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Hausdorff dimension of the boundary of the immediate basin of infinity of McMullen maps

XIAOGUANG WANG[1] and FEI YANG[2]

[1]Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China
[2]Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China
E-mail: wxg688@163.com; yangfei_math@163.com

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Abstract. We give an asymptotic formula of the Hausdorff dimension of the boundary of the immediate basin of infinity of McMullen maps $f_\lambda(z) = z^d + \lambda/z^d$, where $d \geq 3$ and λ is small.

Keywords. Hausdorff dimension; Julia sets; McMullen maps.

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1. Introduction

The study of the Hausdorff dimension of the Julia sets of rational maps is one of the important topics in complex dynamics. Most of the time, one can only obtain the numerical estimation on the Hausdorff dimension of the Julia sets of rational maps since they are not strictly self-similar fractals. The first heart-stirring formula on the Hausdorff dimension of Julia sets was due to Ruelle [12]. He proved that for polynomials $P_c(z) = z^d + c$ with degree $d \geq 2$, if c is small, then the Hausdorff dimension of the Julia set J_c of P_c is given by

$$\dim_H(J_c) = 1 + \frac{|c|^2}{4\log d} + \mathcal{O}(c^3).$$

Later, the Hausdorff dimension formula of J_c was recalculated in [14] and [1], where the formula was expanded to the third order and fourth order in c, respectively. In theory, terms of higher orders can be calculated successively. However, the calculation becomes more complicated as the order increases.

For rational maps, Osbaldestin [10] gave an asymptotic formula of the Hausdorff dimension on the Julia sets of the renormalization transformation of the standard diamond lattice as the parameter tends to infinity.

Note that in both cases stated above, the asymptotic formulas of the Hausdorff dimension were calculated when the Julia sets are quasicircles. In this paper, we adopt the method in [14] to calculate the asymptotic formula of the Hausdorff dimension of a subset of the Julia sets of a family of rational maps when the subset is a quasicircle.

A subset of the Riemann sphere $\bar{\mathbb{C}}$ is called a *Cantor set of circles* (or *Cantor circles* in short) if it consists of uncountably many closed Jordan curves which is homeomorphic to $\mathcal{C} \times \mathbb{S}^1$, where $\mathcal{C}$ is the middle third Cantor set and $\mathbb{S}^1$ is the unit circle. It was known that if $\lambda \neq 0$ is small enough, then the Julia set of

$$f_\lambda(z) = z^d + \lambda/z^d \tag{1.1}$$

is a Cantor set of circles, where $d \geq 3$ (see [4, 7] and figure 1). This family is commonly referred to as the *McMullen maps*, which has been studied extensively (see [2, 3, 11, 13] and references therein).

If $\lambda \neq 0$ is small, then f_λ can be viewed as a perturbation of the simple polynomial $f_0(z) = z^d$. In this case, all Fatou components are attracted by ∞. We denote by B_λ the immediate attracting basin of ∞, then the boundary ∂B_λ is a Jordan curve (actually quasicircle by Lemma 2.3). In fact, it was proven in [11] that ∂B_λ is always a Jordan curve if $J(f_\lambda)$ is not a Cantor set.

In this paper, we calculate the Hausdorff dimension of the boundary ∂B_λ and prove the following main theorem.

Main Theorem. *Let $d \geq 3$. For small λ such that $J(f_\lambda)$ is a Cantor set of circles, the Haudorff dimension of ∂B_λ is given by*

$$\dim_H(\partial B_\lambda) = 1 + \frac{|\lambda|^2}{\log d} + \mathcal{O}(\lambda^3). \tag{1.2}$$

In particular, if $d \neq 4$, then the higher order $\mathcal{O}(\lambda^3)$ in (1.2) can be replaced by $\mathcal{O}(\lambda^4)$.

We give the proof of the main theorem in §2 and leave the completed calculations to the last section as an Appendix. As a remark, we would like to remind the reader that the main theorem can be derived by the thermodynamic formalism in [9], [12] and [16].

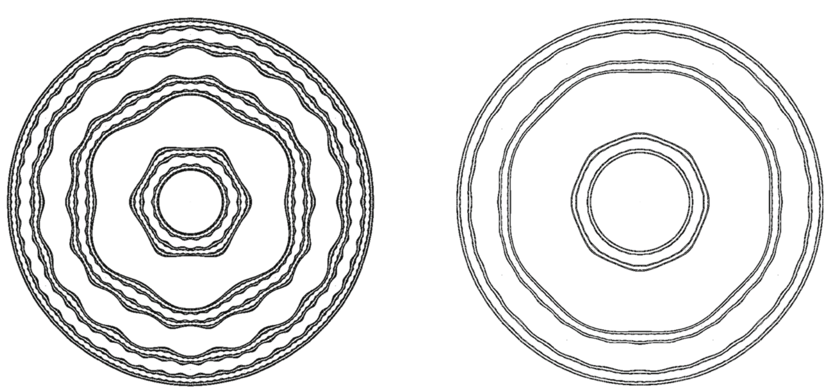

Figure 1. The Julia sets of $f_\lambda(z) = z^d + \lambda/z^d$, where $\lambda = 0.005$ and $d = 3, 4$. Both are Cantor circles.

2. Proof of the main theorem

Unlike the polynomials $P_c(z) = z^d + c$, the parameter space of McMullen family has a special point at $\lambda = 0$. The whole Julia set $J(f_\lambda)$ does not converge to $J(f_0)$ (the unit circle $\mathbb{S}^1$) in the Hausdorff topology when λ tends to 0 (see [3]). However, the boundary of the immediately attracting basin of infinity ∂B_λ does. In fact, we can show (Lemma 2.3) that ∂B_λ is a holomorphic motion of the unit circle $\mathbb{S}^1$.

In the following, we always assume that λ is small ($\lambda = 0$ is allowed). All details of complicated calculations will be included in the next section. Firstly, we recall the definition of holomorphic motion.

DEFINITION 2.1 (Holomorphic motion [6])

Let E be a subset of $\bar{\mathbb{C}}$, a map $h : \mathbb{D} \times E \to \bar{\mathbb{C}}$ is called a *holomorphic motion* of E parametrized by the unit disk $\mathbb{D}$ and with base point 0 if

(1) for every $z \in E$, $\lambda \mapsto h(\lambda, z)$ is holomorphic for λ in $\mathbb{D}$;
(2) for every $\lambda \in \mathbb{D}$, $z \mapsto h(\lambda, z)$ is injective on E; and
(3) $h(0, z) = z$ for all $z \in E$.

The unit disk $\mathbb{D}$ in Definition 2.1 can be replaced by any other topological disk.

Theorem 2.2 (The λ-lemma [6]). *A holomorphic motion $h : \mathbb{D} \times E \to \bar{\mathbb{C}}$ of E has a unique extension to a holomorphic motion $h : \mathbb{D} \times \bar{E} \to \bar{\mathbb{C}}$ of the closure $\bar{E}$. The extension is a continuous map. For each $\lambda \in \mathbb{D}$, the map $h(\lambda, \cdot) : E \to \bar{\mathbb{C}}$ extends to a quasiconformal map of the Riemann sphere to itself.*

It is known from [13] that in the parameter space of f_λ, the *McMullen domain* $\mathcal{M} := \{\lambda \in \mathbb{C} \setminus \{0\};\ J(f_\lambda) \text{ is a Cantor set of circles}\}$ is a deleted neighborhood of the origin. It turns out that $\mathcal{V} = \mathcal{M} \cup \{0\}$ is a topological disk containing 0.

Lemma 2.3. *There is a holomorphic motion $h : \mathcal{V} \times \mathbb{S}^1 \to \mathbb{C}$ parametrized by $\mathcal{V}$ and with a base point* 0 *such that $h(\lambda, \mathbb{S}^1) = \partial B_\lambda$ for all $\lambda \in \mathcal{V}$.*

Proof. We first prove that every repelling periodic point of $f_0(z) = z^d$ moves holomorphically in $\mathcal{V}$. Let $z_0 \in \mathbb{S}^1$ be such a point with period k. For small λ, the map f_λ is a small perturbation of f_0. By implicit function theorem, there is a neighborhood U_0 of 0 such that z_0 becomes a repelling point z_λ of f_λ with the same period k, for all $\lambda \in U_0$. On the other hand, for all $\lambda \in \mathcal{M}$, since f_λ has no non-repelling cycles, each repelling cycle of f_λ moves holomorphically throughout $\mathcal{M}$ (see Theorem 4.2 of [8]).

Since $\mathcal{V}$ is simply connected, there is a holomorphic map $Z : \mathcal{V} \to \mathbb{C}$ such that $Z(\lambda) = z_\lambda$ for $\lambda \in U_0$. Let $\mathrm{Per}(f_0)$ be all-repelling periodic points of f_0. Then the map $h : \mathcal{V} \times \mathrm{Per}(f_0) \to \mathbb{C}$ defined by $h(\lambda, z_0) = Z(\lambda)$ is a holomorphic motion. Notice that $\mathbb{S}^1 = \overline{\mathrm{Per}(f_0)}$. By Theorem 2.2, there is an extension of h, say $h : \mathcal{V} \times \mathbb{S}^1 \to \mathbb{C}$. It is obvious that $h(\lambda, \mathbb{S}^1)$ is a connected component of $J(f_\lambda)$.

To complete, we show that $h(\lambda, \mathbb{S}^1) = \partial B_\lambda$ for all $\lambda \in \mathcal{V}$. By the uniqueness of the holomorphic motion of hyperbolic Julia sets, it suffices to show that $h(\lambda, \mathbb{S}^1) = \partial B_\lambda$ for small and real parameter $\lambda \in (0, \epsilon)$, where $\epsilon > 0$.

Under the small perturbation f_λ with $\lambda \in (0, \epsilon)$, the fixed point $z_0 = 1$ of f_0 becomes the repelling fixed points z_λ of f_λ, which is real and close to 1. The map f_λ has exactly two real and positive fixed points. One is z_λ and the other is z_λ^*, which is near 0. It is obvious that z_λ is the landing point of the zero external ray of f_λ. So $h(\lambda, 1) = z_\lambda \in \partial B_\lambda$. This implies that $h(\lambda, \mathbb{S}^1) = \partial B_\lambda$ for all $\lambda \in (0, \epsilon)$. The proof is complete. □

DEFINITION 2.4 [12]

Let M be a real analytic manifold of finite dimension, J a compact subset of M and V an open neighborhood of J in M. The set J is called a *repeller* for the real analytic map $f : V \to M$ if

(1) there exist $C > 0$, $\alpha > 0$ such that $\|(T_x f^{\circ n})u\| \geq C\alpha^n \|u\|$ for all $x \in J$, $u \in T_x M$ and $n \geq 1$;
(2) $J = \{x \in V : f^{\circ n}(x) \in V \text{ for all } n > 0\}$;
(3) for every non-empty open set O intersecting J, there exists $n > 0$ such that $J \subset f^{\circ n}(O)$.

The boundary ∂B_λ is a 'repeller' of the map f_λ by Definition 2.4.

Theorem 2.5 [12]. *If the repeller J_λ of a family of real analytic holomorphic maps f_λ depends analytically on λ, then the Hausdorff dimension of J_λ depends real analytically on λ.*

We define a function $H : \mathcal{V} \to \mathbb{R}^+$ by $H(\lambda) = \dim_H(\partial B_\lambda)$ and derive some basic properties of H. The fact is that $\partial B_0 = \mathbb{S}^1$ implies $H(0) = 1$. By Theorem 2.5, it follows that H is a real analytic function. Thus when λ is near 0, we have

$$H(\lambda) = \sum_{s,t \geq 0} a_{st} \lambda^s \bar{\lambda}^t, \quad \text{where } a_{00} = 1. \tag{2.1}$$

By the formula (1.1) of f_λ, it follows that $\overline{f_\lambda(\bar{z})} = f_{\bar{\lambda}}(z)$ and $f^{\circ 2}_{\rho^2\lambda}(\rho z) = \rho f^{\circ 2}_\lambda(z)$, where $\rho = \mathrm{e}^{\frac{\pi i}{d-1}}$. This means that

$$\overline{H(\lambda)} = H(\lambda) = H(\bar{\lambda}) \quad \text{and} \quad H(\rho^2 \lambda) = H(\lambda).$$

So the coefficients in (2.1) satisfy

$$a_{st} = \bar{a}_{st} = a_{ts} \quad \text{and} \quad a_{st} = a_{st}\rho^{2(s-t)}.$$

In particular, if $s - t \not\equiv 0 \mod (d-1)$, then $a_{st} = 0$. Thus we have

$$H(\lambda) = \begin{cases} 1 + a_{20}(\lambda^2 + \bar{\lambda}^2) + a_{11}|\lambda|^2 + \mathcal{O}(\lambda^4), & \text{if } d = 3, \\ 1 + a_{11}|\lambda|^2 + \mathcal{O}(\lambda^3), & \text{if } d = 4, \\ 1 + a_{11}|\lambda|^2 + \mathcal{O}(\lambda^4), & \text{if } d \geq 5. \end{cases} \tag{2.2}$$

Let Ω be a closed subset of $\mathbb{R}^n$. A map $S : \Omega \to \Omega$ is called a *contraction* on Ω if there exists a real number $c \in (0, 1)$ such that $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in \Omega$. A finite family of contractions $\{S_1, S_2, \ldots, S_m\}$ defined on $\Omega \subset \mathbb{R}^n$ with $m \geq 2$ is called an *iterated function system* (or IFS in short).

To compute the Hausdorff dimension of ∂B_λ with $\lambda \in \mathcal{M}$, we need the following results (see Theorem 9.1, Propositions 9.6 and 9.7 of [5]).

Theorem 2.6 [5]. *Let $\{S_1, \ldots, S_m\}$ be an IFS on a closed set $\Omega \subset \mathbb{R}^n$ such that $|S_i(x) - S_i(y)| \le c_i|x-y|$ with $0 < c_i < 1$. Then*

(1) *there exists a unique non-empty compact set J such that $J = \bigcup_{i=1}^m S_i(J)$;*
(2) *the Hausdorff dimension $\dim_H(J)$ of J satisfies $\dim_H(J) \le s$, where $\sum_{i=1}^m c_i^s = 1$;*
(3) *if we require further $|S_i(x) - S_i(y)| \ge b_i|x-y|$ for $0 < b_i < 1$, then $\dim_H(J) \ge s'$, where $\sum_{i=1}^m b_i^{s'} = 1$.*

The non-empty compact set J appearing in Theorem 2.6(1) is called the *attractor* of the IFS $\{S_1, \ldots, S_m\}$. Let Fix(f) be the collection of all the repelling fixed points of a rational map f. The following lemma is a similar statement of Lemma 6.3 of [15]. For completeness, we include the proof here.

Lemma 2.7. *For any $\lambda \in \mathcal{V}$, the Hausdorff dimension $D := \dim_H(\partial B_\lambda)$ of ∂B_λ is determined by $A_n(D) = \mathcal{O}(1)$ as $n \to \infty$, where*

$$A_n(D) = \sum_{z \in Fix(f_\lambda^{\circ n}) \cap \partial B_\lambda} |(f^{\circ n})'(z)|^{-D}. \tag{2.3}$$

Proof. For simplicity, we use f and J to replace f_λ and ∂B_λ in the proof. Since f is hyperbolic and J is a quasicircle, there exist a pair of closed annular neighborhoods W_1, W_2 of J and a quasiconformal mapping $\phi : W_1 \to \mathbb{A}_\varepsilon$ which conjugates $f : W_1 \to W_2$ to $z \mapsto z^d$, where $\mathbb{A}_\varepsilon := \{z : 1-\varepsilon \le |z| \le 1+\varepsilon\}$ and $\varepsilon > 0$ is small enough.

In order to define IFS, we lift J and f under the exponential map. Note that J separates 0 and ∞. We define a curve $\gamma := \phi^{-1}([(1-\varepsilon)^d, (1+\varepsilon)^d]) \subset W_2$. Fix a component of $\exp^{-1}(W_2 \setminus \gamma)$ and denote it by U. Then U is homeomorphic to a strip and $\exp : U \to W_2 \setminus \gamma$ is conformal in the interior of U. We use $\log : W_2 \setminus \gamma \to U$ to denote the inverse of this exponential map (see figure 2).

For each $n \ge 1$, the map $f^{\circ n} : W_1 \to W_2$ has d^n inverse branches, say $T_1, \ldots, T_{d^n}$, each maps $W_2 \setminus \gamma$ onto a half open quadrilateral such that their images are arranged in anticlockwise order one by one. Let $S_i := \log \circ T_i \circ \exp$ be an inject map defined in U,

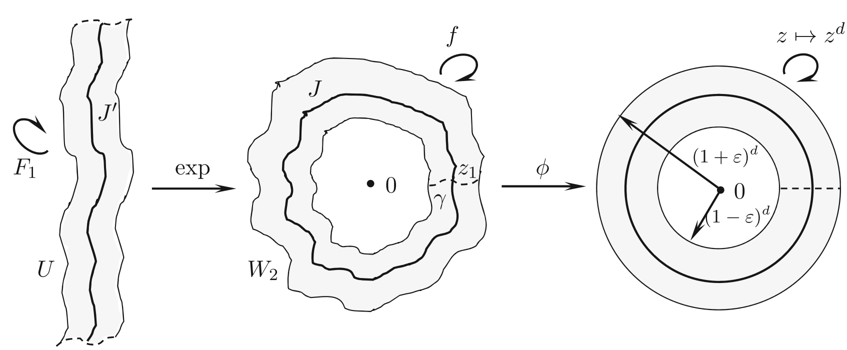

Figure 2. Sketch illustration of the construction of the IFS.

where $1 \le i \le d^n$. It is easy to see each S_i is conformal in the interior of U and can be conformally extended to an open neighborhood of $\bar{U}$.

By definition, it can be seen that $\{S_1, \ldots, S_{d^n}\}$ is an IFS defined on $\bar{U}$ since f is strictly expanding on W_1. The attractor J' of $\{S_1, \ldots, S_{d^n}\}$ is a closed set which satisfies $J = \exp(J')$. Further, $J \setminus \{z_1\}$ is the conformal image of J' with two ends moved, where $z_1 \in J \cap \gamma$ is a fixed point of f. This means that J' and J have the same Hausdorff dimension.

Let $F_n|_U := \bigsqcup_{i=1}^{d^n} S_i^{-1}|_{S_i(U)}$ be the lift of $f^{\circ n}$ under the exponential map. Then each $S_i(U)$ contains exactly one fixed point $\zeta_i \in J'$ of F_n in its interior for $1 < i < d^n$ and on its boundary for $i = 1$ and d^n. Since S_i can be conformally extended to an open neighborhood of $\bar{U}$, by Koebe's distortion theorem, there exist two constants $0 < C_1 \le 1 \le C_2$ both independent of n such that

$$\frac{C_1}{|F_n'(\zeta_i)|} \le \frac{|S_i(x) - S_i(y)|}{|x-y|} \le \frac{C_2}{|F_n'(\zeta_i)|}, \quad \forall\, 1 \le i \le d^n,\ x, y \in \bar{U}.$$

By Theorem 2.6, the Hausdorff dimension $D = \dim_H(J') = \dim_H(J)$ satisfies $s_1 \le D \le s_2$, where $\sum_{i=1}^{d^n} C_j^{s_j} |F_n'(\zeta_i)|^{-s_j} = 1$ and $j = 1, 2$. Then we have

$$\begin{aligned} \frac{1}{C_2^D} \le \frac{1}{C_2^{s_2}} \le \sum_{i=1}^{d^n} \frac{1}{|F_n'(\zeta_i)|^{s_2}} &\le \sum_{i=1}^{d^n} \frac{1}{|F_n'(\zeta_i)|^{D}} \\ &\le \sum_{i=1}^{d^n} \frac{1}{|F_n'(\zeta_i)|^{s_1}} = \frac{1}{C_1^{s_1}} \le \frac{1}{C_1^D}. \end{aligned} \tag{2.4}$$

The $d^n - 1$ fixed points of $f^{\circ n}$ in the Julia set J are $\{z_i = \exp(\zeta_i) : 1 \le i < d^n\}$. In particular, $z_1 = \exp(\zeta_1) = \exp(\zeta_{d^n})$. Since F_n is conformally conjugate to $f^{\circ n}$ in the interior of each $S_i(U)$, we have $F_n'(\zeta_i) = (f^{\circ n})'(z_i)$ for $1 \le i < d^n$. Therefore, by (2.4) and $|F_n'(\zeta_{d^n})| \to \infty$ as $n \to \infty$, we have

$$\begin{aligned} \sum_{z \in \mathrm{Fix}(f^{\circ n}) \cap \partial B_\lambda} \frac{1}{|(f^{\circ n})'(z)|^D} &= \sum_{i=1}^{d^n} \frac{1}{|(f^{\circ n})'(z_i)|^D} \\ &= \sum_{i=1}^{d^n} \frac{1}{|F_n'(\zeta_i)|^D} - |F_n'(\zeta_{d^n})|^{-D} = \mathcal{O}(1). \end{aligned}$$

The proof is complete. □

Proof of the main theorem. Note that ∂B_λ is the unit circle if $\lambda = 0$. For $z \in \partial B_0 = \mathbb{S}^1$, we have $f_0(z) = z^d$. By Lemma 2.3, there exists a holomorphic motion $\phi_\lambda : \partial B_0 \to \bar{\mathbb{C}}$ of ∂B_0 parametrized by $\mathcal{V}$ and with a base point 0 such that $\phi_\lambda(\partial B_0) = \partial B_\lambda$ and

$$f_\lambda \circ \phi_\lambda(z) = \phi_\lambda \circ f_0(z) = \phi_\lambda(z^d) \tag{2.5}$$

for all $z \in \partial B_0$ (see Chap. 4 of [8]). Since every point on ∂B_0 moves holomorphically, we can write $\phi_\lambda(z)$ in a power series of λ as

$$\phi_\lambda(z) = z\,(1 + u_1(z)\lambda + u_2(z)\lambda^2 + \mathcal{O}(\lambda^3)), \tag{2.6}$$

where $z \in \partial B_0$.

Substituting (2.6) and (1.1) into (2.5), then comparing the terms to the second order in λ, we obtain the following equations:

$$u_1(z^d) - du_1(z) = z^{-2d}, \tag{2.7}$$

$$u_2(z^d) - du_2(z) = \frac{d(d-1)}{2} u_1^2(z) - dz^{-2d} u_1(z). \tag{2.8}$$

For each non-zero integer $l \in \mathbb{Z}$, the functional equation

$$u(z^d) - du(z) = z^l \tag{2.9}$$

has the formal solution

$$u(z) = -\sum_{k=0}^{+\infty} \frac{z^{ld^k}}{d^{k+1}}. \tag{2.10}$$

Note that the solution (2.10) is convergent if $|z| \leq 1$. This means that the solutions of (2.7) is

$$u_1(z) = -\sum_{k=0}^{+\infty} \frac{z^{-2d^{k+1}}}{d^{k+1}}. \tag{2.11}$$

Therefore, equation (2.8) can be reduced to

$$u_2(z^d) - du_2(z) = \frac{d(d-1)}{2} \left(\sum_{l=0}^{+\infty} \frac{z^{-2d^{l+1}}}{d^{l+1}} \right)^2 + \sum_{l=0}^{+\infty} \frac{z^{-2d(d^l+1)}}{d^l}. \tag{2.12}$$

By (2.9), (2.10) and (2.12), the solution of u_2 is

$$u_2(z) = -\frac{d-1}{2} \sum_{k=0}^{+\infty} \frac{1}{d^k} \left(\sum_{l=0}^{+\infty} \frac{z^{-2d^{k+l+1}}}{d^{l+1}} \right)^2 - \sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} \frac{z^{-2d^{k+1}(d^l+1)}}{d^{k+l+1}}. \tag{2.13}$$

For each $n \geq 1$, the collection of the fixed points of $f_\lambda^{\circ n}$ on the boundary ∂B_λ forms the finite set

$$\mathrm{Fix}(f_\lambda^{\circ n}) \cap \partial B_\lambda = \left\{ \phi_\lambda(\mathrm{e}^{2\pi i t_j}) : t_j = \frac{j}{d^n - 1}, 1 \leq j \leq d^n - 1 \right\}. \tag{2.14}$$

By (2.5) and the chain rule, we have $(f_\lambda^{\circ n})'(\phi_\lambda(\mathrm{e}^{2\pi i t_j})) = \prod_{m=0}^{n-1} f_\lambda'(\phi_\lambda(\mathrm{e}^{2\pi i q^m t_j}))$. The calculation in Appendix shows that for every $D > 0$ and all sufficiently large n, the following holds:

$$\frac{1}{d^n - 1} \sum_{j=1}^{d^n - 1} \prod_{m=0}^{n-1} |f_\lambda'(\phi_\lambda(\mathrm{e}^{2\pi i q^m t_j}))|^{-D} = d^{-nD}(1 + D^2 n |\lambda|^2 + \mathcal{O}(\lambda^3)). \tag{2.15}$$

Let $D_\lambda := \dim_H(\partial B_\lambda)$ be the Hausdorff dimension of ∂B_λ. By (2.15), one can write the corresponding (2.3) of f_λ in Lemma 2.7 as

$$(d^n - 1)\, d^{-nD_\lambda}(1 + D_\lambda^2 n|\lambda|^2 + \mathcal{O}(\lambda^3)) = \mathcal{O}(1). \tag{2.16}$$

Fix some large n when λ is small enough. Then (2.16) is equivalent to

$$\exp\,(n(D_\lambda^2|\lambda|^2 - (D_\lambda - 1)\log d) + \mathcal{O}(\lambda^3)) = \mathcal{O}(1). \tag{2.17}$$

Substituting (2.2) into (2.17) and comparing the corresponding coefficients, we have

$$a_{20} = 0 \quad \text{and} \quad a_{11} = 1/\log d. \tag{2.18}$$

This means that

$$D_\lambda = 1 + \frac{|\lambda|^2}{\log d} + \mathcal{O}(\lambda^3). \tag{2.19}$$

By comparing with (2.2), the proof of the Main Theorem is complete. □

A. Appendix

This section is devoted to proving (2.15). From (1.1), we have

$$f_\lambda'(z) = dz^{d-1} - d\lambda/z^{d+1}. \tag{A.1}$$

Substituting (2.6) into (A.1), we have

$$\begin{aligned} f_\lambda'(\phi_\lambda(z)) \;=\;& dz^{d-1} + \frac{d}{z^{d+1}}((d-1)z^{2d}u_1(z) - 1)\,\lambda \\ &+ \frac{d}{z^{d+1}}\left(\frac{(d-1)(d-2)}{2} z^{2d}u_1^2(z) + (d+1)u_1(z)\right. \\ &\left. + (d-1)z^{2d}u_2(z)\right)\lambda^2 + \mathcal{O}(\lambda^3). \end{aligned} \tag{A.2}$$

Define $\sigma := \sigma(t) = \mathrm{e}^{2\pi i t} \in \mathbb{S}^1$. Then $\sigma\bar{\sigma} = 1$. For $0 \le m \le n-1$, by (A.2), we have

$$\begin{aligned} |f_\lambda'(\phi_\lambda(\sigma^{d^m}))|^2 \;=\;& f_\lambda'(\phi_\lambda(\sigma^{d^m}))\,\overline{f_\lambda'(\phi_\lambda(\sigma^{d^m}))} \\ =\;& d^2 + A_m\lambda + \bar{A}_m\bar{\lambda} + A_m\bar{A}_m|\lambda|^2/d^2 \\ & + B_m\lambda^2 + \bar{B}_m\bar{\lambda}^2 + \mathcal{O}(\lambda^3), \end{aligned} \tag{A.3}$$

where

$$A_m = d^2(d-1)\,u_1(\sigma^{d^m}) - d^2\,\sigma^{-2d^m} \tag{A.4}$$

and

$$\begin{aligned} B_m \;=\;& \frac{d^2(d-1)(d-2)}{2}u_1^2(\sigma^{d^m}) + d^3(d+1)\sigma^{-2d^m}u_1(\sigma^{d^m}) \\ & + d^2(d-1)u_2(\sigma^{d^m}). \end{aligned} \tag{A.5}$$

For every $D > 0$, by (A.3), we have

$$\begin{aligned}
\prod_{m=0}^{n-1}|f_\lambda'(\phi_\lambda(\sigma^{d^m}))|^{-D} &= \prod_{m=0}^{n-1}(|f_\lambda'(\phi_\lambda(\sigma^{d^m}))|^2)^{-\frac{D}{2}} \\
&= d^{-nD}\prod_{m=0}^{n-1}\Bigg(1+\frac{A_m\lambda+\bar{A}_m\bar{\lambda}+B_m\lambda^2+\bar{B}_m\bar{\lambda}^2}{d^2} \\
&\quad +\frac{A_m\bar{A}_m|\lambda|^2}{d^4}+\mathcal{O}(\lambda^3)\Bigg)^{-\frac{D}{2}} \\
&= d^{-nD}-\frac{D}{2}d^{-nD-2}\sum_{m=0}^{n-1}(A_m\lambda+\bar{A}_m\bar{\lambda}+B_m\lambda^2+\bar{B}_m\bar{\lambda}^2) \\
&\quad -\frac{D}{2}d^{-nD-4}\Big(\sum_{0\le m_1<m_2\le n-1}(A_{m_1}A_{m_2}\lambda^2+\bar{A}_{m_1}\bar{A}_{m_2}\bar{\lambda}^2) \\
&\quad +\sum_{0\le m_1,m_2\le n-1}A_{m_1}\bar{A}_{m_2}|\lambda|^2\Big) \\
&\quad +\frac{D(D+2)}{8}d^{-nD-4}\Big(\sum_{m=0}^{n-1}(A_m\lambda+\bar{A}_m\bar{\lambda})\Big)^2+\mathcal{O}(\lambda^3).
\end{aligned} \tag{A.6}$$

Lemma A.1. Let $m, m_1, m_2 \in \mathbb{N}$. If $n \geq 3$, then

(1) $2d^m \not\equiv 0 \bmod d^n - 1$;
(2) $2d^{m_1}(d^{m_2}+1) \not\equiv 0 \bmod d^n - 1$;
(3) $2(d^{m_1} - d^{m_2}) \equiv 0 \bmod d^n - 1$ *if and only if* $m_1 - m_2 = kn$ *for some* $k \in \mathbb{Z}$.

Proof. Since $(d, d^n - 1) = 1$, it follows that $(d^m, d^n - 1) = 1$. This means that $2d^m$ cannot be divided by $d^n - 1$ since $0 < 2 < d^n - 1$ for $n \geq 2$.

For the second assertion, set $m_1 + m_2 = kn + r$, where $k \geq 0$ and $0 \leq r \leq n - 1$. We have

$$2d^{m_1}(d^{m_2}+1) = 2(d^{kn+r} - d^r) + 2(d^r + 1) \equiv 2(d^r + 1) \not\equiv 0 \bmod d^n - 1$$

since $0 < 2(d^r + 1) \leq 2(d^{n-1} + 1) < d^n - 1$.

For Lemma A.1(3), set $m = kn + r$, where $k \geq 0$ and $0 \leq r \leq n - 1$. We have

$$2(d^m - 1) = 2(d^{kn+r} - d^r + d^r - 1) \equiv 2(d^r - 1) \bmod d^n - 1.$$

Note that $2(d^r - 1) < d^n - 1$, this means that $2(d^{m_1} - d^{m_2}) \equiv 0 \bmod d^n - 1$ if and only if $r = 0$. □

Following §2 of [14], it is convenient to introduce the *average notation*

$$\langle G(t)\rangle_n := \frac{1}{d^n - 1}\sum_{j=1}^{d^n-1} G(t_j), \tag{A.7}$$

where G is a continuous function defined on the interval $[0, 1)$ and $t_j = j/(d^n - 1)$.

In order to prove (2.15), we only need to prove that for every $D > 0$ and sufficiently large n, the following holds:

$$\left\langle \prod_{m=0}^{n-1} |f_\lambda'(\phi_\lambda(\sigma^{d^m}))|^{-D} \right\rangle_n = d^{-nD}(1 + D^2 n|\lambda|^2 + \mathcal{O}(\lambda^3)). \tag{A.8}$$

For each $n \geq 1$ and any $k \in \mathbb{Z}$, it is straightforward to verify that the average in (A.7) has following useful property:

$$\langle \sigma^k \rangle_n = \langle \mathrm{e}^{2\pi ikt} \rangle_n = \begin{cases} 1, & \text{if } k \equiv 0 \bmod d^n - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{A.9}$$

Lemma A.2. For $0 \leq m, m_1, m_2 \leq n-1$, *we have* $\langle \sigma^{-2d^m} \rangle_n = 0$, $\langle \sigma^{-2(d^{m_1}+d^{m_2})} \rangle_n = 0$, $\langle u_1(\sigma^{d^m}) \rangle_n = 0$, $\langle \sigma^{-2d^{m_1}} u_1(\sigma^{d^{m_2}}) \rangle_n = 0$, $\langle u_1(\sigma^{d^{m_1}}) u_1(\sigma^{d^{m_2}}) \rangle_n = 0$ *and* $\langle u_2(\sigma^{d^m}) \rangle_n = 0$.

Proof. By (2.11) and (2.13), the equations stated in the average property (A.9) and Lemma A.1(1)–(2) can be verified directly. □

As an immediate Corollary of Lemma A.2, from (A.4) and (A.5), we have

COROLLARY A.3

$\langle A_m \rangle_n = \langle \bar{A}_m \rangle_n = 0$, $\langle B_m \rangle_n = \langle \bar{B}_m \rangle_n = 0$, $\langle A_{m_1} A_{m_2} \rangle_n = \langle \bar{A}_{m_1} \bar{A}_{m_2} \rangle_n = 0$ *for* $0 \leq m, m_1, m_2 \leq n-1$.

By (A.6) and Corollary A.3, we have

$$\left\langle \prod_{m=0}^{n-1} |f'_\lambda(\phi_\lambda(\sigma^{d^m}))|^{-D} \right\rangle_n = d^{-nD} \left(1 + \frac{D^2}{4d^4} \sum_{0 \leq m_1, m_2 \leq n-1} \langle A_{m_1} \bar{A}_{m_2} \rangle_n |\lambda|^2 \right) + \mathcal{O}(\lambda^3). \tag{A.10}$$

By (A.4) and (A.5), we have

$$\begin{aligned} \langle A_{m_1} \bar{A}_{m_2} \rangle_n = {} & d^4(d-1)^2 \langle u_1(\sigma^{d^{m_1}}) \overline{u_1(\sigma^{d^{m_2}})} \rangle_n + d^4 \langle \sigma^{-2(d^{m_1}-d^{m_2})} \rangle_n \\ & - d^4(d-1) \langle u_1(\sigma^{d^{m_1}}) \sigma^{2d^{m_2}} + \overline{u_1(\sigma^{d^{m_2}})} \sigma^{-2d^{m_1}} \rangle_n. \end{aligned} \tag{A.11}$$

Since $0 \leq m_1, m_2 \leq n-1$, it follows that $m_1 - m_2 = kn$ for $k \in \mathbb{Z}$ if and only if $m_1 = m_2$. By Lemma A.1(3), we have

$$\langle \sigma^{-2(d^{m_1}-d^{m_2})} \rangle_n = \begin{cases} 1, & \text{if } m_1 = m_2, \\ 0, & \text{otherwise.} \end{cases} \tag{A.12}$$

This means that

$$\sum_{0 \leq m_1, m_2 \leq n-1} \langle \sigma^{-2(d^{m_1}-d^{m_2})} \rangle_n = n. \tag{A.13}$$

Similarly, by Lemma A.1(3), we have

$$\begin{aligned} \langle u_1(\sigma^{d^{m_1}}) \sigma^{2d^{m_2}} \rangle_n &= -\sum_{k=1}^{+\infty} \frac{\langle \sigma^{-2(d^{k+m_1}-d^{m_2})} \rangle_n}{d^k} \\ &= \begin{cases} -\sum_{k=0}^{+\infty} \frac{1}{d^{n-(m_1-m_2)+kn}} = -\frac{d^{m_1-m_2}}{d^n-1}, & \text{if } m_1 \geq m_2, \\ -\sum_{k=0}^{+\infty} \frac{1}{d^{m_2-m_1+kn}} = -\frac{d^{n-(m_2-m_1)}}{d^n-1}, & \text{if } m_1 < m_2. \end{cases} \end{aligned} \tag{A.14}$$

This means that

$$\sum_{0\le m_1,m_2\le n-1}\langle u_1(\sigma^{d^{m_1}})\sigma^{2d^{m_2}}\rangle_n = -\sum_{0\le m_2\le m_1\le n-1}\frac{d^{m_1-m_2}}{d^n-1} - \sum_{0\le m_1<m_2\le n-1}\frac{d^{n-(m_2-m_1)}}{d^n-1} = -\frac{n}{d^n-1}(1+d+\cdots+d^{n-1}) = \frac{n}{1-d}. \tag{A.15}$$

Moreover, by Lemma A.1(3), we have

$$\langle u_1(\sigma^{d^{m_1}})\overline{u_1(\sigma^{d^{m_2}})}\rangle_n = \sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty}\frac{\langle\sigma^{-2(d^{k_1+m_1}-d^{k_2+m_2})}\rangle_n}{d^{k_1+k_2}}. \tag{A.16}$$

Similar to the reduction process of (A.15), we have

$$\sum_{0\le m_1,m_2\le n-1}\langle u_1(\sigma^{d^{m_1}})\overline{u_1(\sigma^{d^{m_2}})}\rangle_n = \frac{n}{(d-1)^2}. \tag{A.17}$$

By substituting (A.13), (A.15) and (A.17) into (A.11), we have

$$\sum_{0\le m_1,m_2\le n-1}\langle A_{m_1}\bar{A}_{m_2}\rangle_n = 4nd^4. \tag{A.18}$$

By (A.10) and (A.18), it follows that (A.8) holds. The proof of (2.15) is complete. □

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References

[1] Collet P, Dobbertin R and Moussa P, Multifractal analysis of nearly circular Julia set and thermodynamical formalism, *Ann. Inst. H Poincaré*, **56** (1992) 91–122

[2] Devaney R, Baby Mandelbrot sets adorned with halos in families of rational maps, Complex dynamics, 37–50, Contemp. Math., 396, Amer. Math. Soc. (2006) (Providence, RI)

[3] Devaney R and Garijo A, Julia sets converging to the unit disk, *Proc. AMS* **136** (2008) 981–988

[4] Devaney R, Look D and Uminsky D, The Escape Trichotomy for Singularly Perturbed Rational Maps *Indiana Univ. Math. J.* **54 (6)** (2005) 1621–1634

[5] Falconer K, Fractal geometry: Mathematical foundations and applications, second edition (2003) (England: John Wiley & Sons)

[6] Mañé R, Sad P and Sullivan D, On the dynamics of rational maps, *Ann. Sci. École Norm. Sup.* **16(4)** (1983) 193–217

[7] McMullen C, Automorphisms of rational maps, holomorphic functions and moduli I (Mathematical Sciences Research Institute Publications, 10) (1988) (New York: Springer)

[8] McMullen C, Complex Dynamics and Renormalization, Ann. of Math. Studies 135 (1994) (Princeton, NJ: Princeton Univ. Press)

[9] McMullen C, Thermodynamics, dimension and the Weil–Peterson metric, *Invent. Math.* **173(2)** (2008) 365–425

[10] Osbaldestin A, $1/s$-expansion for generalized dimensions in a hierarchical s-state Potts model, *J. Phys. A: Math. Gen.* **28** (1995) 5951–5962

[11] Qiu W, Wang X and Yin Y, Dynamics of McMullen maps, *Adv. Math*. **229** (2012) 2525–2577

[12] Ruelle D, Repellers for real analytic maps, *Ergodic Theory Dyn. Syst.* **2** (1982) 99–107

[13] Steinmetz N, On the dynamics of the McMullen family $R(z) = z^m + \lambda/z^\ell$, *Conform. Geom. Dyn.* **10** (2006) 159–183

[14] Widom M, Bensimon D, Kadanoff L and Shenker S, Strange objects in the complex plane, *J. Stat. Phys.* **32** (1983) 443–454

[15] Yang F and Zeng J, On the dynamics of a family of generated renormalization tranformations, *J. Math. Anal. Appl*. **413** (2014) 361–377.

[16] Zinsmeister M, Formalisme thermodynamique et systèmes dynamiques holomorphes, Panoramas et synthèses, Société Mathématique de France (1996)