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Meromorphic connections on vector bundles over curves

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Abstract. We give a criterion for filtered vector bundles over curves to admit a filtration preserving meromorphic connection that induces a given meromorphic connection on the corresponding graded bundle.

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1. Introduction

Let E be a holomorphic vector bundle of rank r over a compact connected Riemann surface X. A holomorphic connection on E is a $\mathbb{C}$-linear homomorphism $D : E \longrightarrow E \otimes \Omega_X^1$ satisfying the Leibniz rule which says that $D(f \cdot s) = s \otimes \mathrm{d}f + f \cdot D(s)$ for any holomorphic function $f \in \mathcal{O}_X(U)$ and any holomorphic section s of $E|_U$, where U is any open subset of X. According to Weil's criterion, E admits a holomorphic connection if and only if E is a direct sum of indecomposable vector bundles of degree zero [1, 6].

A meromorphic connection on E is a $\mathbb{C}$-linear homomorphism $D : E \longrightarrow E \otimes \Omega_X^1 \otimes \mathcal{O}_X(\Delta)$ satisfying the Leibniz rule, where Δ is an effective divisor on X. Any holomorphic vector bundle can be endowed with a meromorphic connection. Indeed, any algebraic vector bundle on X is Zariski locally trivial. Hence using the holomorphic connection on the trivial vector bundle over a Zariski open subset U defined by the de Rham differential, and an algebraic isomorphism of $E|_U$ with this trivial vector bundle, we can construct a meromorphic connection on E.

A meromorphic connection on E is said to be logarithmic if its polar divisor Δ is reduced. A proof for the (classical) fact that *any holomorphic vector bundle can be endowed with a logarithmic connection* is given in § 5.

Our aim here is to investigate when E can be endowed with a meromorphic connection satisfying some given conditions. In particular, we are interested in the following question.

Question 1.1. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$ be a filtration of holomorphic subbundles of a holomorphic vector bundle E on X. For each $i \in \{1, \ldots, \ell\}$, let D_i be a meromorphic connection with polar divisor (at most) Δ on E_i/E_{i-1}. Does there exist a meromorphic connection D on E with polar divisor (at most) Δ such that

- D preserves the subbundle E_i for every $i \in [1, \ell]$, and
- the connection on E_i/E_{i-1} induced by D coincides with the given connection D_i?

In § 3, we give the following answer to this question.

Theorem 1.2. *A connection D as in Question* 1.1 *exists if*

$$\mathrm{H}^0(X, \mathrm{End}(E)/\mathrm{End}(E_\bullet) \otimes \mathcal{O}_X(-\Delta)) = 0,$$

where $\mathrm{End}(E_\bullet) \subset \mathrm{End}(E)$ *is the subbundle defined by the sheaf of filtration preserving endomorphisms.*

In particular, such a connection exists if $\deg(\Delta)$ is sufficiently large or if the filtration is *rigid*, i.e. $\mathrm{H}^0(X, \mathrm{End}(E)/\mathrm{End}(E_\bullet)) = 0$ (see [3]). A weaker sufficient condition in the case of a filtration of length $\ell = 2$ is given in § 4. We thereby generalize results of [3] and [4] on the holomorphic connections.

Throughout the article, X will denote a compact connected Riemann surface. We shall say $\Delta' \geq \Delta$ for two divisors Δ' and Δ on X if $\Delta' - \Delta \geq 0$, i.e., if the divisor $\Delta' - \Delta$ is effective. In order to simplify notations, we shall not distinguish between a reduced effective divisor and its support on X. Moreover, we make no difference in notation between a holomorphic vector bundle over X and the sheaf of its holomorphic sections.

2. Meromorphic Atiyah bundle

Let E be a holomorphic vector bundle over X of rank r. Let Δ be an effective reduced divisor on X. A logarithmic connection on E singular over Δ is a holomorphic splitting of the Atiyah exact sequence associated to E and Δ. Since it may not be standard, let us give the construction of the Atiyah bundle $\mathrm{At}_\Delta(E)$ (see [1] for $\Delta = \emptyset$).

For every analytic open subset $U \subset X$, consider all pairs of the form (D_U, h_U), where h_U is a holomorphic function on U, and

$$D_U : E|_U \longrightarrow (E \otimes K_X \otimes \mathcal{O}_X(\Delta))|_U$$

is a holomorphic differential operator satisfying the identity

$$D_U(f \cdot s) = f \cdot D_U(s) + h_U \cdot d(f) \otimes s, \tag{2.1}$$

where f is any holomorphic function on U and s is any holomorphic section of E over U. For any such pair (D_U, h_U) and any holomorphic function h on U, the pair $(h \cdot D_U, h \cdot h_U)$ clearly satisfies the condition in (2.1). Let $\mathrm{At}_\Delta(E)$ be the torsionfree coherent analytic sheaf on X whose sections over any U are all pairs (D_U, h_U) satisfying (2.1). This $\mathrm{At}_\Delta(E)$ defines a holomorphic vector bundle over X of rank r^2+1. This holomorphic vector bundle will be denoted by $\mathrm{At}_\Delta(E)$.

We have a short exact sequence of holomorphic vector bundles on X,

$$0 \longrightarrow \mathrm{End}(E) \otimes K_X \otimes \mathcal{O}_X(\Delta) \stackrel{\mu}{\longrightarrow} \mathrm{At}_\Delta(E) \stackrel{\nu}{\longrightarrow} \mathcal{O}_X \longrightarrow 0, \tag{2.2}$$

where μ sends Θ_U to the pair $(\Theta_U, 0)$ and ν sends a pair (D_U, h_U) to the function h_U.

Let

$$D \,:\, \mathcal{O}_X \longrightarrow \mathrm{At}_\Delta(E)$$

be an $\mathcal{O}_X$-linear homomorphism such that $\nu \circ D = \mathrm{Id}_{\mathcal{O}_X}$, where ν is the homomorphism in (2.2). The constant function 1 on X will be denoted by 1_X. Note that $D(1_X)$ is a logarithmic connection on E whose singular locus is contained in Δ. Conversely, if $\mathcal{D}$ is a logarithmic connection on E whose singular locus is contained in Δ, then sending 1_X to $\mathcal{D}$ we construct an $\mathcal{O}_X$-linear homomorphism $D \,:\, \mathcal{O}_X \longrightarrow \mathrm{At}_\Delta(E)$ such that $\nu \circ D = \mathrm{Id}_{\mathcal{O}_X}$.

3. Meromorphic connections on filtered bundles

Let E be a holomorphic vector bundle over X. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E \tag{3.1}$$

be a filtration of holomorphic subbundles of E. Define the vector bundle $\mathrm{End}(E) := E \otimes E^*$. So we have the evaluation homomorphism $\sigma \,:\, \mathrm{End}(E) \otimes E \longrightarrow E$. Let

$$\mathrm{End}(E_\bullet) \subset \mathrm{End}(E)$$

be the unique maximal subbundle such that $\sigma(\mathrm{End}(E_\bullet) \otimes E_i) \subset E_i$ for all $i \in [1\,,\ell]$. In other words, $\mathrm{End}(E_\bullet)$ is the subbundle of $\mathrm{End}(E)$ that preserves the filtration in (3.1).

Theorem 3.1. *Let Δ be an effective divisor on X. Take a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E \tag{3.2}$$

as in (3.1) *such that*

$$\mathrm{H}^0(X,\, \mathrm{End}(E)/\mathrm{End}(E_\bullet) \otimes \mathcal{O}_X(-\Delta)) = 0\,. \tag{3.3}$$

For each $i \in \{1, \ldots, \ell\}$, let D_i be a meromorphic connection on E_i/E_{i-1} with polar divisor (*at most*) *Δ. Then there is a meromorphic connection D on E with polar divisor* (*at most*) *Δ such that*

- *D preserves each subbundle E_i, $i \in [1\,,\ell]$, and*
- *the connection on E_i/E_{i-1} induced by D coincides with D_i.*

Proof. Consider $\mathrm{At}_\Delta(E)$ constructed in § 2. Let

$$\mathrm{At}_\Delta(E_\bullet) \subset \mathrm{At}_\Delta(E)$$

be the subbundle given by all pairs $(D_U\,, h_U)$, where h_U is a holomorphic function on an open subset $U \subset X$, and

$$D_U \,:\, E|_U \longrightarrow (E \otimes K_X \otimes \mathcal{O}_X(\Delta))|_U$$

is a holomorphic differential operator satisfying (2.1) such that for every $i \in [1\,,\ell]$, the operator D_U sends any section of $E_i|_U$ to a section of $(E_i \otimes K_X \otimes \mathcal{O}_X(\Delta))|_U$. Therefore, we have a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathrm{End}(E_\bullet) \otimes K_X \otimes \mathcal{O}_X(\Delta) \longrightarrow \mathrm{At}_\Delta(E_\bullet) \xrightarrow{\nu_0} \mathcal{O}_X \longrightarrow 0\,. \tag{3.4}$$

A holomorphic splitting of the exact sequence in (3.4) corresponds to a filtration preserving meromorphic connection on E with polar divisor (at most) Δ. In the same way, we construct a short exact sequence

$$0 \longrightarrow \mathrm{End}(\mathrm{Gr}(E_\bullet)) \otimes K_X \otimes \mathcal{O}_X(\Delta) \longrightarrow \mathrm{At}_\Delta(\mathrm{Gr}(E_\bullet)) \xrightarrow{\nu_1} \mathcal{O}_X \longrightarrow 0\,, \tag{3.5}$$

where $\mathrm{Gr}(E_\bullet) := \bigoplus_{i=1}^{\ell} E_i/E_{i-1}$ is the graded bundle associated to the filtration in (3.2) and $\mathrm{End}(\mathrm{Gr}(E_\bullet)) := \bigoplus_{i=1}^{\ell} \mathrm{End}(E_i/E_{i-1})$ is the sheaf of endomorphisms of the graded bundle. A splitting of this sequence corresponds to a meromorphic connection on the graded bundle preserving the decomposition with polar divisor (at most) Δ.

These exact sequences fit into a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{V}_0 \otimes K_X \otimes \mathcal{O}_X(\Delta) & = & \mathcal{V}_0 \otimes K_X \otimes \mathcal{O}_X(\Delta) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{End}(E_\bullet) \otimes K_X \otimes \mathcal{O}_X(\Delta) & \longrightarrow & \mathrm{At}_\Delta(E_\bullet) & \xrightarrow{\nu_0} & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow p & & \| & & \\
0 & \longrightarrow & \mathrm{End}(\mathrm{Gr}(E_\bullet)) \otimes K_X \otimes \mathcal{O}_X(\Delta) & \longrightarrow & \mathrm{At}_\Delta(\mathrm{Gr}(E_\bullet)) & \xrightarrow{\nu_1} & \mathcal{O}_X & \longrightarrow & 0\,, \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.6}$$

where $\mathcal{V}_0$ is the sheaf of nilpotent endomorphisms with respect to the filtration in (3.2).

Let

$$D_{\mathrm{Gr}} := \bigoplus_{i=1}^{\ell} D_i$$

be the meromorphic connection on $\mathrm{Gr}(E_\bullet)$, where D_i are the given connections. It defines a holomorphic homomorphism

$$D_{\mathrm{Gr}} : \mathcal{O}_X \longrightarrow \mathrm{At}_\Delta(\mathrm{Gr}(E_\bullet)) \tag{3.7}$$

such that $\nu_1 \circ D_{\mathrm{Gr}} = \mathrm{Id}_{\mathcal{O}_X}$, where ν_1 is the projection in (3.5). Therefore, we have a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{V}_0 \otimes K_X \otimes \mathcal{O}_X(\Delta) \longrightarrow \mathcal{V} := p^{-1}(D_{\mathrm{Gr}}(\mathcal{O}_X)) \xrightarrow{\nu_0} \mathcal{O}_X \longrightarrow 0\,, \tag{3.8}$$

where p is the projection in (3.6). A splitting of this exact sequence provides a splitting of the exact sequence (3.4) which corresponds to a meromorphic connection D on E with polar divisor (at most) Δ, preserving the filtration in (3.1) and restricting to the connection D_{Gr} on $\mathrm{Gr}(E_\bullet)$. The obstruction for splitting of (3.8) is a cohomology class

$$\psi \in \mathrm{H}^1(X, \mathcal{V}_0 \otimes K_X \otimes \mathcal{O}_X(\Delta)) = \mathrm{H}^0(X, \mathcal{V}_0^* \otimes \mathcal{O}_X(-\Delta))^* \tag{3.9}$$

(by Serre duality).

We have seen in [3] that

$$\mathcal{V}_0^* \simeq \mathrm{End}(E)/\mathrm{End}(E_\bullet). \tag{3.10}$$

Now the given condition (3.3) is equivalent to the statement that

$$\mathrm{H}^0(X, \mathcal{V}_0^* \otimes \mathcal{O}_X(-\Delta)) = 0\,.$$

Therefore, ψ in (3.9) vanishes.

For completeness, let us recall the proof of equation (3.10). Denote by $\mathcal{W}_0$ the cokernel of $\mathrm{End}(E_\bullet)$ in $\mathrm{End}(E)$:

$$\mathcal{W}_0 := \mathrm{End}(E)/\mathrm{End}(E_\bullet). \tag{3.11}$$

Since $\mathrm{End}(E)$ and $\mathrm{End}(\mathrm{Gr}(E_\bullet))$ are self-dual, the short exact sequences associated to $\mathcal{V}_0$ and $\mathcal{W}_0$ fit into a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{V}_0 & \overset{\varphi}{\dashrightarrow} & \mathcal{W}_0^* & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{End}(E_\bullet) & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathcal{W}_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{End}(\mathrm{Gr}(E_\bullet)) & \longrightarrow & \mathrm{End}(E_\bullet)^* & \longrightarrow & \mathcal{V}_0^* & \longrightarrow & 0\,, \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.12}$$

and one can easily check (by restriction to a field), that it is commutative. We thereby obtain a morphism φ from $\mathcal{V}_0$ to $\mathcal{W}_0^*$, and again we can easily check that this is an isomorphism. Indeed, φ is injective and the vector bundles $\mathcal{V}_0$ and $\mathcal{W}_0^*$ have the same rank. □

4. Meromorphic connections on extensions

PROPOSITION 4.1

Let Δ be an effective divisor on X. Let E and F be holomorphic vector bundles on X endowed with meromorphic connections D_E and D_F respectively, having polar divisor *at most* Δ. *Assume that every holomorphic section of* $\mathrm{Hom}(E, F) \otimes \mathcal{O}_X(-\Delta)$ *is flat with respect to the meromorphic connection on* $\mathrm{Hom}(E, F)$ *induced by* D_E *and* D_F. *Then for any holomorphic extension*

$$0 \longrightarrow E \longrightarrow W \longrightarrow F \longrightarrow 0\,,$$

the holomorphic vector bundle W *admits a meromorphic connection with polar divisor (at most)* Δ *preserving the subbundle* E *and inducing* D_E *and* D_F *on* E *and* $W/E = F$ *respectively.*

Proof. The holomorphic connection $D_E \oplus D_F$ on $E \oplus F$ gives a section D_{Gr} as in (3.7), and the vector bundle $\mathcal{V}_0$ in (3.8) is $E \otimes F^*$. Consider

$$\begin{aligned}\psi \in \mathrm{H}^1(X,\, \mathcal{V}_0 \otimes K_X \otimes \mathcal{O}_X(\Delta)) &= \mathrm{H}^0(X,\, \mathcal{V}_0^* \otimes \mathcal{O}_X(-\Delta))^* \\ &= \mathrm{H}^0(X,\, \mathrm{Hom}(E\,,F) \otimes \mathcal{O}_X(-\Delta))^*\end{aligned}$$

as in (3.9). Given any $T \in \mathrm{H}^0(X,\, \mathrm{Hom}(E\,, F) \otimes \mathcal{O}_X(-\Delta))$, we will explicitly describe the evaluation $\psi(T) \in \mathbb{C}$.

Fix a C^∞ splitting

$$\eta \,:\, F \longrightarrow W$$

of the short exact sequence in the proposition. We will identity F with $\eta(F) \subset W$. Let $\bar\partial_E$ (respectively, $\bar\partial_F$) be the Dolbeault operator defining the holomorphic structure of E (respectively, F). Using the C^∞ isomorphism

$$W \longrightarrow E \oplus F$$

given by η, the Dolbeault operator of W is

$$\begin{pmatrix} \bar\partial_E & A \\ 0 & \bar\partial_F \end{pmatrix},$$

where A is a smooth section of $\mathrm{Hom}(F\,, E) \otimes \Omega^{0,1}_X$.

Let $D_{F,E}$ be the meromorphic connection on $\mathrm{Hom}(F\,, E)$ given by D_E and D_F. We have

$$D_{F,E}(A) \in C^\infty(X;\; \mathrm{Hom}(F\,, E) \otimes \Omega^{1,1}_X \otimes \mathcal{O}(\Delta))\,.$$

It is straightforward to check that

$$\psi(T) = \int_X \mathrm{trace}(D_{F,E}(A) \circ T) \in \mathbb{C}\,. \tag{4.1}$$

Note that $D_{F,E}(A) \circ T$ is an element of $C^\infty(X;\; \mathrm{Hom}(F\,, E) \otimes \Omega^{1,1}_X)$.

Let $D_{E,E}$ be the meromorphic connection on $\mathrm{End}(E)$ induced by D_E. Let $D_{E,F}$ be the meromorphic connection on $\mathrm{Hom}(E\,, F)$ induced by D_E and D_F. Note that

$$D_{E,F}(T) = 0$$

by the condition given in the proposition. Therefore, we have

$$D_{F,E}(A) \circ T = D_{F,E}(A) \circ T + A \circ D_{E,F}(T) = D_{E,E}(A \circ T)\,.$$

On the other hand,

$$\int_X \mathrm{trace}(D_{E,E}(A \circ T)) = \int_X \partial(\mathrm{trace}(A \circ T)) = 0\,.$$

Combining these, from (4.1) it follows that $\psi = 0$. The holomorphic vector bundle W thus admits a meromorphic connection D with polar divisor (at most) Δ that preserves the subbundle E and induces D_E and D_F on E and $W/E = F$ respectively. □

5. Existence of logarithmic connections on vector bundles

A connection $D : E \longrightarrow E \otimes \Omega^1_X \otimes \mathcal{O}_X(\Delta)$ is said to be *reducible* if there is a non-trivial subbundle F of E such that

$$D(F) \subset F \otimes \Omega^1_X \otimes \mathcal{O}_X(\Delta).$$

Any semi-stable vector bundle E of degree 0 over a compact connected Riemann surface X can be endowed with a holomorphic connection D preserving the Jordan–Hölder filtration of E. Moreover, if Δ is a reduced divisor of degree at least 3, then there is a logarithmic Higgs-field $\Theta \in \mathrm{H}^0(\mathrm{Hom}(E, E \otimes \Omega^1_X \otimes \mathcal{O}_X(\Delta)))$ such that the logarithmic connection $D' = D + \Theta$ on E is *irreducible* (see [5]). Recall that *any line bundle L over X can be endowed with a logarithmic connection.* Indeed, let s be a non-trivial meromorphic section of L. Then $\mathrm{d} - s^{-1}\mathrm{d}s$ defines a logarithmic connection $L \to L \otimes \Omega(\Delta)$, where Δ is the reduced effective divisor associated to $\mathrm{div}(s)$.

PROPOSITION 5.1

Any vector bundle E with $\mathrm{rank}(E) > 1$ *over a compact connected Riemann surface X can be endowed with a reducible logarithmic connection.*

Proof. This result follows from Theorem 3.1 by induction on the rank r of the vector bundle E, if we allow a sufficiently large polar divisor. Indeed, any vector bundle E has a line subbundle and any line bundle and can be endowed with a logarithmic connection. □

It would be interesting to find an upper bound to the minimal number of necessary poles in order to endow a given vector bundle E with a reducible logarithmic connection. As an example, by the proof of Theorem 3.1, we get the following estimation:

Example 5.2. Let L_1 and L_2 be effective line bundles over a compact connected Riemann surface X. Let E be a rank 2 vector bundle given by an extension

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0.$$

Then E can be endowed with a logarithmic connection preserving L_1 whose number of poles is less than or equal to $\deg(E) + 1$.

More generally, recall the following.

Theorem 5.3. *Let E be a vector bundle of rank r over X. Let x be a point of X. Then E can be endowed with a logarithmic connection with polar divisor $\Delta = [x]$.*

Proof. This statement is classical. Since we could not find a reference, let us recall the proof. By induction, we may assume that E is indecomposable. Further, we may assume the genus of X to be non-zero, since the statement is clear for $X = \mathbb{P}^1$. Denote by d the degree of the vector bundle E. Since the Jacobian of X has the structure of a complex torus, there is a line bundle $L \to X$ of *r-torsion*:

$$L^{\otimes r} \simeq \mathcal{O}_X,$$

but $L^{\otimes k} \not\simeq \mathcal{O}_X$ for each $k \in \{1, \ldots, r-1\}$. Let s be a non-vanishing holomorphic section of $L^{\otimes r}$. Denote by Y_L the r-sheeted étale cover of X given by

$$Y_L := \{z \in L \mid z^{\otimes r} \in s\}.$$

Denote the covering map by

$$\pi : Y_L \longrightarrow X$$

and its (cyclic) Galois group by Γ. The vector bundle $\pi^* E$ over Y_L is indecomposable of degree rd. Choose a point $y \in \pi^{-1}(\{x\})$. As before, the line bundle $\eta = \mathcal{O}_{Y_L}(d[y])$ can be endowed with a logarithmic connection with polar divisor $[y]$. On the other hand, $\eta^* \otimes \pi^* E$ is irreducible of degree 0 and can be endowed with a holomorphic connection according to Weil's criterion. Thus $\pi^* E$ can be endowed with a logarithmic connection $\tilde{D}_{\pi^* E}$ with polar divisor $[y]$. We can associate the Galois-invariant connection

$$D_{\pi^* E} := \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \gamma^* \tilde{D}_{\pi^* E}$$

on $\pi^* E$. We have

$$\pi_*(\pi^* E) = E \oplus L \otimes E \oplus \cdots \oplus L^{r-1} \otimes E.$$

and it is straightforward to check that the logarithmic connection $\pi_* D_{\pi^* E}$ on $\pi_*(\pi^* E)$ keeps each direct summand invariant. In particular, it induces a logarithmic connection D_E with polar divisor $[x]$ on E. □

This result can be used to further refine the upper bound for the minimal number of necessary poles in order to construct a reducible connection on a given vector bundle.

Example 5.4. Let E be a rank 2 vector bundle over X which is *unstable*, i.e. which possesses a line subbundle L such that $\deg(E) - 2\deg(L) < 0$. Then E can be endowed with a logarithmic connection preserving L whose number of poles is equal to 1.

6. A criterion for flatness of endomorphisms

Let φ be an endomorphism of E and suppose there is a meromorphic connection $D_E : E \longrightarrow E \otimes \Omega^1_X \otimes \mathcal{O}(\Delta)$ such that $D_{E,E}(\varphi) \equiv 0$. Let x be an element of $X \setminus (\Delta)_{\text{red}}$ and let $U \simeq \mathbb{D}$ be a neighborhood of x in $X \setminus (\Delta)_{\text{red}}$. Since any holomorphic connection over a Riemann surface is integrable, there is a local trivialization of E over U where D_E is given by the trivial connection. In particular, any flat section of $D_{E,E}$ is constant with respect to this trivialization. This implies that $\overline{\varphi(x)} \in \mathrm{M}(r, \mathbb{C})/\mathrm{GL}(r, \mathbb{C})$ is constant with respect to $x \in X \backslash (\Delta)_{\mathrm{red}}$. It may not be constant with respect to $x \in X$, as we see in the following example.

Example 6.1. Let $X = \mathbb{P}^1$ and $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Let e_1 and e_2 be holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(1)$ respectively over $\mathbb{C}$. Denote by (e_1', e_2') the dual basis of E^* over $\mathbb{C}$, i.e. $e_i'(e_j) = \delta_{ij}$. Let $\varphi \in \mathrm{H}^0(\mathbb{P}^1, \mathrm{End}(E)) = \mathrm{H}^0(\mathbb{P}^1, E^* \otimes E)$ be given with respect to the basis $\begin{pmatrix} e_1' \otimes e_1 & e_1' \otimes e_2 \\ e_2' \otimes e_1 & e_2' \otimes e_2 \end{pmatrix}$ by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Let D_E be the logarithmic connection on E given with respect to the basis (e_1, e_2) by $\mathrm{d} + A$ where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{\mathrm{d}x}{x}.$$

We then have

$$D_{E,E}(\varphi) = \mathrm{d}\varphi + A \cdot \varphi - \varphi \cdot A \equiv 0,$$

but the conjucacy class of the matrix $\varphi(x)$ depends on $x \in \mathbb{P}^1$.

Proof of the following theorem is identical to the proof in [2].

Theorem 6.2. *Let E be a vector bundle of rank r over X. Let*

$$\varphi \in \mathrm{H}^0(X, \mathrm{End}(E))$$

be a holomorphic endomorphism. Assume that

(1) $\varphi(x) \in \mathrm{End}(E_x)$ *is regular for each $x \in X$, and*
(2) *the conjugacy class*

$$\overline{\varphi(x)} \in \mathrm{M}(r, \mathbb{C})/\mathrm{GL}(r, \mathbb{C})$$

of the matrix $\varphi(x)$ acting on $E_x \simeq \mathbb{C}^r$ does not depend on $x \in X$.

Then for each point $x_0 \in X$ there is a logarithmic connection $D_E : E \longrightarrow E \otimes \Omega_X^1 \otimes \mathcal{O}(\Delta)$ with $\Delta = [x_0]$ such that

$$D_{E,E}(\varphi) \equiv 0,$$

where $D_{E,E}$ is the induced connection on $\mathrm{End}(E)$.

Remark 6.3. Note that (1) implies that the property (2) is satisfied for almost every $x \in X$. Indeed, up to considering a finite cover $Y \to X$, the characteristic polynomial of φ splits. The eigenvalues of φ can then be seen as holomorphic functions on Y, hence they are constant. In restriction to an eigenspace, for each $x \in X$, $\varphi(x)$ then is of the form $\lambda_i \mathrm{Id}_{E_x} + N_i$, where N_i is either zero or nilpotent of maximal order.

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