Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 4, November 2014, pp. 511–516.

A complete classification of minimal non-*PS*-groups

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MS received 16 February 2013; revised 31 July 2013;

Abstract. Let G be a finite group. A subgroup H of G is called s-permutable in G if it permutes with every Sylow subgroup of G, and G is called a *PS*-group if all minimal subgroups and cyclic subgroups with order 4 of G are s-permutable in G. In this paper, we give a complete classification of finite groups which are not *PS*-groups but their proper subgroups are all *PS*-groups.

Keywords. *PS*-groups; minimal non-*PS*-groups; supersolvable groups; power automorphisms.

2010 Mathematics Subject Classification. 20D10, 20E34.

1. Introduction

In this paper, only finite groups are considered and our notation is standard.

Two subgroups H and K of a group G are said to permute if $HK = KH$. A subgroup H of G is called quasinormal (or permutable) in G if it permutes with every subgroup of G. Kegel [7] called a subgroup H of G s-quasinormal (or s-permutable) in G if it permutes with every Sylow subgroup of G. We call a group G a *PS*-group if all minimal subgroups and cyclic subgroups with order 4 of G are s-permutable in G, and a group G is a minimal non-*PS*-group if all proper subgroups of G are *PS*-groups but G itself is not.

Schmidt [13] determined the structure of minimal non-nilpotent groups, and Doerk [5] determined the structure of minimal non-supersolvable groups. However these results are not complete classifications. Later, Rédei completely classified the minimal non-nilpotent groups in [10], and Ballester-Bolinches and Esteban-Romero [2] gave complete classifications of the minimal non-p-supersolvable groups in the p-solvable universe and the minimal non-supersolvable groups. Buckley [4] called a group G a PN-group if every minimal subgroup of G is normal in G and gave some properties of PN-groups. Sastry [12] investigated the structure of those groups which are not PN-groups but whose proper subgroups are all PN-groups and called such groups minimal non PN-groups. Asaad [1] call a group G an MS-group if each minimal subgroup of G is s-permutable in G and investigated the structure of minimal non MS-groups (non MS-groups all of whose proper subgroups are MS-groups). Li and Zhao [16] investigated the structure of a group G by considering some weakly normal subgroups of prime power order of G (a subgroup H of a group G is weakly normal in G if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$). But these classifications are not complete.

The purpose of this paper is to give a complete classification of minimal non-*PS* -groups. The specific result is as follows.

Theorem 1.1. *Let p and q be distinct primes. Then, G is a minimal non-PS-group if and only if G is one of the following types*:

(I) $G = \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^r \rangle$, *where* $r^q \equiv 1 (\mathrm{mod}\ p)$, $q \mid p-1$ *and* $1 < r < p$;

(II) $G = \langle x, y \mid x^p = y^4 = 1, y^{-1}xy = x^{p-1} \rangle$;

(III) $G = P \rtimes Q$, *where* $P = \langle a, b \rangle$ *is an elementary abelian p-group of order* p^2, *and* $Q = \langle y \rangle$ *is cyclic of order* q^r *with* $r \geq 2$. *The action of Q on P is defined by* $a^y = a^i, b^y = b^j$, $p \equiv 1 (\mathrm{mod}\ q)$, *where* $i \not\equiv j (\mathrm{mod}\ p)$, $i^{q^{r-1}} \equiv j^{q^{r-1}} \equiv 1 (\mathrm{mod}\ p)$;

(IV) $G = P \rtimes Q$, *where* $P = \langle a, b \rangle$ *is an elementary abelian p-group of order* p^2, *and* $Q = \langle y \rangle$ *is cyclic of order* 2^r *with* $r \geq 3$. *The action of Q on P is defined by* $a^y = a^i, b^y = b^j$, *where* $i \not\equiv j (\mathrm{mod}\ p)$, $i^{2^{r-2}} \equiv j^{2^{r-2}} \equiv 1 (\mathrm{mod}\ p)$;

(V) $G = P \rtimes Q$, *where* $Q = \langle y \rangle$ *is cyclic of order* $q^r > 1$, *with* $q \nmid p-1$, *and P is an irreducible Q-module over the field of p elements with kernel* $\langle y^q \rangle$ *in Q*;

(VI) $G = P \rtimes Q$, *where P is a non-abelian special p-group of rank* $2m$, *the order of p modulo q being* $2m$, $Q = \langle y \rangle$ *is cyclic of order* $q^r > 1$, *y induces an automorphism in P such that* $P/\Phi(P)$ *is a faithful and irreducible Q-module, and y centralizes* $\Phi(P)$. *Furthermore,* $|P/\Phi(P)| = p^{2m}$ *and* $|P'| \leq p^m$;

(VII) $G = P \rtimes Q$, *where* $P = \langle a_0, a_1, \ldots, a_{q-1} \rangle$ *is an elementary abelian p-group of order* p^q, $Q = \langle y \rangle$ *is cyclic of order* q^r, q^f *is the highest power of q dividing* $p-1$ *and* $r-1 \geq f \geq 1$. *The action of Q on P is defined by* $a_j^y = a_{j+1}$ *for* $0 \leq j < q-1$ *and* $a_{q-1}^y = a_0^i$, *where i is a primitive q^f-th root of unity modulo p.*

(VIII) $G = P \rtimes Q$, *where* $P = \langle a_0, a_1 \rangle$ *is an elementary abelian p-group of order* p^2, $Q = \langle y \rangle$ *is cyclic of order* 2^r, 2^f *is the highest power of 2 dividing* $p-1$ *and* $r-2 \geq f \geq 1$. *The action of Q on P is defined by* $a_0^y = a_1$ *and* $a_1^y = a_0^i$, *where i is a primitive 2^f-th root of unity modulo p.*

(IX) $G = P \rtimes Q$, *where* $P = \langle a_0, a_1 \rangle$ *is an extraspecial group of order* p^3 *with exponent p*, $Q = \langle y \rangle$ *is a cyclic group of order* 2^r, *with* 2^f *the largest power of 2 dividing* $p-1$ *and* $r-2 \geq f \geq 1$. *The action of Q on P is defined by* $a_0^y = a_1$ *and* $a_1^y = a_0^i x$, *where* $x \in \langle [a_0, a_1] \rangle$ *and i is a primitive 2^f-th root of unity modulo p.*

2. Preliminary results

We collect some lemmas which will be frequently used in the sequel.

The first lemma is a special case of Theorem 3.4 of [14].

Lemma 2.1. *If a group G is a PS-group, then G is supersolvable.*

Lemma 2.2. *Let H be an s-permutable subgroup of a group G. Then*

(1) *if* $H \leq K \leq G$, *then H is s-permutable in K*;

(2) *if H is s-permutable in G, then H is subnormal in G*;

(3) *if H is an s-permutable Hall subgroup of G, then H is normal in G;*
(4) *if H and K are s-permutable in G, then $\langle H, K\rangle$ is s-permutable in G.*

Proof. For cases (1), (2) and (3), see [7]. For case (4), see 2.2(d) of [1]. □

Lemma 2.3 [5]. *Let G be a minimal non-supersolvable group. Then*
(1) *G has a unique normal Sylow p-subgroup P;*
(2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, and $P/\Phi(P)$ is non-cyclic;*
(3) *if $p \neq 2$, then the exponent of P is p;*
(4) *if P is non-abelian and $p = 2$, then the exponent of P is at most* 4;
(5) *if P is abelian, then the exponent of P is p.*

Lemma 2.4. *Let G be a minimal non-PS-group. Then there exists a normal Sylow p-subgroup P of G and a non-normal Sylow q-subgroup Q of G with $p \neq q$ such that $G = PQ$.*

Proof. Since every proper subgroup of G is a *PS*-group, it follows that every subgroup of G is supersolvable by Lemma 2.1. Thus, G is either supersolvable or minimal non-supersolvable and so G is solvable.

Since G is solvable, it has a Sylow basis $\mathcal{A} = \{P_1, P_2, \ldots, P_s\}$ where P_i is a Sylow p_i-subgroup and $p_1 < p_2 < \cdots < p_s$. Suppose that $s \geq 3$.

Since $P_i P_s < G$, $P_i P_s$ is supersolvable. Thus, P_s is normalized by each P_i, $i = 1, 2, \ldots, s$. Hence, $P_s \trianglelefteq G$. Since P_s has a complement which must be supersolvable, it follows that G has a Sylow tower. We will now assume that $\mathcal{A}$ is a Sylow tower for G.

Let N be a cyclic subgroup of G with $|N|$ a prime or 4 and let P be a Sylow subgroup of G. Further, let K be a normal p_1'-subgroup of G, since G has a Sylow tower. If $N \leq K$, as K is a *PS*-group, we have $N \leq O_{p_j}(K)$ for some appropriate $p_j \in \pi(G)$. Now $O_{p_j}(K) \trianglelefteq G$, so $PO_{p_j}(K)$ is a *PS*-group and $NP = PN$, as required.

It follows that N is a p_1-group and $|N| = p_1$ or $|N| = 4$. Thus, for some $g \in G$, $N^g \leq P_1$. If $N^g = P_1$, then $N^g \trianglelefteq \langle P_1, P_2, \ldots, P_s\rangle = G$ by Lemma 2.2(3). It follows that $N \trianglelefteq G$ and $NP = PN$, as required.

Thus, we have to assume that $N^g < P_1$. It follows that there exists a maximal subgroup T of P_1 such that $N^g \leq T$ and $TK \trianglelefteq G$. Now $N \leq TK$ and it follows as before that $N \leq O_{p_1}(TK) \trianglelefteq G$ and $NP = PN$, as required. Thus, we have shown that if $s \geq 3$, then G is a *PS*-group. Hence, if G is not a *PS*-group, then $s = 2$. □

Lemma 2.5 (*Theorem* 6 *of* [15]). *A group G is an MNP-group (all maximal subgroups of the Sylow subgroups of G are normal in G) if and only if $G = H \rtimes \langle x\rangle$, where*
(i) *H is a normal nilpotent Hall subgroup of G;*
(ii) *every generator of every Sylow subgroup of $\langle x\rangle$ induces a power automorphism of order dividing a prime in $H/\Phi(H)$.*

Lemma 2.6 [6]. *Let $\{P_1, P_2, \ldots, P_r\}$ be a Sylow basis of a solvable group G. Then the following statements are equivalent:*
(a) *Every subgroup of P_i permutes with every subgroup of P_j for $i \neq j$.*
(b) *The nilpotent residual N of G is an abelian Hall subgroup of G and every element of G induces a power automorphism in N.*

Lemma 2.7 (*Theorem* 13.4.3 *of* [11]). *Let α be a power automorphism of an abelian group A. If A is a p-group of finite exponent, then there is a positive integer l such that $a^{\alpha} = a^{l}$ for all a in A. If α is non-trivial and has order prime to p, then α is fixed-point-free.*

3. The proof of Theorem 1.1

Proof. If G is a minimal non-*PS*-group, then we may assume $G = PQ$ with $P \trianglelefteq G$ and $Q \ntrianglelefteq G$ by Lemma 2.4, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$.
(1) Assume that there exists a cyclic subgroup $\langle y\rangle$ of Q with order q or 4 (if $q = 2$) such that $\langle y\rangle$ is not s-permutable in G. If $P\langle y\rangle < G$, then $P\langle y\rangle$ is a *PS*-group and so $\langle y\rangle \trianglelefteq P\langle y\rangle$ by Lemma 2.2(3). It follows that $[y, P] = 1$. Thus, $\langle y\rangle^G \leq Q$ and $\langle y\rangle \leq Q^g$ for all $g \in G$. Thus, $\langle y\rangle$ is s-permutable, contrary to hypotheses. Hence, $G = P\langle y\rangle$.

Case 1. Assume $P = \langle x\rangle$ with $|x| = p^m$. It is easy to see that G is a minimal non-cyclic group. By [8], G is one of types (I) and (II).

Case 2. Assume that P is non-cyclic. It is easy to see that G is minimal non-nilpotent. Applying Theorem 3 of [3], G belongs to either type (V) or type (VI).
(2) Assume that Q has property that all its minimal subgroups and cyclic subgroups of order 4 (if $q = 2$) are s-permutable in G but P does not have this property.

Let S be an arbitrary Sylow q-subgroup of G. If S is non-cyclic, then PS_1 and PS_2 are *PS*-groups for two maximal subgroups S_1 and S_2 of S, and so N is s-permutable in not only PS_1 but also PS_2, where N is an arbitrary cyclic subgroup of P with order p or 4 (if $p = 2$). Hence N is s-permutable in G, a contradiction. Thus, Q is cyclic and we let $Q = \langle y\rangle$.

Case 1. *G is supersolvable*. If $\Omega_1(P) < P$, then $\Omega_1(P)S$ is a *PS*-group for every Sylow q-subgroup S of G. Hence every minimal subgroup of P is s-permutable in G, a contradiction. Thus, $\Omega_1(P) = P$.

In view of the supersolvability of G, we have easily that P has at least two maximal subgroups R and K which are normal in G when Maschke's theorem (Theorem 8.1.2 of [11]) and Schreier's refinement theorem (Theorem 3.1.2 of [11]) are applied.

Now we prove $d(P) = k = 2$, where $d(P)$ is the rank of P. If $k \geq 3$, then we can let $P/\Phi(P) = \langle \bar{a_1}\rangle \times \langle \bar{a_2}\rangle \times \cdots \times \langle \bar{a_s}\rangle$, where $a_1, a_2, \ldots, a_{s-1} \in R$ and $a_2, a_3, \ldots, a_s \in K$. Since every minimal subgroup N of R is s-permutable in $R\langle y\rangle$ and $\Omega_1(P) = P$, it follows that every maximal subgroup R_1 of R is normal in $R\langle y\rangle$ when Lemma 2.2(4) is applied. By Lemma 2.5, $(r\Phi(R))^y = r^l\Phi(R)$, and so $(r\Phi(P))^y = r^l\Phi(P)$, where $r \in R$ and l is a positive integer. Similarly, $(k\Phi(P))^y = k^m\Phi(P)$, where $k \in K$ and m is a positive integer. It follows that $a_2^l\Phi(P) = (a_2\Phi(P))^y = a_2^m\Phi(P)$, and so $l \equiv m (\mathrm{mod}\ p)$. Hence $(a_i\Phi(P))^y = a_i^l\Phi(P)$ for $i = 1, 2, \ldots, s$. It is easy to see that y induces a power automorphism in $P/\Phi(P)$. By Lemma 2.5, every maximal subgroup of P is normal in G. By induction, every minimal subgroup of P is s-permutable in G, a contradiction. Hence $k = 2$.

Let $P/\Phi(P) = R/\Phi(P) \times K/\Phi(P) = \langle \bar{a_1}\rangle \times \langle \bar{a_2}\rangle$, where $a_1 \in R, a_2 \in K$, $\bar{a_1}^y = \bar{a_1}^{k_1}$, $\bar{a_2}^y = \bar{a_2}^{k_2}$, and k_1, k_2 are two positive integers. If $k_1 = k_2$, then every maximal subgroup of P is normal in G by Lemma 2.5. By induction, every minimal subgroup of P is s-permutable in G, a contradiction. Hence $k_1 \neq k_2$ and P has only two maximal subgroups which are normal in G. Clearly, at least one action of which y acts on R and K is non-trivial. Without loss of generality, we may assume that y induces a non-trivial automorphism α in R. By hypothesis, we have that every subgroup of R permutes with every subgroup of $\langle y\rangle$. By Lemma 2.6, R is abelian and α is a power automorphism of order prime to p in R. Lemma 2.7 implies that α is fixed-point-free. Hence we have either $K \cap R = 1$ if $K\langle y\rangle = K \times \langle y\rangle$ or $K\langle y\rangle \neq K \times \langle y\rangle$. If $K \cap R = 1$ and $K\langle y\rangle = K \times \langle y\rangle$, then P is an elementary abelian group of order p^2. If $K\langle y\rangle \neq K \times \langle y\rangle$, similar arguments can be made as above, i.e. K is abelian, and y induces a power automorphism of order prime to p in K. Thus, $\Phi(P) = R \cap K \leq Z(P)$. If $|P : Z(P)| \leq p$, then P is abelian of order p^2. It is clear that every cyclic subgroup of Q with order q or 4 is normal in G by Lemma 2.2. Letting $P = \langle a, b\rangle$, we have easily that G is one of types (III) and (IV).

If $|P : Z(P)| = p^2$, then $\Phi(P) = R \cap K = Z(P)$, and so P is a minimal non-abelian of order p^3, i.e. $P = \langle a, b\rangle, a^p = b^p = c^p = 1, ba = abc, ca = ac, cb = bc$, where $\Phi(P) = Z(P) = \langle c\rangle$. Without loss of generality, we assume $R = \langle a, c\rangle$ and $K = \langle b, c\rangle$. Clearly, y induces power automorphisms in R and K by hypothesis, respectively. Therefore, there exists a positive integer r such that $a^y = a^r, c^y = c^r, b^y = b^r$. Furthermore, y induces a power automorphism on $P/\Phi(P)$, and so every maximal subgroup of P is normal in G, a contradiction.

Case 2. Assume that G is minimal non-supersolvable. We prove that: if $Q \leq M \lessdot G$, then $\Phi(P)$ is a Sylow p-subgroup of M.

Denote $M = P_3 Q$, where P_3 is a Sylow p-subgroup of M. By the subnormality of P_3 in G, we have that P_3 is normal in M, and so P_3 is normal in G. By Lemma 2.3, $P_3\Phi(P) = P_3 = \Phi(P)$ is the Sylow p-subgroup of M.

(i) If G is also a minimal non-nilpotent group, then by Theorem 2.8 of [9], P is non-cyclic. Applying Theorem 3 of [3], G is of either type (V) or type (VI).

(ii) If G is not a minimal non-nilpotent group and P is abelian, applying Theorem 9, 10 of [2], we assume that $G = PQ$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle y\rangle$ is cyclic of order q^r, q^f is the highest power of q dividing $p - 1$ and $r > f \geq 1$. Define $a_j^y = a_{j+1}$ for $0 \leq j < q - 1$ and $a_{q-1}^y = a_0^i$, where i is a primitive q^f-th root of unity modulo p.

If $q > 2$, it is clear that $P\langle y^{q^{r-1}}\rangle$ is nilpotent and so $f \leq r - 1$. Hence G is of type (VII).

Similarly, if $q = 2$, then $P\langle y^{2^{r-2}}\rangle$ is nilpotent and so $f \leq r - 2$. Hence G is of type (VIII).

(iii) Assume that G is not a minimal non-nilpotent group and P is non-abelian. Applying Theorem 9, 10 of [2], we may assume that $G = PQ$ such that $P = \langle a_0, a_1\rangle$ is an extraspecial group of order p^3 with exponent p, $Q = \langle y\rangle$ is a cyclic group of order 2^r with 2^f the largest power of 2 dividing $p - 1$ and $r > f \geq 1$, and $a_0^y = a_1$ and $a_1^y = a_0^i x$, where $x \in \langle [a_0, a_1]\rangle$ and i is a primitive 2^f-th root of unity modulo p.

Since $P\langle y^{2^{r-2}}\rangle$ is a PS-group, it follows that $P\langle y^{2^{r-2}}\rangle$ is nilpotent and so $f \leq r - 2$. Therefore, G is of type (IX).

Conversely, it is clear that all types are not isomorphic, and types (I)–(IV) are all minimal non-*PS*-groups.

For types (V) and (VI), it follows that G is non-supersolvable from Theorem 2.8 of [9]. By Lemma 2.1, G is not a *PS*-group, and so G is a minimal non-*PS*- group.

For type (VII), (VIII) and (IX), by Lemma 2.1 again, G is not a *PS*-group. It is easy to see that G is a minimal non-*PS*-group. □

Acknowledgements

The authors express their deep sense for gratitude to the anonymous referee for his/her valuable suggestions and comments which refined this paper, and also for providing a concise and original proof for Lemma 2.4. This work was supported by the National Natural Science Foundation of China (No. 11371237), Jiangsu Overseas Research & Training Program for University Prominent Young & Middle-Aged Teachers and Presidents, Qing Lan Project of Jiangsu Province, '333' Project of Jiangsu Province.

References

[1] Asaad M, On minimal subgroups of finite groups, *Glasgow Math. J.* **51** (2009) 359–366

[2] Ballester-Bolinches A and Esteban-Romero R, On minimal non-supersoluble groups, *Rev. Mat. Iberoamericana* **23** (2007) 127–142

[3] Ballester-Bolinches A, Esteban-Romero R and Robinson D J S, On finite minimal non-nilpotent groups, *Proc. Amer. Math. Soc.* **133** (2005) 3455–3462

[4] Buckley J, Finite groups whose minimal subgroups are normal, *Math. Z.* **116** (1970) 15–17

[5] Doerk K, Minimal nicht überauflösbare, endliche Gruppen, *Math. Z.* **91** (1966) 198–205

[6] Huppert B, Zur sylowstruktur auflösbarer gruppen, *Arch. Math.* **12** (1961) 161–169

[7] Kegel O H, Sylow Gruppen and subnormalteiler endlicher Gruppen, *Math. Z.* **78** (1962) 205–221

[8] Miller G A and Moreno H C, Non-abelian groups in which every subgroup is abelian, *Trans. Amer. Math. Soc.* **4** (1903) 398–404

[9] Mingyao Xu and Qinhai Zhang, On conjugate-permutable subgroups of a finite group, *Algebra Colloq.* **12** (2005) 669–676

[10] Rédei L, Die endlichen einstufig nichtnilpotenten Gruppen, *Publ. Math. Debrecen* **4** (1956) 303–324

[11] Robinson D J S, *A Course in the Theory of Groups* (1980) (Springer-Verlag, New York-Heidelberg-Berlin)

[12] Sastry N S N, On minimal nonPN-groups, *J. Algebra* **65** (1980) 104–109

[13] Schmidt O J, Über Gruppen, deren sämtliche Teiler spezielle Gruppen sind, *Mat. Sbornik* **31** (1924) 366–372

[14] Shaalan A, The influence of π-quasinormality of some subgroups on the structure of a finite group, *Acta Math. Hung.* **56** (1990) 287–293

[15] Walls G L, Groups with maximal subgroups of Sylow subgroups normal, *Israel J. Math.* **43** (1982) 166–168

[16] Xianhua Li and Tao Zhao, Weakly normal subgroups of finite groups, *Indian J. Pure Appl. Math.* **41**(6) (2010) 745–753