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Zeros and uniqueness of Q-difference polynomials of meromorphic functions with zero order

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Abstract. In this paper, we investigate the value distribution of q-difference polynomials of meromorphic function of finite logarithmic order, and study the zero distribution of difference-differential polynomials $[f^n(z)f(qz+c)]^{(k)}$ and $[f^n(z)(f(qz+c)-f(z))]^{(k)}$, where $f(z)$ is a transcendental function of zero order. The uniqueness problem of difference-differential polynomials is also considered.

Keywords. Meromorphic functions; Nevanlinna theory; logarithmic order; uniqueness problem; difference-differential polynomial.

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1. Introduction and main results

In this paper, we assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory (see for example [14]). A meromorphic function $f(z)$ means meromorphic in the complex plane $\mathbb{C}$. If no poles occur, then $f(z)$ reduces to an entire function. A meromorphic function $\alpha(z)$ is called a small function with respect to $f(z)$, if $T(r,\alpha)=S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set E of finite logarithmic measure. If $f-a$ and $g-a$ have the same zeros, then we say f and g share the value a ignoring multiplicities (IM), if $f-a$ and $g-a$ have the same zeros with the same multiplicities, then f and g share the value a counting multiplicities (CM). We denote by $N_p(r,\frac{1}{f-a})$ the counting function of the zeros of $f-a$, where an m-fold zero is counted m times if $m\le p$ and p times if $m>p$. The difference operators for a meromorphic function f are defined as

$$\triangle_c f(z)=f(z+c)-f(z)\quad (c\neq 0),$$

$$\nabla_q f(z)=f(qz)-f(z)\quad (q\neq 0,1).$$

The order and the exponent of convergence of zeros of meromorphic function $f(z)$ is respectively defined as

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}.$$

A Borel exceptional value of $f(z)$ is any value a satisfying $\lambda(f-a) < \sigma(f)$.

The logarithmic order of a meromorphic function $f(z)$ is defined as

$$\sigma_{\log}(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log\log r}.$$

$f(z)$ is said to be of finite logarithmic order if the above limit superior is finite. It is clear that if a meromorphic function $f(z)$ has finite logarithmic order, then the order of $f(z)$ is zero. From the definition of logarithmic order, it is easily seen that the logarithmic order of a constant function is zero and of a non-constant rational function is 1. For a transcendental meromorphic function $f(z)$ the logarithmic order is at least 1. There is no meromorphic function with logarithmic order strictly between 0 and 1.

If $f(z)$ is a meromorphic function of finite positive logarithmic order $\sigma_{\log}(f)$, then $T(r, f)$ has proximate logarithmic order $\sigma_{\log}(r)$. The logarithmic-type function of $T(r, f)$ is defined as

$$U(r, f) = (\log r)^{\sigma_{\log}(r)}.$$

We have $T(r, f) \le U(r, f)$ for sufficiently larger r. The logarithmic exponent of convergence of a-points of $f(z)$ is equal to the logarithmic order of $n(r, f=a)$ which is defined as

$$\lambda_{\log}(a) = \limsup_{r\to\infty} \frac{\log n\left(r, \frac{1}{f-a}\right)}{\log\log r}.$$

It is known that for any meromorphic function $f(z)$ of finite positive order and for any $a \in \mathbb{C}$, the counting functions $N(r, f=a)$ and $n(r, f=a)$ both have the same order, the situation is different for functions of finite logarithmic order, that is the logarithmic order of $N(r, f=a)$ is $\lambda_{\log}(a)+1$, where $\lambda_{\log}(a)$ is the logarithmic order of $n(r, f=a)$ (see [2]).

In the following, we assume that a, q are nonzero complex constants, m, n and k are positive integers, unless otherwise specified.

Recently, the difference variant of the Nevanlinna theory has been established independently in [1, 3–5]. With the development of difference analogue of Nevanlinna theory, many authors paid attention to the value distribution of difference polynomials. Liu and Qi [8] proved the following theorem which considered q-difference polynomials which can be seen as a different version of Hayman conjecture.

Theorem A (Theorems 1.1 and 1.2 of [8]). *If $f(z)$ is a transcendental meromorphic function of zero order, a, q are nonzero complex constants. If $n \ge 6$, then $f^n(z)f(qz+c)$ assumes every nonzero value $b \in \mathbb{C}$ infinitely often. If $n \ge 8$, then $f^n(z) + a[f(qz+c) - f(z)]$ assumes every nonzero value $b \in \mathbb{C}$ infinitely often.*

In [10], Liu *et al.* extended this to consider zero distributions of q-difference products $f^n(z)(f^m(z)-a)f(qz+c)$ and $f^n(z)(f^m(z)-a)[f(qz+c)-f(z)]$ for meromorphic function f of zero order.

Theorem B (Theorems 1.1 and 1.3 of [10]). *If $f(z)$ is a transcendental meromorphic function of zero order, a, q are nonzero complex constants, $\alpha(z)$ is a nonzero small function with respect to f. If $n \geq 6$, then $f^n(z)(f^m(z) - a)f(qz + c) - \alpha(z)$ has infinitely many zeros. If $n \geq 7$, then $f^n(z)(f^m(z) - a)[f(qz + c) - f(z)] - \alpha(z)$ has infinitely many zeros.*

Recently, using the idea of finite logarithmic order, Xu and Zhang [12] obtained the following theorems.

Theorem C (Theorem 2.1 of [12]). *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\sigma_{\log}(f)$, with the logarithmic exponent of convergence of poles less than $\sigma_{\log}(f)-1$ and q, c are nonzero complex constants, then for $n \geq 2$, $f^n(z)f(qz+c)$ assumes every value $b \in \mathbb{C}$ infinitely often.*

Theorem D (Theorem 2.2 of [12]). *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\sigma_{\log}(f)$, with the logarithmic exponent of convergence of poles less than $\sigma_{\log}(f) - 1$ and a, q are nonzero complex constants, then for $n \geq 5$, $f^n(z) + a[f(qz + c) - f(z)]$ assumes every value $b \in \mathbb{C}$ infinitely often.*

The main purpose of this paper is to also adopt the idea of finite logarithmic order, and consider the value distribution of meromorphic functions of finite logarithmic order. The first theorem (Theorem 1.1 below) is of the value distribution of the difference polynomials $af^n(qz + c) + f(z)$ for a meromorphic function $f(z)$ of finite logarithmic order.

Theorem 1.1. *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\sigma_{\log}(f)$, with the logarithmic exponent of convergence of poles less than $\sigma_{\log}(f)-1$ and a, q are nonzero complex constants. If $n \geq 3$, then $af^n(qz + c) + f(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a small function with respect to $f(z)$.*

The following two main theorems will consider difference polynomials in Theorem B.

Theorem 1.2. *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\sigma_{\log}(f)$, with the logarithmic exponent of convergence of poles less than $\sigma_{\log}(f)-1$ and a, q are nonzero complex constants, m, n are positive integers. If $n \geq 2$, then $f^n(z)(f^m(z) - a)f(qz + c) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a small function with respect to $f(z)$.*

Theorem 1.3. *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\sigma_{\log}(f)$, with the logarithmic exponent of convergence of poles less than $\sigma_{\log}(f)-1$ and a, q are nonzero complex constants, m, n are positive integers. If $n \geq 4$, then $f^n(z)(f^m(z) - a)[f(qz + c) - f(z)] - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a small function with respect to $f(z)$.*

Recall that Zhang and Korhonen [15] proved that for a transcendental meromorphic function $f(z)$ of zero order and a nonzero complex constant q and $n \geq 6$, both $f^n(z)f(qz)$ and $f^n(z)(f(z) - 1)f(qz)$ assume every nonzero value $a \in \mathbb{C}$ infinitely often. Next, we consider the value distribution of $f^n(z)(f(z) - 1)f(qz)$ for a meromorphic function $f(z)$ of finite logarithmic order.

Theorem 1.4. *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\sigma_{\log}(f)$, with the logarithmic exponent of convergence of poles less than $\sigma_{\log}(f)-1$ and q is a nonzero complex constant, n is a positive integer, then $f^n(z)(f(z)-1)f(qz)-\alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a small function with respect to $f(z)$.*

The zero distribution of differential polynomials is a classical topic in the theory of meromorphic functions. Hayman (Theorem 10 of [6]) firstly considered the value distribution of $f^n f'-1$, where f is a transcendental function. Wang and Fang [11] improved this result and proved that for a transcendental entire function $f(z)$, n, k are two positive integers with $n \geq k+1$, then $(f^n)^{(k)}-1$ has infinitely many zeros. For the difference analogues of this problem, Liu *et al.* [9] investigated the zeros of $[f^n(z)f(z+c)]^{(k)}-\alpha(z)$ and $[f^n(z)\Delta_c f]^{(k)}-\alpha(z)$, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.

Theorem E (Theorems 1.1 and 1.3 of [9]). *Let f be a transcendental entire function of finite order and $\alpha(z)$ be a nonzero small function with respect to $f(z)$. If $n \geq k+2$, then $[f^n(z)f(z+c)]^{(k)}-\alpha(z)$ has infinitely many zeros. If f is not a periodic function with period c and $n \geq k+3$, then $[f^n(z)\Delta_c f]^{(k)}-\alpha(z)$ has infinitely many zeros.*

We will consider zeros of q-difference differential polynomials and obtain the following theorems.

Theorem 1.5. *If f is a transcendental meromorphic function of zero order and $\alpha(z)$ is a nonzero small function with respect to $f(z)$. If $n \geq k+6$, then $[f^n(z)f(qz+c)]^{(k)}-\alpha(z)$ has infinitely many zeros.*

If f is a transcendental entire function of zero order, from the proof of Theorem 1.5, one can immediately get the following corollary.

COROLLARY 1.1

If f is a transcendental entire function of zero order and $\alpha(z)$ is a nonzero small function with respect to $f(z)$. If $n \geq k+4$, then $[f^n(z)f(qz+c)]^{(k)}-\alpha(z)$ has infinitely many zeros.

Theorem 1.6. *If f is a transcendental meromorphic function of zero order and $\alpha(z)$ is a nonzero small function with respect to $f(z)$, $f(qz+c) \not\equiv f(z)$. If $n \geq k+8$, then $[f^n(z)(f(qz+c)-f(z))]^{(k)}-\alpha(z)$ has infinitely many zeros.*

Similarly, when we consider a transcendental entire function of zero order, the following result is immediately true from the proof of Theorem 1.6.

COROLLARY 1.2

If f is a transcendental entire function of zero order and $\alpha(z)$ is a nonzero small function with respect to $f(z)$, $f(qz+c) \not\equiv f(z)$. If $n \geq k+4$, then $[f^n(z)(f(qz+c)-f(z))]^{(k)}-\alpha(z)$ has infinitely many zeros.

By using the idea of finite logarithmic order and similar discussions as Theorem 1.2, one can obtain that Corollaries 1.1 and 1.2 are reduced to the following corollaries which are true for any positive integers n and k.

COROLLARY 1.3

If f is a transcendental entire function of finite logarithmic order and $\alpha(z)$ is a small function with respect to $f(z)$. Then for any positive integers n and k, $[f^n(z)f(qz+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.

COROLLARY 1.4

If f is a transcendental entire function of finite logarithmic order and $\alpha(z)$ is a small function with respect to $f(z)$, $f(qz+c) \not\equiv f(z)$. Then for any positive integers n and k, $[f^n(z)(f(qz+c) - f(z))]^{(k)} - \alpha(z)$ has infinitely many zeros.

Liu *et al.* [9] also considered the uniqueness problem of two c-difference differential polynomials sharing a common value and obtained the following result.

Theorem F (Theorems 1.5 of [9]). *Let f and g be transcendental entire functions of finite order, $n \geq 2k+6$. If $[f^n(z)f(z+c)]^{(k)}$ and $[g^n(z)g(z+c)]^{(k)}$ share the value 1 CM, then either $f(z) = c_1 e^{Cz}$, $g(z) = c_2 e^{-Cz}$, where c_1, c_2 and C are constants satisfying $(-1)^k(c_1c_2)^{n+1}[(n+1)C]^{2k} = 1$ or $f = tg$, where $t^{n+1} = 1$.*

Similarly, we will extend Theorem F to q-difference differential polynomials and obtain the following result for transcendental entire function of zero order.

Theorem 1.7. *Let f and g be transcendental entire functions of zero order, $n \geq 2k+m+6$. If $[f^n(z)(f^m(z)-a)f(qz+c)]^{(k)}$ and $[g^n(z)(g^m(z)-a)g(qz+c)]^{(k)}$ share the value 1 CM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

From the proof of Theorem 1.7, one can immediately get the following corollary.

COROLLARY 1.5

Let f and g be transcendental entire functions of zero order, $n \geq 2k+6$. If $[f^n(z)f(qz+c)]^{(k)}$ and $[g^n(z)g(qz+c)]^{(k)}$ share the value 1 CM, then $f = tg$, where $t^{n+1} = 1$.

2. Some lemmas

The first lemma is the characteristic function relationship between $f(z)$ and $f(qz+c)$, provided that $f(z)$ is a transcendental meromorphic function of finite logarithmic order.

Lemma 2.1 [12]. *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order and $q \in \mathbb{C} \setminus \{0\}$, then*

$$T(r, f(qz+c)) = (1+o(1))T(r, f) \tag{2.1}$$

on a set of lower logarithmic density 1.

Lemma 2.2 [12]. *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order and $q \in \mathbb{C} \setminus \{0\}$, then*

$$N(r, f(qz+c)) = (1+o(1))N(r, f) \tag{2.2}$$

on a set of lower logarithmic density 1.

Lemma 2.3 [8]. *Let $f(z)$ be a nonconstant meromorphic function of zero order, and let $q \in \mathbb{C} \setminus \{0\}$, then*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S(r, f) \tag{2.3}$$

on a set of logarithmic density 1.

Chern [2] obtained the following lemma in which a and b are two distinct extended complex values. From the proof of Theorem 7.1 of [2], one can easily see that complex values a and b can be changed into $a(z)$ and $b(z)$, where $a(z)$ and $b(z)$ are two distinct small functions with respect to $f(z)$.

Lemma 2.4. *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\sigma_{\log}(f)$, then for any two distinct small functions $a(z)$ and $b(z)$ with respect to $f(z)$, we have*

$$T(r, f) \le N(r, f=a) + N(r, f=b) + o(U(r, f)), \tag{2.4}$$

where $U(r, f) = (\log r)^{\sigma_{\log}(r)}$ is a logarithmic-type function of $T(r, f)$. Furthermore, if $T(r, f)$ has a finite lower logarithmic order

$$\mu = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log\log r},$$

with $\sigma_{\log}(f) - \mu < 1$, then

$$T(r, f) \le N(r, f=a) + N(r, f=b) + o(T(r, f)).$$

Lemma 2.5 [14]. *Let $f(z)$ be a transcendental meromorphic function and k be a positive integer. Then*

$$T(r, f^{(k)}) \le T(r, f) + k\bar{N}(r, f) + S(r, f). \tag{2.5}$$

Lemma 2.6 [7]. *Let $f(z)$ be a nonconstant meromorphic function and p, k be positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \le T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f), \tag{2.6}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \le kN(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.7}$$

Lemma 2.7 [13]. *Let F and G be nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

(i) $T(r) \leq N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G) + S(r)$, *where* $T(r) = \max\{T(r, F), T(r, G)\}$ *and* $S(r) = \max\{S(r, F), S(r, G)\}$,
(ii) $F = G$,
(iii) $F \cdot G = 1$.

Lemma 2.8 [10]. *Let $f(z)$ be a transcendental entire function of zero order, a, q be nonzero complex constants, m, n be positive integers. Set*

$$F = f^n(z)(f^m(z) - a)f(qz + c),$$

then

$$T(r, F) = (n + m + 1)T(r, f) + S(r, f) \tag{2.8}$$

on a set of logarithmic density 1. *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order, then*

$$(n+m-1)T(r, f)+S(r, f) \leq T(r, F) \leq (n+m+1)T(r, f)+S(r, f) \tag{2.9}$$

on a set of logarithmic density 1.

Lemma 2.9. *Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order, a, q be nonzero complex constants and m, n be positive integers. If*

$$[f^n(z)(f^m(z) - a)f(qz + c)]^{(k)} = [g^n(z)(g^m(z) - a)g(qz + c)]^{(k)} \tag{2.10}$$

and $n \geq m + 5$, then $f = tg$, where $t^{n+1} = t^m = 1$.

Proof. By (2.10), we get $f^n(z)(f^m(z)-a)f(qz+c) = g^n(z)(g^m(z)-a)g(qz+c)+P(z)$, where $P(z)$ is a polynomial of degree at most $k - 1$. If $P(z) \not\equiv 0$, then

$$\frac{f^n(z)(f^m(z) - a)f(qz + c)}{P(z)} = \frac{g^n(z)(g^m(z) - a)g(qz + c)}{P(z)} + 1.$$

Since $f(z)$ and $g(z)$ are transcendental entire functions of zero order, from the second main theorem and (2.8), we get

$$\begin{aligned}
&(n + m + 1)T(r, f) \\
&\quad = T\left(r, \frac{f^n(z)(f^m(z) - a)f(qz + c)}{P(z)}\right) + S(r, f) \\
&\quad \leq \bar{N}\left(r, \frac{f^n(z)(f^m(z)-a)f(qz+c)}{P(z)}\right) + \bar{N}\left(r, \frac{P(z)}{f^n(z)(f^m(z)-a)f(qz+c)}\right) \\
&\qquad + \bar{N}\left(r, \frac{P(z)}{g^n(z)(g^m(z) - a)g(qz + c)}\right) + S(r, f) \\
&\quad \leq \bar{N} + \bar{N}\left(r, \frac{1}{f^m(z) - a}\right) + \bar{N}\left(r, \frac{1}{f(qz + c)}\right) + \bar{N}\left(r, \frac{1}{g^n(z)}\right)
\end{aligned}$$

$$+\bar{N}\left(r, \frac{1}{g^m(z)-a}\right)+\bar{N}\left(r, \frac{1}{g(qz+c)}\right)+S(r, f)$$
$$\leq (m+2)[T(r, f)+T(r, g)]+S(r, f)+S(r, g).$$

Similarly as above, we obtain

$$(n+m+1)T(r, g) \leq (m+2)[T(r, f)+T(r, g)]+S(r, f)+S(r, g).$$

So

$$(n+m+1)[T(r, f)+T(r, g)] \leq 2(m+2)[T(r, f)+T(r, g)]+S(r, f)+S(r, g),$$

which contradicts with $n \geq m+5$. Hence, we get $P(z) \equiv 0$. Then

$$f^n(z)(f^m(z)-a)f(qz+c) = g^n(z)(g^m(z)-a)g(qz+c).$$

Let $h(z) = \frac{f(z)}{g(z)}$, we get

$$g^m(z)[h^{n+m}(z)h(qz+c)-1] = a[h^n(z)h(qz+c)-1]. \tag{2.11}$$

In the following, we will prove $h(z)$ to be a constant.

Case 1. Suppose $h(z)$ is a nonconstant rational function. Since $g(z)$ is a transcendental entire function, from (2.11) we get

$$h^{n+m}(z)h(qz+c) \equiv 1, h^n(z)h(qz+c) \equiv 1.$$

So $h^m(z) \equiv 1$, which contradicts with $h(z)$ is a nonconstant.

Case 2. Suppose $h(z)$ is a transcendental function. We claim $h^{n+m}(z)h(qz+c)$ is not a constant, but suppose on the contrary, $h^{n+m}(z)h(qz+c)$ is a constant c, then $h^{n+m}(z) = c/h(qz+c)$. Thus

$$(n+m)T(r, h(z)) = T(r, h^{n+m}(z)) \leq T(r, h(z)) + S(r, h),$$

which contradicts that n, m are positive integers.

Denote $H(z) = h^{n+m}(z)h(qz+c)$.

Subcase 2.1. Suppose 1 is a Picard exceptional value of $H(z)$. Using the second main theorem to $H(z)$, we obtain

$$\begin{aligned}
(n+m)T &= T(r, h^{n+m}(z)) \\
&= T\left(r, \frac{H(z)}{h(qz+c)}\right) \\
&\leq T(r, H(z)) + T(r, h(qz+c)) + S(r, h) \\
&\leq \bar{N}(r, H(z)) + \bar{N}\left(r, \frac{1}{H(z)}\right) + \bar{N}\left(r, \frac{1}{H(z)-1}\right) \\
&\quad + T(r, h(qz+c)) + S(r, h) \\
&\leq 4T(r, h(z)) + T(r, h(qz+c) + S(r, h) \\
&\leq 5T(r, h(z)) + S(r, h),
\end{aligned}$$

which contradicts with $n \geq m+5$.

Subcase 2.2. Suppose that there exists a point z_0 such that $h^{n+m}(z_0)h(qz_0+c) = 1$. Since $g(z)$ is a transcendental entire function, from (2.11), we have $h^n(z_0)h(qz_0+c) = 1$. Hence $h^m(z_0) = 1$, and

$$\bar{N}\left(r, \frac{1}{h^{n+m}(z)h(qz+c)-1}\right) \leq \bar{N}\left(r, \frac{1}{h^m(z)-1}\right) \leq mT(r,h).$$

From the above inequality, we get

$$\begin{aligned}
&(n+m)T(r,h(z)) = T(r,h^{n+m}(z)) \\
&\quad = T\left(r, \frac{H(z)}{h(qz+c)}\right) \\
&\quad \leq T(r,H(z)) + T(r,h(qz+c)) + S(r,h) \\
&\quad \leq \bar{N}(r,H(z)) + \bar{N}\left(r,\frac{1}{H(z)}\right) + \bar{N}\left(r,\frac{1}{H(z)-1}\right) + T(r,h(qz+c)) + S(r,h) \\
&\quad \leq (m+4)T(r,h(z)) + T(r,h(qz+c) + S(r,h) \\
&\quad \leq (m+5)T(r,h(z)) + S(r,h),
\end{aligned}$$

which contradicts with $n \geq m+5$.

Therefore, $h(z)$ is a constant t and substituting this into (2.11), we have

$$g^m(z)[t^{n+m+1}-1] = a[t^{n+1}-1].$$

Since $g(z)$ is a transcendental entire function, we deduce from the above equation that $t^{n+1} = t^m = 1$. Hence $f = tg$, where $t^{n+1} = t^m = 1$. □

3. The proofs

3.1 *Proof of Theorem* 1.1

Denote

$$F = \frac{\alpha(z) - f(z)}{af^n(qz+c)}.$$

Since $f(z)$ is a transcendental meromorphic function of finite logarithmic order, by Lemma 2.1, we obtain

$$\begin{aligned}
nT(r,f) &= T(r,af^n(qz+c)) + S(r,f) \\
&= T\left(r, \frac{F}{\alpha(z)-f(z)}\right) + S(r,f) \\
&\leq T(r,F) + T(r,f) + S(r,f),
\end{aligned}$$

which implies that

$$(n-1)T(r,f) + S(r,f) \leq T(r,F) \tag{3.1}$$

on a set of lower logarithmic density 1. From the above inequality, we get that $F(z)$ is transcendental as $f(z)$ is a transcendental meromorphic function and $n \geq 3$. On the other hand, we can easily get

$$T(r,F) \leq (n+1)T(r,f) + S(r,f). \tag{3.2}$$

By (3.1), (3.2) and $n \geq 3$, we get

$$T(r, F) = O(T(r, f)).$$

By Lemma 2.2, we obtain

$$N\left(r, \frac{1}{F-1}\right) = N\left(r, \frac{af^n(qz+c))}{af^n(qz+c) + f(z) - \alpha(z)}\right)$$
$$\leq N\left(r, \frac{1}{af^n(qz+c)+f(z) - \alpha(z)}\right)+nN(r, f) + S(r, f).$$

Suppose $af^n(qz+c) + f(z) - \alpha(z)$ has finitely many zeros, then

$$N\left(r, \frac{1}{F-1}\right) \leq nN(r, f) + S(r, f). \tag{3.3}$$

Since the logarithmic exponent of convergence of poles of $f(z)$ is less than $\sigma_{\log}(f) - 1$, we have

$$\limsup_{r\to\infty} \frac{\log N(r, f)}{\log\log r} < \sigma_{\log}(f).$$

By Lemmas 2.2, 2.4 and eq. (3.3), we obtain

$$\begin{aligned} T(r, F) &\leq N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F-1}\right) + o(U(r, F)) \\ &\leq N\left(r, \frac{1}{\alpha(z) - f(z)}\right) + 2nN(r, f) + S(r, f) + o(U(r, f)) \\ &\leq T(r, f) + 2nN(r, f) + S(r, f) + o(U(r, f)). \end{aligned}$$

Combining this inequality and (3.1), we get

$$(n-2)T(r, f) \leq 2nN(r, f) + S(r, f) + o(U(r, f)).$$

Since $n \geq 3$,

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{\log\log r} \leq \limsup_{r\to\infty} \frac{\log N(r, f)}{\log\log r} < \sigma_{\log}(f),$$

which contradicts that $T(r, f)$ has logarithmic order $\sigma_{\log}(f)$.

Hence, $af^n(qz+c) + f(z) - \alpha(z)$ has infinitely many zeros.

3.2 *Proof of Theorem* 1.2

Denote

$$F(z) = f^n(z)(f^m(z) - a)f(qz+c).$$

By Lemma 2.8, we get

$$(n+m-1)T(r, f)+S(r, f) \leq T(r, F) \leq (n+m+1)T(r, f)+S(r, f). \tag{3.4}$$

From (3.4), we obtain $T(r, F) = O(T(r, f))$, $F(z)$ is transcendental as $f(z)$ is a transcendental meromorphic function and $n \geq 2$. Suppose that $F - \alpha(z)$ has finitely many zeros, then

$$N\left(r, \frac{1}{F - \alpha(z)}\right) = S(r, F) = S(r, f). \tag{3.5}$$

Since the logarithmic exponent of convergence of poles of f is less than $\sigma_{\log}(f) - 1$, we have

$$\limsup_{r\to\infty} \frac{\log N(r, f)}{\log\log r} < \sigma_{\log}(f).$$

By Lemmas 2.2 and 2.4 and eqs (3.4) and (3.5) we obtain

$$\begin{aligned}
&(n + m - 1)T(r, f) \leq T(r, F) + S(r, f)\\
&\leq N(r, F) + N\left(r, \frac{1}{F - \alpha(z)}\right) + S(r, f) + o(U(r, F))\\
&\leq N(r, f^n(z)) + N(r, f^m(z) - a) + N(r, f(qz + c)) + S(r, f) + o(U(r, F))\\
&\leq (n + 1)N(r, f) + mT(r, f) + S(r, f) + o(U(r, F)).
\end{aligned}$$

Hence, we get

$$(n - 1)T(r, f) \leq (n + 1)N(r, f) + S(r, f) + o(U(r, f)).$$

Since $n \geq 2$, the above inequality implies that

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{\log\log r} \leq \limsup_{r\to\infty} \frac{\log N(r, f)}{\log\log r} < \sigma_{\log}(f),$$

which contradicts that $T(r, f)$ has logarithmic order $\sigma_{\log}(f)$. Hence,

$$N\left(r, \frac{1}{F - \alpha(z)}\right) \neq S(r, F).$$

So $F - \alpha(z)$ has infinitely many zeros, that is, $f^n(z)(f^m(z) - a)f(qz + c) - \alpha(z)$ has infinitely many zeros.

3.3 *Proofs of Theorems* 1.3 *and* 1.4

Using similar argument as in the proof of Theorem 1.2, one can deduce Theorems 1.3 and 1.4. We omit the details here.

3.4 *Proof of Theorem* 1.5

Denote $F(z) = f^n(z)f(qz + c)$, then by Lemma 2.1, we get

$$\begin{aligned}
nT(r, f) = T(r, f^n(z)) &= T\left(r, \frac{F(z)}{f(qz + c)}\right)\\
&\leq T(r, F) + T(r, f(qz + c))\\
&\leq T(r, F) + T(r, f) + S(r, f),
\end{aligned}$$

so

$$(n-1)T(r,f)+S(r,f)\le T(r,F) \tag{3.6}$$

on a set of lower logarithmic density 1. From the above inequality, we get that $F(z)$ is transcendental as $f(z)$ is a transcendental meromorphic function and $n\ge k+6$. On the other hand, we can easily get

$$T(r,F)\le (n+1)T(r,f)+S(r,f). \tag{3.7}$$

By (3.6), (3.7) and $n\ge k+6$, we get

$$T(r,F)=O(T(r,f)).$$

Suppose $F(z)^{(k)}-\alpha(z)$ has only finitely many zeros, and since f is a transcendental meromorphic function of zero order, by the classical result of Nevanlinna theory, we easily get $S(r,F^{(k)})=S(r,F)=S(r,f)$. From the second main theorem and (2.6), we obtain

$$\begin{aligned}
T(r,F^{(k)}) &\le \bar{N}(r,F^{(k)})+\bar{N}\left(r,\frac{1}{F^{(k)}}\right)+\bar{N}\left(r,\frac{1}{F^{(k)}-\alpha(z)}\right)+S(r,F^{(k)})\\
&\le N_1\left(r,\frac{1}{F^{(k)}}\right)+\bar{N}(r,F^{(k)})+S(r,F^{(k)})\\
&\le T(r,F^{(k)})-T(r,F)+N_{k+1}\left(r,\frac{1}{F}\right)+\bar{N}(r,F)+S(r,F^{(k)}).
\end{aligned}$$

Combining this inequality with (3.6), we obtain

$$\begin{aligned}
&(n-1)T(r,f)+S(r,f)\le T(r,F)\\
&\quad\le N_{k+1}\left(r,\frac{1}{F}\right)+\bar{N}(r,F)+S(r,f)\\
&\quad\le N_{k+1}\left(r,\frac{1}{f^n(z)f(qz+c)}\right)+\bar{N}(r,f^n(z)f(qz+c))+S(r,f)\\
&\quad\le (k+1)\bar{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f(qz+c)}\right)+\bar{N}(r,f^n(z))\\
&\qquad+\bar{N}(r,f(qz+c))+S(r,f)\\
&\quad\le (k+4)T(r,f)+S(r,f),
\end{aligned}$$

which contradicts $n\ge k+6$.

3.5 *Proof of Theorem* 1.6

Denote $F(z)=f^n(z)[f(qz+c)-f(z)]$. Since $f(z)$ is a transcendental meromorphic function, using the first main theorem and Lemmas 2.1 and 2.3, we get

$$\begin{aligned}
&(n+1)T(r,f(z))=T(r,f^{n+1}(z))=T\left(r,\frac{F(z)}{f(qz+c)-f(z)}\cdot f(z)\right)\\
&\quad\le T(r,F)+T\left(r,\frac{f(qz+c)-f(z)}{f(z)}\right)+S(r,f)
\end{aligned}$$

$$
\begin{aligned}
&\leq T(r,F) + m\left(r, \frac{f(qz+c)-f(z)}{f(z)}\right) \\
&\quad + N\left(r, \frac{f(qz+c)-f(z)}{f(z)}\right) + S(r,f) \\
&\leq T(r,F) + N\left(r, \frac{f(qz+c)}{f(z)}\right) + S(r,f) \\
&\leq T(r,F) + 2T(r,f) + S(r,f).
\end{aligned}
$$

So

$$
(n-1)T(r,f) + S(r,f) \leq T(r,F) \tag{3.8}
$$

on a set of lower logarithmic density 1. From the above inequality, we get that $F(z)$ is transcendental as $f(z)$ is a transcendental meromorphic function and $n \geq k+8$. On the other hand, we can easily get

$$
T(r,F) \leq (n+2)T(r,f) + S(r,f). \tag{3.9}
$$

By (3.8), (3.9) and $n \geq k+8$, we get

$$
T(r,F) = O(T(r,f)).
$$

Suppose $F(z)^{(k)} - \alpha(z)$ has only finitely many zeros and since f is a transcendental meromorphic function of zero order, from the second main theorem and (2.6), we obtain

$$
\begin{aligned}
T(r,F^{(k)}) &\leq \bar{N}(r,F^{(k)}) + \bar{N}\left(r, \frac{1}{F^{(k)}}\right) + \bar{N}\left(r, \frac{1}{F^{(k)} - \alpha(z)}\right) + S(r,F^{(k)}) \\
&\leq N_1\left(r, \frac{1}{F^{(k)}}\right) + \bar{N}(r,F^{(k)}) + S(r,F^{(k)}) \\
&\leq T(r,F^{(k)}) - T(r,F) + N_{k+1}\left(r, \frac{1}{F}\right) + \bar{N}(r,F) + S(r,F^{(k)}).
\end{aligned}
$$

Combining this inequality with (3.8), we obtain

$$
\begin{aligned}
&(n-1)T(r,f) + S(r,f) \leq T(r,F) \\
&\quad\leq N_{k+1}\left(r, \frac{1}{F}\right) + \bar{N}(r,F) + S(r,f) \\
&\quad\leq N_{k+1}\left(r, \frac{1}{f^n(z)[f(qz+c)-f(z)]}\right) \\
&\qquad + \bar{N}(r, f^n(z)[f(qz+c)-f(z)]) + S(r,f) \\
&\quad\leq (k+1)\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f(qz+c)-f(z)}\right) \\
&\qquad + \bar{N}(r, f^n(z)) + \bar{N}(r, f(qz+c)-f(z)) + S(r,f) \\
&\quad\leq (k+6)T(r,f) + S(r,f),
\end{aligned}
$$

which contradicts $n \geq k+8$.

3.6 *Proof of Theorem* 1.7

Denote $F = [f^n(z)(f^m(z) - a)f(qz + c)]^{(k)}$, $G = [g^n(z)(g^m(z) - a)g(qz + c)]^{(k)}$. Since f is a transcendental entire function of zero order, from Lemma 2.5 we get

$$T(r, F) \leq T(r, f^n(z)(f^m(z) - a)f(qz + c)) + S(r, f^n(z)(f^m(z) - a)f(qz + c)). \tag{3.10}$$

Combining (3.10) and (2.8), we have $S(r, F) = S(r, f)$, similarly $S(r, G) = S(r, g)$. Combining (2.6) and (2.8), we obtain

$$\begin{aligned}(n + m + 1)T(r, f) &= T(r, f^n(z)(f^m(z) - a)f(qz + c)) + S(r, f)\\ &\leq T(r, F) - N_2\left(r, \frac{1}{F}\right)\\ &\quad + N_{k+2}\left(r, \frac{1}{f^n(z)(f^m(z) - a)f(qz + c)}\right) + S(r, f).\end{aligned}$$

From (2.7), we get

$$\begin{aligned}N_2\left(r, \frac{1}{F}\right) &\leq N_{k+2}\left(r, \frac{1}{f^n(z)(f^m(z) - a)f(qz + c)}\right) + S(r, f)\\ &\leq (k + 2)\bar{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f^m(z) - a}\right)\\ &\quad + N\left(r, \frac{1}{f(qz + c)}\right) + S(r, f)\\ &\leq (k + m + 3)T(r, f) + S(r, f).\end{aligned}$$

Similarly as above, we also obtain

$$\begin{aligned}(n + m + 1)T(r, g) &\leq T(r, G) - N_2\left(r, \frac{1}{G}\right)\\ &\quad + N_{k+2}\left(r, \frac{1}{g^n(z)(g^m(z) - a)g(qz + c)}\right) + S(r, g),\end{aligned}$$

$$N_2\left(r, \frac{1}{G}\right) \leq (k + m + 3)T(r, g) + S(r, g).$$

Suppose F, G satisfy Lemma 2.7(1). Then

$$\begin{aligned}\max\{T(r, F), T(r, G)\} &\leq N_2\left(r, \frac{1}{F}\right) + N_2(r, F)\\ &\quad + N_2\left(r, \frac{1}{G}\right) + N_2(r, G) + S(r, F) + S(r, G).\end{aligned}$$

Thus, combining the above inequalities, we obtain

$$(n + m + 1)[T(r, f) + T(r, g)]$$

$$\leq 2\left[N_{k+2}\left(r, \frac{1}{f^n(z)(f^m(z)-a)f(qz+c)}\right)\right.$$
$$\left.+ N_{k+2}\left(r, \frac{1}{g^n(z)(g^m(z)-a)g(qz+c)}\right)\right]$$
$$+ S(r, f) + S(r, g)$$
$$\leq 2(k+m+3)[T(r, f) + T(r, g)] + S(r, f) + S(r, g),$$

which contradicts $n \geq 2k + m + 6$. Hence $F = G$ or $F \cdot G = 1$.

Case 1. $F = G$, from Lemma 2.9, then $f = tg$, where $t^{n+1} = t^m = 1$.

Case 2. $F \cdot G = 1$ implies that

$$[f^n(z)(f^m(z)-a)f(qz+c)]^{(k)} \cdot [g^n(z)(g^m(z)-a)g(qz+c)]^{(k)} = 1. \quad (3.11)$$

Since $n \geq 2k + m + 6$, from (3.11) and $f(z)$, $g(z)$ are transcendental entire functions, we get $f(z)$ and $g(z)$ have no zeros. Then $f(z) = \mathrm{e}^{s(z)}$, $g(z) = \mathrm{e}^{t(z)}$, where $s(z), t(z)$ are nonzero polynomials. Since $f(z)$ and $g(z)$ are zero order, $s(z), t(z)$ are constants, and so $f(z)$ and $g(z)$ are constants, which contradicts the assumption that $f(z)$ and $g(z)$ are transcendental entire functions.

Hence, the proof of Theorem 1.7 is completed.

4. Remark

When f is a transcendental meromorphic function, the composite function $G = af^n(qz+c) + f(z) - \alpha(z)$ would not only have an infinite number but also a logarithmic number of (integrated) zeros. Furthermore, it is possible that the integrated counting function $N(r, 1/G)$ for zeros of G could be a logarithm to some power or more when the logarithmic order of f is large. Based on these, we remark that the conclusion 'has infinitely many zeros' in Theorem 1.1 which can be improved as follows:

$$\limsup_{r\to\infty} \frac{N(r, 1/G)}{T(r, f)} \geq 1.$$

Suppose contrary to the claim suggested above,

$$\limsup_{r\to\infty} \frac{N(r, 1/G)}{T(r, f)} < 1,$$

then there is some constant $\delta \in (0, 1)$ such that

$$N(r, 1/G) \leq \delta T(r, f)$$

for all r large enough. Then from the proof of Theorem 1.1, one can see that eq. (3.3) becomes

$$N\left(r, \frac{1}{F-1}\right) \leq \delta T(r, f) + nN(r, f) + S(r, f).$$

Hence, combining Lemmas 2.2, 2.4 and eq. (3.1), the above inequality implies that

$$(n-2-\delta)T(r, f) \leq 2nN(r, f) + S(r, f) + o(U(r, f)),$$

where $n-2-\delta > 0$ since $n \geq 3$. Therefore, we also obtain

$$\sigma_{\log}(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log\log r} \leq \limsup_{r\to\infty} \frac{\log N(r, f)}{\log\log r} < \sigma_{\log}(f),$$

which is a contradiction.

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