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Irrational factor races

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Abstract. We investigate the behavior of the sum of the irrational factor function over arithmetic progressions. We first establish a general asymptotic formula for such a sum, and then obtain some further results in the case of arithmetic progressions $3n \pm 1$.

Keywords. Irrational factor function; races; Dirichlet L-functions.

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1. Introduction

By the prime number theorem on arithmetic progressions, the prime numbers are distributed evenly among the residue classes $[a]$ modulo q, with a relatively prime to q. Chebyshev observed that there are more prime numbers congruent to 3 modulo 4 than congruent to 1 modulo 4. In other words, he observed that the difference

$$\sum_{\substack{p \le x \\ p \text{ prime} \\ p \equiv 3 \bmod 4}} 1 - \sum_{\substack{p \le x \\ p \text{ prime} \\ p \equiv 1 \bmod 4}} 1$$

appears to be always positive. A definite result by Littlewood [9] states that there exists an unbounded set of values of x for which the above difference is positive, and there also is an unbounded set of values of x for which the above difference is negative. One may rewrite the above difference in terms of sums of an arithmetic function over all positive integers n less than x, for example

$$\sum_{\substack{n \le x \\ n \equiv 3 \bmod 4}} \frac{\Lambda(n)\mu^2(n)}{\log n} - \sum_{\substack{n \le x \\ n \equiv 1 \bmod 4}} \frac{\Lambda(n)\mu^2(n)}{\log n},$$

where $\Lambda(n)$ and $\mu(n)$ are the von Mangoldt function and Möbius function respectively. In the present paper, we consider a similar difference for another arithmetic function $I(n)$, the irrational factor function introduced by Atanassov in [2]. If $n = \prod_{j=1}^{k} p_j^{\alpha_j}$ is the prime factorization of n, then one defines $I(n) := \prod_{j=1}^{k} p_j^{1/\alpha_j}$.

Alkan *et al.* [1] have obtained the following asymptotic formula:

$$\sum_{n\le x} I(n) = c_2 x^2 + \mathrm{O}(x^{3/2}(\log x)^{9/4}) \qquad \text{for some absolute constant } c_2 > 0.$$

We start by following the same argument in order to obtain a similar asymptotic formula for

$$S(x;q,a) := \sum_{\substack{n \le x \\ n \equiv a \bmod q}} I(n).$$

Theorem 1.1. *Let a and q be relatively prime positive integers. Then*

$$S(x;q,a) = \frac{1}{\phi(q)} \frac{c_{q,a}}{2\zeta(2)} x^2 + \mathrm{O}_q(x^{3/2}(\log x)^{9/4})$$

for some constant $c_{q,a} > 0$, depending on q and a.

In particular, when $q = 3$, one finds that $c_{3,1} = c_{3,2}$ and $c_{3,0} > c_{3,1}$. As a consequence, the difference $S(x;3,0) - S(x;3,1)$ is strictly positive for x sufficiently large. We are interested in the behavior of the sign changes of the difference $S(x;3,1) - S(x;3,2)$. Similar to Littlewood's theorem, we prove the following result:

Theorem 1.2. *The set of values of x for which the difference $S(x;3,1) - S(x;3,2)$ is strictly positive and the set of values of x for which the difference $S(x;3,1) - S(x;3,2)$ is strictly negative are unbounded.*

2. Proof of Theorem 1.1

In this section we provide a proof of Theorem 1.1. The proof below follows the work from Theorem 2(i) of [1], so we will merely provide the details where there are differences in the two proofs.

Proof. Let a and q be relatively prime. For any $n \equiv a \bmod q$, $\sum_{\chi} \bar{\chi}(a)\chi(n) = \phi(q)$, where the sum is over all Dirichlet characters modulo q. For all other integers n, the above sum vanishes. As a consequence,

$$S(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{n\le x} \chi(n) I(n).$$

Next, we focus on the inner sum $\sum_{n\le x}\chi(n)I(n)$. Note that the arithmetic function $\chi(n)I(n)$ is multiplicative. Thus the associated Dirichlet series admits an Euler product, which is easily seen to converge for $\Re(s) > 2$. One has

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\chi(n)I(n)}{n^s} &= \prod_p\left(1+\frac{\chi(p)}{p^{s-1}}+\frac{\chi(p^2)}{p^{2s-1/2}}+\cdots+\frac{\chi(p^m)}{p^{ms-1/m}}+\cdots\right)\\ &= \frac{L(s-1,\chi)}{L(2s-2,\chi^2)}K_\chi(s),\end{aligned}\tag{2.1}$$

where

$$K_\chi(s)=\prod_p(1+A_{p,\chi}(s))$$

and

$$\begin{aligned}A_{p,\chi}(s) = {} & \frac{\chi(p)^2}{(1+\chi(p)/p^{s-1})(p^{2s-1/2})}\\ & \times\left(1+\frac{\chi(p)}{p^{s+1/6}}+\frac{\chi(p^2)}{p^{2s+1/4}}+\cdots+\frac{\chi(p^{m-2})}{p^{(m-2)s+1/2-1/m}}+\cdots\right).\end{aligned}$$

For any complex number s we write $s=\sigma+it$. On the question of convergence of $K_\chi(s)$, observe that

$$\left|\frac{\chi(p)^2}{(1+\chi(p)/p^{s-1})(p^{2s-1/2})}\right|\le\left(1-\frac{1}{2^{\sigma-1}}\right)^{-1}\frac{1}{p^{2\sigma-1/2}}$$

and

$$\begin{aligned}&\left|1+\frac{\chi(p)}{p^{s+1/6}}+\frac{\chi(p^2)}{p^{2s+1/4}}+\cdots+\frac{\chi(p^{m-2})}{p^{(m-2)s+1/2-1/m}}+\cdots\right|\\ &\quad\le 1+\left|\frac{\chi(p)}{p^{s+1/6}}\right|+\left|\frac{\chi(p^2)}{p^{2s+1/4}}\right|+\cdots+\left|\frac{\chi(p^{m-2})}{p^{(m-2)s+1/2-1/m}}\right|+\cdots\\ &\quad\le 1+\frac{1}{p^{\sigma+1/6}}+\frac{1}{p^{2\sigma+1/4}}+\cdots+\frac{1}{p^{(m-2)\sigma+1/2-1/m}}+\cdots.\end{aligned}$$

We deduce that for any character χ modulo q, $K_\chi(s)$ is absolutely convergent and analytic on the half-plane $\sigma > 1$.

Next, by Perron's formula,

$$\sum_{n\le x}\chi(n)I(n)=\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{x^s L(s-1,\chi)}{sL(2s-2,\chi^2)}K_\chi(s)\,\mathrm{d}s+R(x,\alpha,T),$$

where $R(x,\alpha,T)$ is obtained by integrating the above integrand along the line $\alpha-i\infty$ to $\alpha-iT$ and along $\alpha+iT$ to $\alpha+i\infty$.

Let $2 \leq T \leq x^2$ and $\alpha = 2 + \frac{c}{\log x}$ for some absolute constant $c > 0$. Since $|\chi(n)I(n)| \leq I(n)$, the bound for $R(x, \alpha, T)$ obtained on page 303 of [1] also works in our case. Therefore,

$$|R(x, \alpha, T)| \leq \frac{x^\alpha}{T} \sum_{n=1}^{\infty} \frac{|\chi(n)I(n)|}{n^\alpha |\log x/n|} = \mathrm{O}\left(\frac{x^2 \log x}{T}\right).$$

We now focus on the integral $\frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{x^s L(s-1, \chi)}{s L(2s-2, \chi^2)} K_\chi(s)\, \mathrm{d}s$, which we evaluate by shifting the path of integration to the rectangular path with vertices $\alpha \pm iT$ and $3/2 \pm iT$.

We first consider the principal character χ_0 modulo q. In this case,

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} (1 - p^{-s}),$$

and by the residue theorem,

$$\frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{x^s L(s-1, \chi_0)}{s L(2s-2, \chi_0^2)} K_\chi(s)\, \mathrm{d}s = \prod_{p|q} (1 - p^{-1}) \frac{K_{\chi_0}(2)}{2L(2, \chi_0)} x^2 + \sum_{m=1}^{3} J_m, \tag{2.2}$$

where J_1 is the integral along the line from $\alpha + iT$ to $3/2 + iT$, J_2 is the integral along the line from $3/2 + iT$ to $3/2 - iT$, and J_3 is the integral along the line from $3/2 - iT$ to $\alpha - iT$.

Note that the first term in equation (2.2), which is obtained from the residue of the pole at $s = 2$, gives rise to the main term in the asymptotic formula in the statement of Theorem (1.1).

To estimate the J_m's, one finds that the bounds provided on page 304 of [1], for similar integrals, apply to our case as well, modulo multiplication by constants depending on q. Therefore,

$$|J_1|, |J_3| = \mathrm{O}_q\left(\frac{x^2 (\log x)^2}{T}\right)$$

and

$$|J_2| = \mathrm{O}_q(x^{3/2} \log x (\log T)^{5/4}).$$

Next, we consider the case when we have a non-principal character modulo q. We continue with the rectangular contour defined above. We use the following bounds to estimate $|J_m|$:

$$|L(s-1, \chi)| = \begin{cases} \mathrm{O}_\epsilon\left(T^{1/6(1-\sigma)+\epsilon} q^{1-\sigma}\right), & \text{for } 3/2 \leq \sigma \leq 2, \\ \mathrm{O}\left((\log T)^{2/3}\right), & \text{for } 2 \leq \sigma \leq \alpha. \end{cases}$$

The first bound is standard, and the second follows from the bound

$$|L(s, \chi)| = \mathrm{O}(\tau^{100(1-\sigma)3/2} q^{1-\sigma} (\log \tau)^{2/3})$$

for $1/2 \leq \sigma \leq 1$ and $\tau = |t| + 2$ provided by Kolesnik in [7]. We will also use the bound

$$\left|\frac{1}{L(2\sigma - 2 + i2T, \chi^2)}\right| = \mathrm{O}_q\left(\log T\right),$$

for $3/2 \leq \sigma \leq \alpha$. Lastly, note that $|K_\chi(s)|$ is bounded on the region $3/2 \leq \sigma \leq \alpha$.

Then

$$\begin{aligned}
|J_1|, |J_3| &= \mathrm{O}_q\left(\int_{3/2}^{\alpha} \frac{|x^{\sigma+iT}||L(\sigma - 1 + iT, \chi)|}{|\sigma + iT||L(2\sigma - 2 + i2T, \chi^2)|}|K_\chi(\sigma + iT)|\,\mathrm{d}\sigma\right) \\
&= \mathrm{O}_{q,\epsilon}\left(\int_{3/2}^{2} \frac{x^\sigma T^{1/6(1-\sigma)+\epsilon} q^{1-\sigma}}{T} \log T\,\mathrm{d}\sigma\right. \\
&\quad \left. + \int_{2}^{\alpha} \frac{x^\sigma (\log T)^{2/3}}{T} \log T\,\mathrm{d}\sigma\right) \\
&= \mathrm{O}_{q,\epsilon}\left((\log x) \int_{3/2}^{2} \frac{x^\sigma T^{1/6(1-\sigma)+\epsilon} q^{1-\sigma}}{T}\,\mathrm{d}\sigma\right. \\
&\quad \left. + (\log x)^{5/3} \int_{2}^{\alpha} \frac{x^\sigma}{T}\,\mathrm{d}\sigma\right) \\
&= \mathrm{O}_q\left(\frac{(\log x)^{5/3} x^\alpha}{T}\right).
\end{aligned}$$

On the line segment $s = 3/2 + it$ for $|t| \leq T$, we have $|\frac{1}{L(1+i2t,\chi^2)}| = \mathrm{O}_q\left(\log t\right)$. Then

$$\begin{aligned}
|J_2| &= \mathrm{O}_q\left(\int_{-T}^{T} \frac{|x^{3/2+it}||L(3/2 + it, \chi)|}{|3/2 + it||L(1 + i2t, \chi^2)|}|K_\chi(3/2 + it)|\,\mathrm{d}t\right) \\
&= \mathrm{O}_q\left(x^{3/2} \log t \int_{-T}^{T} \frac{|L(3/2 + it, \chi)|}{|t|}\,\mathrm{d}t\right) \\
&= \mathrm{O}_q\left(x^{3/2} \log t \left[1 + \sum_{0 \leq k \leq [\log T/\log 2]} \frac{1}{2^k}\right.\right. \\
&\quad \left.\left. \int_{2^k}^{2^{k+1}} (|L(3/2 + it, \chi)| + |L(3/2 - it, \chi)|)\,\mathrm{d}t\right]\right) \\
&= \mathrm{O}_q\left(x^{3/2} \log t \left(1 + \sum_{0 \leq k \leq [\log T/\log 2]} k^{1/4}\right)\right) \\
&= \mathrm{O}_q(x^{3/2} \log t (\log T)^{5/4})
\end{aligned}$$

by Ramachandra's mean value theorem (Theorem 2 and Remark 2 of [10]). Collecting all the above estimates and selecting $T = x$ one obtains a bound for the error term of the form $\mathrm{O}_q(x^{3/2}(\log x)^{9/4})$. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

In this section we provide a proof of Theorem 1.2. We begin with the following remark.

Remark. The constant $c_{3,a}$ which appears in the main term in the asymptotic formula for an arithmetic progression $3n + a$, where $a \in \{1, 2\}$ is given by $c_{3,a} = \dfrac{K_{\chi_0}(2)}{\phi(3)2\zeta(2)}$. Combining this with the result from [1] concerning the sum over all $n \leq x$, one obtains an asymptotic formula corresponding to the arithmetic progression $3n + a$ with $a = 0$. In particular, the corresponding constant in [1] equals $\frac{K(2)}{2\zeta(2)}$, for some specific function $K(s)$, hence

$$c_{3,0} = \frac{K(2)}{2\zeta(2)} - 2\frac{K_{\chi_0}(2)}{\phi(3)2\zeta(2)}.$$

As mentioned in the Introduction, a numerical comparison of the two constants reveals that $\frac{c_{3,0}}{c_{3,1}} > 1$. So it follows that the sum of irrational factor grows quicker over the progression $3n$. Theorem 1.2 compares its growth over the arithmetic progressions $3n + 1$ and $3n + 2$.

Proof. In order to prove Theorem 1.2, we use the approach by Knapowski and Turan in [6]. We begin with the integral

$$\int_1^\infty (S(x; k, l_1) - S(x; k, l_2) \pm c x^{\frac{5}{4}-\epsilon}) x^{-s-1} \mathrm{d}x, \qquad \text{for } \epsilon > 0.$$

As noted before,

$$S(x; q, l) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(l) \sum_{n \leq x} \chi(n) I(n),$$

hence

$$S(x; q, l_1) - S(x; q, l_2) = \left(\frac{1}{\phi(q)} \sum_{\chi \bmod q} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \right) \sum_{n \leq x} \chi(n) I(n).$$

Therefore

$$\begin{aligned} &\int_1^\infty \frac{S(x; q, l_1) - S(x; q, l_2)}{x^{s+1}} \mathrm{d}x \pm \int_1^\infty \frac{c x^{\frac{5}{4}-\epsilon}}{x^{s+1}} \mathrm{d}x \\ &\quad = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \left((\bar{\chi}(l_1) - \bar{\chi}(l_2)) \int_1^\infty \frac{\sum_{n \leq x} \chi(n) I(n)}{x^{s+1}} \mathrm{d}x \right) \\ &\quad \pm \frac{c}{s - \frac{5}{4} + \epsilon}. \end{aligned} \tag{3.1}$$

Note that the integral on the right-hand side above represents the Dirichlet series $\sum_{n=1}^\infty \chi(n) I(n) n^{-s}$ as a Mellin transform. Combining this with (2.1), we find that

$$\int_1^\infty \frac{\sum_{n \leq x} \chi(n) I(n)}{x^{s+1}} \mathrm{d}x = \frac{L(s-1, \chi)}{s L(2s-2, \chi^2)} K_\chi(s), \qquad \text{for } \sigma > 2,$$

therefore

$$\int_1^\infty \frac{S(x;q,l_1) - S(x;q,l_2) \pm cx^{\frac{5}{4}-\epsilon}}{x^{s+1}}\mathrm{d}x$$
$$= \frac{1}{\phi(q)} \sum_{\chi \bmod q} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{L(s-1,\chi)}{sL(2s-2,\chi^2)} K_\chi(s)$$
$$\pm \frac{c}{s - \frac{5}{4} + \epsilon}.$$

Note that the contribution of the principal character χ_0 on the right-hand side above vanishes as $\bar{\chi}_0(l_1) = \bar{\chi}_0(l_2) = 1$, hence

$$\int_1^\infty \frac{S(x;q,l_1) - S(x;q,l_2) \pm cx^{\frac{5}{4}-\epsilon}}{x^{s+1}}\mathrm{d}x$$
$$= \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{L(s-1,\chi)}{sL(2s-2,\chi^2)} K_\chi(s)$$
$$\pm \frac{c}{s - \frac{5}{4} + \epsilon}. \tag{3.2}$$

Next we use the following classical theorem of Landau [8] on the location of singularities of the Mellin transforms of a non-negative function.

Suppose $A(x)$ is a real valued function in one variable, and $A(x)$ does not change its sign for $x > x_0$, where x_0 is a sufficiently large real number. Suppose also for some real number $\beta < \gamma$, that the Mellin transform $g(s) := \int_1^\infty A(x)x^{-s-1}\mathrm{d}x$ is analytic for $\Re(s) > \gamma$, and can be analytically continued to the real segment $(\beta, \gamma]$. Then $g(s)$ represents an analytic function in the half plane $\Re(s) > \beta$.

We apply Landau's result by taking

$$A(x) = S(x;3,l_1) - S(x;3,l_2) \pm cx^{\frac{5}{4}-\epsilon}.$$

Hence from eq. (3.2),

$$g(s) = \frac{1}{s\phi(3)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \frac{L(s-1,\chi)}{L(2s-2,\chi^2)} K_\chi(s) \pm \frac{c}{s - \frac{5}{4} - \epsilon}.$$

Clearly $g(s)$ is regular in the half plane $\sigma \geq \frac{3}{2}$ as none of the denominators $L(2s-2,\chi^2)$ have zeros in that region. Also, unless at least one of these denominators $L(2s-2,\chi^2)$ has nontrivial real zeros, the function $g(s)$ will be regular on the real segment $\frac{5}{4} - \epsilon < \sigma \leq \frac{3}{2}$.

We now return to our particular case, where $q = 3$. In this case, there is only one non-principal character χ, which is a quadratic character. So we have only one denominator $L(2s-2,\chi^2) = L(2s-2,\chi_0)$, which does not have any nontrivial real zeros since the Riemann zeta function does not have any. Hence in this case $g(s)$ is regular on the real segment $\frac{5}{4} - \epsilon < \sigma \leq \frac{3}{2}$.

Let us assume now that there exists a real number x_0 such that $A(x)$ does not change sign for $x > x_0$. Then by Landau's theorem, $g(s)$ is regular in the half plane $\sigma > \frac{5}{4} - \epsilon$. Therefore each zero of the denominator $L(2s-2, \chi^2) = L(2s-2, \chi_0)$ in this half plane must also be either a zero of $L(s-1, \chi)$ or a zero of $K_\chi(s)$. But, it is well known that almost all the nontrivial zeros of the Riemann zeta function cluster around the critical line, so almost all the zeros of the denominator $L(2s-2, \chi_0)$ are close to the line $\sigma = 5/4$. Similarly, almost all the nontrivial zeros of $L(s-1, \chi)$ are close to the line $\sigma = 3/2$. Therefore the proportion of zeros of the denominator $L(2s-2, \chi_0)$ that are cancelled by zeros of $L(s-1, \chi)$ is negligible. It remains to consider possible instances when zeros of $L(2s-2, \chi_0)$ are also zeros of $K_\chi(s)$.

To proceed, we will focus on those zeros which lie on the critical line. We know that a positive proportion of zeros of the Riemann zeta function lie on the critical line [3]. Thus, the number of zeros of the denominator $L(2s-2, \chi_0)$ which lie on the line $\sigma = 5/4$ and with imaginary parts bounded by a large parameter T, grows at least like a constant times $T \log T$ as T tends to infinity.

On the other hand, the argument from the previous section used to establish convergence of $K_\chi(s)$ on the half plane $\sigma > 1$ also shows that the only possible zeros of $K_\chi(s)$ in this half plane are zeros of individual Euler factors $1 + A_{p,\chi}(s)$. The same argument also shows that there exists a p_0 such that for each $p > p_0$, the Euler factor $1 + A_{p,\chi}(s)$ does not have any zeros in the half plane $\sigma \geq 5/4$, and in particular, does not have any zeros on the line $\sigma = 5/4$.

Let us fix now a prime p with $p \leq p_0$, and consider the Euler factor $1 + A_{p,\chi}(s)$ on the line $\sigma = 5/4$. As a function of t, $1 + A_{p,\chi}(5/4 + it)$ is periodic with period $2\pi/\log p$, and on the interval $0 \leq t \leq 2\pi/\log p$ it has only finitely many zeros. Therefore, the number of zeros of $1 + A_{p,\chi}(5/4 + it)$ with $|t| \leq T$ grows at most like a constant times T as T tends to infinity. This holds for each $p \leq p_0$, hence the total number of zeros of $K_\chi(s)$ on the line $\sigma = 5/4$ and with imaginary parts bounded by T, grows most like a constant times T as T tends to infinity.

We deduce that there exist zeros of the denominator $L(2s-2, \chi_0)$ on the line $\sigma = 5/4$ which are not zeros of the product $L(s-1, \chi)K_\chi(s)$. Any such zero is a pole of $g(s)$, and this contradicts via Laundau's theorem, the assumption that there exists an x_0 such that $A(x)$ does not change sign for $x > x_0$. This completes the proof of Theorem 1.2. □

Remark. Using the above methods, we do not gain any information about the frequency of sign changes of the function $S(x; 3, 1) - S(x; 3, 2)$. A result by Kaczorowski and Wiertelak (see Lemma 3.1 of [5]), allows us to consider this question and further characterize the behavior of the race. Before we state their result, we provide some necessary notation from Kaczorowski and Pintz in [4].

For any real function $f(x)$ defined for $x > 0$, Kaczorowski and Pintz defined the number $V(f, T)$ of sign changes in the interval $(0, T]$ as

$$V(f, T) := \sup\{N : \exists \{x_i\}_{i=1}^{N}, 0 < x_1 < x_2 < \cdots < x_N \leq T, f(x_i) \neq 0,$$
$$\operatorname{sgn} f(x_i) \neq \operatorname{sgn} f(x_{i+1}), \quad 1 \leq i < N\}.$$

They say $V(f, T) > h(T)$ with combined oscillation of size $g(x)$ if there exists a series $\{x_i\}_{i=1}^{h(T)}$ with $\operatorname{sgn} f(x_i) \neq \operatorname{sgn} f(x_{i+1})$ and $|f(x_i)| > g(x_i)$.

We are now in a position to state Kaczorowski and Wiertelak's result:

Let A(x) be real for $x > 0$ and suppose that the integral $g(s) := \int_0^\infty A(x)x^{-s-1}\mathrm{d}x$ converges absolutely for $\Re(s) > \sigma_0$, and has meromorphic continuation to a half-plane $\Re(s) > \theta$ for certain $\theta < \sigma_0$. Suppose that $g(s)$ is not holomorphic for $\Re(s) > \theta$, but is holomorphic on the segment $(\theta, \sigma_0]$ of the real axis. Then, for sufficiently large T, $A(x)$ has in the interval $(0, T]$ at least $\gg \log T$ oscillations of size x^θ.

Applying this in our case for the function

$$A(x) = S(x; 3, 1) - S(x; 3, 2) \pm cx^{\frac{5}{4}-\epsilon},$$

we obtain a sequence $\{x_i\}_{i=1}^{[\log T]}$ of length $\log T$ in the interval $(1, T]$ such that sgn $A(x_i) \neq$ sgn $A(x_{i+1})$ and $|A(x_i)| > {x_i}^{5/4-\epsilon}$. Thus, in the interval $(1, T]$, $A(x)$ has at least $\gg \log T$ oscillations of size $x^{5/4-\epsilon}$.

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