in \mathbb{R}_0^4 \mid \psi(x, y, u, v) < \frac{1}{4}\}$. Then g satisfies

$$\frac{1}{2}(1+\sqrt{1 - 4\psi(x, y, u, v)}) \le |g(x, y, u, v)| \le \frac{1}{2}(1+\sqrt{1 + 4\psi(x, y, u, v)}) \tag{3.43}$$

for all $(x, y, u, v) \in K$, or else

$$|g(x, y, u, v)| \le \frac{1}{2}(1 - \sqrt{1 - 4\psi(x, y, u, v)}) \tag{3.44}$$

for all $(x, y, u, v) \in K$.

Proof. Let $G = \mathbb{H}$ and let $\sigma(p, q) = qp^*$ or $\sigma(p, q) = pq^*$ in Lemma 3.5. Then we have $\Gamma = \mathbb{F} \setminus \{0\}$. Applying Lemma 3.5 to the inequality (3.19) with $\sigma(p, q) = qp^*$, we get the result for inequality (1.9). Applying Lemma 3.5 to the inequality (3.27) with $\sigma(p, q) = pq^*$, we get the result for inequality (1.10). This completes the proof. □

In particular, if $\phi, \psi \equiv \epsilon < \frac{1}{4}$, a positive constant, then we have the following results (see [1, 2]).

COROLLARY 3.8

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function satisfying (1.7) *for $\phi(x, y) \equiv \epsilon < \frac{1}{4}$. Then f satisfies*

$$\frac{1}{2}(1 + \sqrt{1 - 4\epsilon}) \le |f(x, y)| \le \frac{1}{2}(1 + \sqrt{1 + 4\epsilon}) \tag{3.45}$$

for all $(x, y) \in \mathbb{R}_0^2$, or else

$$|f(x, y)| \le \frac{1}{2}(1 - \sqrt{1 - 4\epsilon}) \tag{3.46}$$

for all $(x, y) \in \mathbb{R}_0^2$.

Remark 3.9. The values of f at the origin may not be contained in the ranges $f(\mathbb{R}_0^2)$; for example, $f(x, y) \approx 1$ for all $(x, y) \in \mathbb{R}_0^2$ and $f(0, 0) \approx 0$.

COROLLARY 3.10

Let $g:\mathbb{R}^4\to\mathbb{R}$ *be a bounded function satisfying* (1.9) *for* $\phi(x,y,u,v)\equiv\epsilon<\frac{1}{4}$. *Then* g *satisfies*

$$\frac{1}{2}(1+\sqrt{1-4\epsilon})\le|g(x,y,u,v)|\le\frac{1}{2}(1+\sqrt{1+4\epsilon}) \tag{3.47}$$

for all $(x,y,u,v)\in\mathbb{R}_0^4$, *or else*

$$|g(x,y,u,v)|\le\frac{1}{2}(1-\sqrt{1-4\epsilon}) \tag{3.48}$$

for all $(x,y,u,v)\in\mathbb{R}_0^4$.

Remark 3.11. Similarly, as in Remark 3.9, the values of g at the origin may not be close to the ranges $g(\mathbb{R}_0^4)$; for example, $g(x,y,u,v)\approx 1$ for all $(x,y,u,v)\in\mathbb{R}_0^4$ and $g(0,0,0,0)\approx 0$.

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