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Some limit theorems for negatively associated random variables

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables. The aim of this paper is to establish some limit theorems of negatively associated sequence, which include the L^p-convergence theorem and Marcinkiewicz–Zygmund strong law of large numbers. Furthermore, we consider the strong law of sums of order statistics, which are sampled from negatively associated random variables.

Keywords. Negatively associated random variables; L^p-convergence; Marcinkiewicz–Zygmund strong law of large numbers; order statistics.

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1. Introduction

Let us consider a sequence $\{X_i, i \geq 1\}$ of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We start with some definitions. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA), if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and any real nondecreasing co-ordinatewise functions f_1 on $\mathbb{R}^A$ and f_2 on $\mathbb{R}^B$,

$$\mathrm{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0$$

whenever f_1 and f_2 are such that covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

The notion of negative association was first introduced by Alam and Saxena [1]. Joag-Dev and Proschan [4] showed that many well known multivariate distributions possess the NA property. Some examples include: (a) multinomial, (b) convolution of unlike multinomial, (c) multivariate hypergeometric, (d) Dirichlet, (e) Dirichlet compound multinomial, (f) negatively correlated normal distribution, (g) permutation distribution, (h) random sampling without replacement, and (i) joint distribution of ranks. Because of its wide applications in multivariate statistical analysis and system reliability, the notion of negative association has received considerable attention recently.

One can refer to Joag-Dev and Proschan [4] and Block *et al*. [2] for some fundamental properties and examples of negatively associated sequences, Newman [6] for the central limit theorem, Matuła [5] for the three series theorems, Su *et al*. [10] for the moment inequality, Roussas [7] for the Hoeffding inequality, Shao [8] for the Rosenthal-type maximal inequality and the Kolmogorov exponential inequality, Shao and Su [9] for the law of the iterated logarithm and, Jing and Liang [3] for strong limit theorems for weighted sums.

In this paper, we shall first establish an L^p-convergence theorem for a negatively associated sequence. The second aim of this paper is to prove Marcinkiewicz–Zygmund strong law of large numbers for negatively associated sequences under the case where $\{X_n, n \geq 1\}$ are uniformly dominated by a random variable X. The third result is to obtain a strong law for order statistics for a negatively associated sequence, which includes the Kolmogorov's strong law of large numbers.

Throughout this paper, C denotes a positive constant, which may take different values whenever it appears in different expressions. 1_A denotes the indicator function of the event A.

2. Main results

We first consider the L^p-convergence theorem for negatively associated sequence.

Theorem 2.1. *Let* $0 < p < 2$. *Suppose that* $\{X_n, n \geq 1\}$ *is a sequence of negatively associated random variables such that* $\sup_n \mathbb{E}|X_n|^p < \infty$. *For a sequence* $\{a_n\}$ *of positive real numbers, assume that*

$$\mathbb{E}(|X_n|^p 1_{A_n}) \to 0, \tag{2.1}$$

where $A_n = \{|X_n| \geq a_n\}$. *If one of the following two conditions holds*:

(1) $0 < p < 1$ *and* $\sum_{k=1}^n a_k^{1-p} = o(n^{1/p})$,
(2) $1 \leq p < 2$, $\sum_{k=1}^n a_k^{2-p} = o(n^{2/p})$ *and* $\mathbb{E}X_n = 0$,

then we have $\mathbb{E}|S_n|^p = o(n)$, *where* $S_n = \sum_{k=1}^n X_k$.

Remark 2.1. If we take $a_k = (k/\log k)^{1/p}$, then the conditions (1) and (2) in Theorem 2.1 will be satisfied. For the sequence $a_k = (k/\log k)^{1/p}$, there are several cases in which condition (2.1) holds:

(i) If $\{X_n, n \geq 1\}$ is a sequence of identically distributed random variables, then condition (2.1) holds.
(ii) If $\{|X_n|^p, n \geq 1\}$ is uniformly integrable for some $0 < p < 2$, then condition (2.1) holds.
(iii) Let $\{X_n, n \geq 1\}$ and X be measurable functions such that $\mathbb{P}(|X_n| > x) \leq \mathbb{P}(|X| > x)$, $x > 0$, with $X \in L^p$, then from Lemma 3.5 below, condition (2.1) holds.

Next, we obtain the Marcinkiewicz–Zygmund strong law of large numbers for the negatively associated sequence. Let us recall that the random variables $\{X_n, n \geq 1\}$ are uniformly dominated by a random variable X (see (iii) in Remark 2.1) if

$$\mathbb{P}(|X_n| > x) \leq \mathbb{P}(|X| > x) \tag{2.2}$$

for all $x > 0$ and $n \geq 1$. This dominated condition means weakly dominated (WD), where weak refers to the fact that domination is distributional. From the proof of the following theorem, we can generalize condition (2.2) as

$$\mathbb{P}(|X_n| > x) \leq C\mathbb{P}(|X| > x) \tag{2.3}$$

for all $x > 0$ and $n \geq 1$, where C is a some constant.

Theorem 2.2. *Let $0 < p < 2$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of negatively associated random variables and X be a random variable possibly defined on a different space satisfying condition (2.2). If $\mathbb{E}|X|^p < \infty$, then we have*

$$n^{-1/p} S_n := n^{-1/p} \sum_{k=1}^{n} X_k \to 0, \quad a.s.$$

The following theorem devotes a strong law of large numbers for order statistics based on sampling from negatively associated random variables. Let $\{X, X_n, n \geq 1\}$ be an identically distributed sequence of negatively associated random variables with common distribution function F. For each positive integer n, $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denotes the order statistics associated with the first n observations $X_1, \ldots, X_n$. Furthermore, the empirical distribution function of $X_1, \ldots, X_n$ is defined as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{X_i \leq x\}}, \quad x \in \mathbb{R}.$$

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables with the common distribution function F and $\mathbb{E}|X_1| < \infty$. If F is absolutely continuous, then for each $k = 0, 1, 2, \ldots,$*

$$\lim_{n\to\infty} \frac{k+1}{n^{k+1}} \sum_{i=1}^{n} i^k X_{i:n} = \frac{1}{k+1} \mathbb{E}Y \quad a.s.,$$

where the distribution function of Y is

$$F_Y(x) = F^{k+1}(x), \quad x \in \mathbb{R}.$$

Remark 2.2. Since $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_{i:n}$ a.s., and if we take $k = 0$, then the Kolmogorov's strong law of large numbers for negatively associated random variables can be derived.

3. Proofs of main results

3.1 *Some useful lemmas*

First we recall the following moment inequality.

Lemma 3.1 (*Theorem* 2 *of* [8]). *Let $1 < p \leq 2$, $\{X_i, 1 \leq i \leq n\}$ be a sequence of negatively associated random variables with $\mathbb{E}X_i = 0$ and $\mathbb{E}|X_i|^p < \infty$ for every $1 \leq i \leq n$. Then we have*

$$\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^p \leq 2^{3-p} \sum_{i=1}^{n} \mathbb{E}|X_i|^p.$$

The following series convergence theorem and Kolmogorov's strong law of large numbers is from Matuła [5].

Lemma 3.2 (*Theorem* 3 *of* [5]). *Let* $\{X_n, n \geq 1\}$ *be a sequence of negatively associated random variables with finite second moments. If*

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty,$$

then $\sum_{n=1}^{\infty}(X_n - \mathbb{E}X_n)$ *converges a.s.*

Lemma 3.3 [5]. *Let* $\{X_n, n \geq 1\}$ *be a sequence of negatively associated random variables with same distribution function* F. *If* $\mathbb{E}|X_1| < \infty$, *then* $n^{-1}\sum_{k=1}^{n} X_k \to \mathbb{E}X_1$.

Lemma 3.4 [11]. *If* $\{X_n\}$ *is a sequence of negatively associated random variables and* $\{f_n\}$ *is a sequence of Borel functions all of which are monotone increasing* (*or all monotone decreasing*), *then* $\{f_n(X_n)\}$ *is a sequence of negatively associated random variables.*

Lemma 3.5. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables dominated by a random variable* X, *i.e.*,

$$\mathbb{P}(|X_n| > x) \leq \mathbb{P}(|X| > x). \tag{3.1}$$

Let $r > 0$ *and for some* $C > 0$,

$$X_i' = X_i 1_{\{|X_i| \leq C\}}, \quad X_i'' = X_i 1_{\{|X_i| > C\}}$$

and

$$X' = X 1_{\{|X| \leq C\}}, \quad X'' = X 1_{\{|X| > C\}}.$$

Then for any k, *we have*

(1) *if* $\mathbb{E}|X|^p < \infty$, *then* $\mathbb{E}|X_k|^p \leq \mathbb{E}|X|^p$,
(2) $\mathbb{E}|X_k'|^p \leq (\mathbb{E}|X'|^p + C^p\mathbb{P}(|X| > C))$ *for any* $C > 0$,
(3) $\mathbb{E}|X_k''|^p \leq \mathbb{E}|X''|^p$.

Proof. We only give the proof of (2) and the others can be obtained by a similar method. Note that the proof is based on the fact that for any random variable Y with $\mathbb{E}|Y|^p < \infty$,

$$\mathbb{E}|Y|^p = p\int_0^{\infty} y^{p-1}\mathbb{P}(|Y| > y)\mathrm{d}y.$$

Hence we have

$$\begin{aligned}
\mathbb{E}|X_k'|^p &= p\int_0^{\infty} y^{p-1}\mathbb{P}(|X_k'| > y)\mathrm{d}y \\
&= p\int_0^{C} y^{p-1}\mathbb{P}(y < |X_k| \leq C)\mathrm{d}y \\
&= p\int_0^{C} y^{p-1}\left[\mathbb{P}(|X_k| > y) - \mathbb{P}(|X_k| > C)\right]\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= p\int_0^C y^{p-1}\mathbb{P}(|X_k| > y)\mathrm{d}y - C^p\mathbb{P}(|X_k| > C) \\
&\le p\int_0^C y^{p-1}\mathbb{P}(|X| > y)\mathrm{d}y.
\end{aligned}
$$

By the same calculation, we have

$$
p\int_0^C y^{p-1}\mathbb{P}(|X| > y)\mathrm{d}y = \mathbb{E}|X'|^p + C^p\mathbb{P}(|X| > C),
$$

which implies (2). □

The following result is Glivenko–Cantelli lemma of negatively associated sequence.

Lemma 3.6. *Let $\{X_n, n \ge 1\}$ be a sequence of negatively associated random variables with the common distribution function F. Then we have*

$$
\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \to 0, \quad a.s.
$$

Proof. Since $\{X_n, n \ge 1\}$ is a sequence of negatively associated random variables, then for any $x \in \mathbb{R}$, by Lemma 3.4, the sequence $\{1_{\{X_n \le x\}}, n \ge 1\}$ is also negatively associated. Let N be a positive integer. For any $0 \le j \le N$, put

$$
x_{j,N} = \inf\{x : F(x) \ge j/N\}.
$$

Then it is obvious that

$$
F(x_{j,N}) \ge j/N \ge F(x_{j,N^-}), \quad F(x_{j,N}) + 1/N \ge F(x_{j+1,N^-}).
$$

For $x \in [x_{j,N}, x_{j+1,N})$, it follows that

$$
\begin{aligned}
F_n(x) \le F_n(x_{j+1,N^-}) &\le F(x_{j+1,N^-}) + |F_n(x_{j+1,N^-}) - F(x_{j+1,N^-})| \\
&\le F(x) + 1/N + \max_{1\le j\le N} |F_n(x_{j+1,N^-}) - F(x_{j+1,N^-})|
\end{aligned}
$$

and

$$
\begin{aligned}
F_n(x) \ge F_n(x_{j,N}) &\ge F(x_{j,N}) - |F_n(x_{j,N}) - F(x_{j,N})| \\
&\ge F(x) - 1/N - \max_{1\le j\le N} |F_n(x_{j,N}) - F(x_{j,N})|.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| &\le \frac{1}{N} + \max_{1\le j\le N} |F_n(x_{j,N}) - F(x_{j,N})| \\
&\quad + \max_{1\le j\le N} |F_n(x_{j+1,N^-}) - F(x_{j+1,N^-})|.
\end{aligned}
$$

Since from Lemma 3.3, for any $0 \le j \le N$,

$$
F_n(x_{j,N}) \to F(x_{j,N}), \text{ a.s. and } F_n(x_{j+1,N^-}) \to F(x_{j+1,N^-}), \text{ a.s.}
$$

we have

$$
\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \le \frac{1}{N}, \text{ a.s.}
$$

Letting $N \to \infty$, the proof of Lemma 3.6 is completed. □

3.2 *Proof of Theorem* 2.1

Define

$$X'_n = X_n 1_{\{|X_n| \le a_n\}} + a_n 1_{\{X_n > a_n\}} - a_n 1_{\{X_n < -a_n\}},\ X''_n = X_n - X'_n,$$

$$S'_n = \sum_{k=1}^{n} [X'_k - \mathbb{E}(X'_k)], \quad S''_n = \sum_{k=1}^{n} [X''_k - \mathbb{E}(X''_k)], \quad M = \sup_n \mathbb{E}|X_n|^p.$$

Then $S_n = S'_n + S''_n$ and we would have $\mathbb{E}|S_n|^p = o(n)$ as soon as we prove that $\mathbb{E}|S'_n| = o(n)$ and $\mathbb{E}|S''_n| = o(n)$. Note that $\{X'_n - \mathbb{E}X'_n, n \ge 1\}$ and $\{X''_n - \mathbb{E}X''_n, n \ge 1\}$ are two negatively associated sequences.

(1) First suppose that $0 < p < 1$ and $\sum_{k=1}^n a_k^{1-p} = o(n^{1/p})$. Since by hypothesis

$$\begin{aligned} \mathbb{E}|X''_n|^p &\le C[\mathbb{E}|X_n|^p 1_{\{|X_n| \ge a_n\}} + a_n^p \mathbb{P}(|X_n| \ge a_n)] \\ &\le C\mathbb{E}|X_n|^p 1_{\{|X_n| \ge a_n\}} \to 0, \end{aligned} \tag{3.2}$$

we have from Lemma 3.1,

$$\mathbb{E}|S''_n|^p \le C \sum_{i=1}^{n} \mathbb{E}|X''_n - \mathbb{E}(X''_n)|^p = o(n).$$

Furthermore,

$$\mathbb{E}|X'_n|^p \le C[\mathbb{E}|X_n|^p 1_{\{|X_n| \le a_n\}} + a_n^p \mathbb{P}(|X_n| \ge a_n)] \le C\mathbb{E}|X_n|^p \le CM. \tag{3.3}$$

Hence, we have

$$\begin{aligned} (\mathbb{E}|S'_n|^p)^{1/p} \le \mathbb{E}|S'_n| &\le \sum_{i=1}^{n} \mathbb{E}|X'_i - \mathbb{E}(X'_i)| \le C \sum_{i=1}^{n} \mathbb{E}|X'_i| \\ &\le C \sum_{i=1}^{n} a_i^{1-p} \mathbb{E}|X'_i|^p \le CM \sum_{i=1}^{n} a_i^{1-p} = o(n^{1/p}), \end{aligned}$$

which yields $\mathbb{E}|S'_n|^p = o(n)$. This proves the assertion for the case $0 < p < 1$.

(2) Suppose that $1 \le p < 2$, $\sum_{k=1}^n a_k^{2-r} = o(n^{2/p})$ and $\mathbb{E}X_n = 0$. By a similar proof of the inequalities (3.2) and (3.3), it follows that

$$\mathbb{E}|X''_n|^p \to 0, \quad \mathbb{E}|X'_n|^p \le CM.$$

From Lemma 3.1, we get

$$\mathbb{E}|S''_n|^p \le C \sum_{i=1}^{n} \mathbb{E}|X''_i - \mathbb{E}(X''_i)|^p = o(n)$$

and

$$\begin{aligned}\mathbb{E}|S_n'|^2 &\le C\sum_{k=1}^{n}\mathbb{E}[X_k' - \mathbb{E}(X_k')]^2 \le C\sum_{k=1}^{n}\mathbb{E}[X_k']^2\\ &\le C\sum_{k=1}^{n}[\mathbb{E}|X_k|^2 1_{\{|X_k|\le a_k\}} + a_k^2\mathbb{P}(|X_k|\ge a_k)]\\ &\le C\sum_{k=1}^{n} a_k^{2-p}[\mathbb{E}|X_k|^p] \le CM\sum_{k=1}^{n} a_k^{2-p} = o(n^{2/p}).\end{aligned}$$

Hence we have

$$\mathbb{E}|S_n'|^p \le (\mathbb{E}|S_n'|^2)^{p/2} = o(n).$$

This proves the assertion for the case $1 \le p < 2$ and the proof of the theorem is completed. □

3.3 *Proof of Theorem* 2.2

Let

$$X_n' = X_n 1_{\{|X_n|\le n^{1/p}\}} + n^{1/p}1_{\{X_n > n^{1/p}\}} - n^{1/p}1_{\{X_n < -n^{1/p}\}},$$

$$X_n'' = X_n - X_n',$$

$$S_n' = \sum_{k=1}^{n}[X_k' - \mathbb{E}(X_k')], \quad S_n'' = \sum_{k=1}^{n}[X_k'' - \mathbb{E}(X_k'')].$$

First under the conditions of Theorem 2.2, we shall prove

$$\sum_{n=1}^{\infty}\mathbb{P}(|X_n|\ge n^{1/p}) < \infty; \tag{3.4}$$

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{-1/p}\mathbb{E}|X_n'| < \infty \quad \text{for } 0 < p < 1,\\ &\sum_{n=1}^{\infty} n^{-1/p}\mathbb{E}|X_n''| < \infty \quad \text{for } 1 < p < 2,\end{aligned} \tag{3.5}$$

$$\lim_{n\to\infty}\mathbb{E}|X_n''| = 0 \quad \text{for } p = 1;$$

$$\sum_{n=1}^{\infty} n^{-2/p}\mathbb{E}|X_n'|^2 < \infty. \tag{3.6}$$

From $\mathbb{E}|X|^p < \infty$ and the weakly dominated condition (2.2), we have

$$\sum_{n=1}^{\infty}\mathbb{P}(|X_n|\ge n^{1/p}) = \sum_{n=1}^{\infty}\mathbb{P}(|X|\ge n^{1/p}) < \infty.$$

By Lemma 3.5, we have for $0 < p < 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1/p}\mathbb{E}|X_n'| &\leq C\sum_{n=1}^{\infty}[n^{-1/p}\mathbb{E}|X_n|1_{\{|X_n|\leq n^{1/p}\}} + \mathbb{P}(|X_n| > n^{1/p})] \\ &\leq C\sum_{n=1}^{\infty}[n^{-1/p}\mathbb{E}|X|1_{\{|X|\leq n^{1/p}\}} + \mathbb{P}(|X| > n^{1/p})] \\ &\leq C\sum_{n=1}^{\infty} n^{-1/p}\sum_{k=1}^{n}\mathbb{E}|X|1_{\{(k-1)^{1/p}<|X|\leq k^{1/p}\}} + C \\ &\leq C\sum_{k=1}^{\infty} k^{1-1/p}\mathbb{E}|X|1_{\{(k-1)^{1/p}<|X|\leq k^{1/p}\}} + C \\ &\leq C\sum_{k=1}^{\infty}\mathbb{E}|X|^p 1_{\{(k-1)^{1/p}<|X|\leq k^{1/p}\}} + C < \infty,\end{aligned}$$

for $1 < p < 2$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1/p}\mathbb{E}|X_n''| &\leq C\sum_{n=1}^{\infty}[n^{-1/p}\mathbb{E}|X_n|1_{\{|X_n|> n^{1/p}\}} + \mathbb{P}(|X_n| > n^{1/p})] \\ &\leq C\sum_{n=1}^{\infty}[n^{-1/p}\mathbb{E}|X|1_{\{|X|> n^{1/p}\}} + \mathbb{P}(|X| > n^{1/p})] \\ &\leq C\sum_{k=1}^{\infty} k^{1-1/p}\mathbb{E}|X|1_{\{k^{1/p}<|X|\leq (k+1)^{1/p}\}} + C \\ &\leq C\sum_{k=1}^{\infty}\mathbb{E}|X|^p 1_{\{k^{1/p}<|X|\leq (k+1)^{1/p}\}} + C < \infty,\end{aligned}$$

and for $p = 1$,

$$\mathbb{E}|X_n''| \leq C[\mathbb{E}|X_n|1_{\{|X_n|>n\}} + n\mathbb{P}(|X_n| > n)] \leq C\mathbb{E}|X|1_{\{|X|>n\}} \to 0.$$

At last, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-2/p}\mathbb{E}|X_n'|^2 &\leq C\sum_{n=1}^{\infty}[n^{-2/p}\mathbb{E}|X_n|^2 1_{\{|X_n|\leq n^{1/p}\}} + \mathbb{P}(|X_n| > n^{1/p})] \\ &\leq C\sum_{n=1}^{\infty}[n^{-2/p}\mathbb{E}|X|^2 1_{\{|X|\leq n^{1/p}\}} + \mathbb{P}(|X| > n^{1/p})] \\ &\leq C\sum_{n=1}^{\infty} n^{-2/p}\sum_{k=1}^{n}\mathbb{E}|X|^2 1_{\{(k-1)^{1/p}<|X|\leq k^{1/p}\}} + C \\ &\leq C\sum_{k=1}^{\infty} k^{1-2/p}\mathbb{E}|X|^2 1_{\{(k-1)^{1/p}<|X|\leq k^{1/p}\}} + C < \infty.\end{aligned}$$

Hence the claims (3.4), (3.5) and (3.6) hold.

Next we prove the theorem. Consider first the case $0 < p < 1$. By the estimate (3.6), we have

$$\sum_{n=1}^{\infty} n^{-2/p}\mathbb{E}[X'_n - \mathbb{E}(X'_n)]^2 \leq \sum_{n=1}^{\infty} n^{-2/p}\mathbb{E}[X'_n]^2 < \infty$$

Then from Lemma 3.2 and Kronecker's lemma, it follows that $n^{-1/p}S'_n \to 0$, a.s. Furthermore, by (3.4), we have

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq X'_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n^{1/p}) < \infty$$

which implies $n^{-1/p}\sum_{k=1}^{n} X''_k \to 0$ a.s. From (3.5), we know $n^{-1/p}\sum_{k=1}^{n} \mathbb{E}X'_k \to 0$. So for the case $0 < p < 1$, we have $n^{-1/p}S_n \to 0$ a.s.

Now consider the case $1 < p < 2$. As before, we have $n^{-1/p}S'_n \to 0$, a.s. and $n^{-1/p}\sum_{k=1}^{n} X''_k \to 0$ a.s. From (3.5), it follows that $n^{-1/p}\sum_{k=1}^{n} \mathbb{E}X''_k \to 0$. So the proof for this case is completed.

Now consider the case $p = 1$. As before, we have $n^{-1/p}S'_n \to 0$, a.s. and $n^{-1/p}\sum_{k=1}^{n} X''_k \to 0$ a.s. From (3.5), we obtain $n^{-1}\sum_{k=1}^{n} \mathbb{E}X'_k \to 0$.

From the above discussions, the proof of this theorem is completed. □

3.4 *Proof of Theorem* 2.3

From Lemmas 3.3 and 3.4, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i F^k(X_i) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_{i:n} F^k(X_{i:n}) = \mathbb{E}[X_1 F^k(X_1)] \text{ a.s.} \tag{3.7}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |X_i| = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |X_{i,n}| = \mathbb{E}|X_1|, \quad \text{a.s.} \tag{3.8}$$

Here we used the decomposition $|X| = X^+ + X^-$. Since F is absolutely continuous, then

$$\frac{\mathrm{d}}{\mathrm{d}x}F^{k+1}(x) = (k+1)F^k(x)\frac{\mathrm{d}}{\mathrm{d}x}F(x) \quad \text{a.s.}$$

So we have

$$\mathbb{E}[X_1 F^k(X_1)] = \frac{1}{k+1}\mathbb{E}Y \tag{3.9}$$

where the distribution function of Y is

$$F_Y(x) = F^{k+1}(x), \quad x \in \mathbb{R}.$$

Let

$$\Delta_n := \frac{1}{n}\sum_{i=1}^{n} \frac{i^k}{n^k} X_{i:n} - \frac{1}{n}\sum_{i=1}^{n} X_{i:n} F^k(X_{i:n}).$$

By (3.7) and (3.9), it is enough to show $\Delta_n \to 0$ a.s. Since

$$\begin{aligned} |\Delta_n| &\le \frac{1}{n}\sum_{i=1}^{n}\left|\frac{i^k}{n^k} - F^k(X_{i:n})\right| |X_{i:n}| \\ &= \frac{1}{n}\sum_{i=1}^{n}\left|F_n^k(X_{i:n}) - F^k(X_{i:n})\right| |X_{i:n}| \ \text{ a.s.,} \end{aligned}$$

then from Lemma 3.6 and eq. (3.8), we have $\Delta_n \to 0$ a.s.

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