Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 3, August 2014, pp. 437–445.

Periodic diffeomorphisms on homotopy $E(4)$ surfaces

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MS received 16 February 2013

Abstract. In this paper, we study the periodic diffeomorphisms on homotopy $E(4)$ surfaces. Under some conditions, we prove the non-existence of periodic diffeomorphisms of odd prime order that act trivially on the cohomology of elliptic surfaces $E(4)$. Besides, we give an application of our main theorem.

Keywords. Homotopy $E(4)$ surfaces; periodic diffeomorphisms; Seiberg–Witten invariants; Adams ψ operation.

2000 Mathematics Subject Classification Primary: 57R55; Secondary: 57S17

1. Introduction

In [9], there is an open problem proposed by Allan Edmonds:

Question 1.1. Do $K3$ surfaces admit a periodic diffeomorphism of prime order acting trivially on cohomology (or homology)?

This question was motivated by the corresponding rigidity for holomorphic actions [1]. It was shown to be true for topological ones [4] and symplectic ones [3]. In smooth case, the homological rigidity was known to be false for involutions [13] and odd prime order periodic diffeomorphism [8].

In this paper, we study Question 1.1 in the case of homotopy elliptic surfaces $E(4)$ and get the following result.

Theorem 1.2. *Let X be a homotopy $E(4)$ surface and admit a periodic diffeomorphism τ of odd prime order p. Suppose the spin number* $\mathrm{Spin}(\hat{\tau}, X)$ *is both rational and negative and $k_0 > -4$, then τ can not act trivially on the self-dual part $H^2_+(X; R)$ of the second cohomology group.*

We organize this paper as follows. In §2, we give some preliminaries about our study. In §3, we show our main results and in §4, we give an application of our main theorem.

2. Preliminaries

In this section, we introduce some preliminaries about the Seiberg–Witten theory, the spin number for a periodic diffeomorphism of odd prime order p and some results about Adams ψ-operation.

2.1 *Seiberg–Witten theory*

We briefly review the new interpretation of the Seiberg–Witten invariants. Note that this part largely depends on the papers of Furuta [6,7], Fang [5] and Kim [8].

Let X be a smooth, closed, 4-manifold such that $b_1(X) = 0$, $b_2^+(X) \geq 2$. Let $\mathcal{C}$ be a Spin^c structure on X, $S^\pm$ denote the positive and negative complex spinor bundle and $L = \det S^+$ denote the determinant line bundle on X. Then Seiberg–Witten monopole equations are as follows:

$$D_A\phi = 0, \qquad F_A^+ = q(\phi),$$

where $D : \Gamma(S^+) \to \Gamma(S^-)$ is the Dirac operator, A is the space of unitary connections on L, $\phi \in \Gamma(S^+)$ is a section, F_A^+ is the self dual part of the curvature F_A and $q(\phi) = \phi \otimes \phi^* - \frac{1}{2}|\phi|^2\mathrm{Id}$.

From the above equations, we get a $\mathcal{G}_L$-equivariant map

$$\begin{aligned} \mathcal{F}: \ & \mathcal{A} \times \Gamma(S^+) \to i\Omega^+ \times \Gamma(S^-) \\ & (A, \phi) \mapsto (F_A^+ - q(\phi), D_A\phi). \end{aligned}$$

Then the moduli space $\mathcal{M}_c = \mathcal{F}^{-1}(0)/\mathcal{G}_L$, where $\mathcal{G}_L$ denotes the gauge group. For convenience, we suppose $\Gamma(S^\pm)$ and $\mathcal{A}$ are all completed with suitable Sobolev norm. And we fix $A_0 \in \mathcal{A}$. Note that $\mathcal{A} = \mathcal{A}_0 + i\Omega^1$ and S^1 is the stablizer of the gauge group action on A_0. Then there is a S^1-equivariant map

$$\begin{aligned} \mathcal{F}_0: \quad & i\Omega_c^1 \times \Gamma(S^+) \to i\Omega^+ \times \Gamma(S^-) \\ & (ia, \phi) \mapsto (d^+(ia) - q(\phi), D_{A_0}\phi + ia \cdot \phi). \end{aligned}$$

Thus the moduli space $\mathcal{M}_c$ is $\mathcal{F}_0^{-1}(\mu_0)/S^1$, where $\mu_0 = -F_{A_0}^+ \in i\Omega^+$.

Now we construct Furuta's finite dimensional approximation. Let $U = \Gamma(S^+)$ and $U' = \Gamma(S^-)$. For each $0 < \lambda \in \mathbf{R}$, U_λ (resp. U'_λ) denotes the vector space spanned by the eigenvectors of the operator $D_{A_0}^* D_{A_0}$ (resp. $D_{A_0} D_{A_0}^*$) with eigenvalues less than or equal to λ. Similarly, we define V_λ and V'_λ by using the operators $d_+^* d_+$ and $d_+ d_+^*$. Note that U_λ, U'_λ, V_λ and V'_λ are all finite dimensional spaces.

Let $p_\lambda : U' \times V' \to U'_\lambda \oplus V'_\lambda$ denote the orthogonal projection. Then from p_λ and the restriction of $\mathcal{F}_0$ onto $U_\lambda \oplus V_\lambda$, we obtain a map

$$\mathcal{F}_\lambda : U_\lambda \oplus V_\lambda \to U'_\lambda \oplus V'_\lambda.$$

Note that $\mathcal{F}_\lambda$ is a S^1-equivariant map, where S^1 acts on Ω^i trivially and on $\Gamma(S^\pm)$ by the complex multiplication.

Using compactness of the moduli space, Furuta showed the following result:

Lemma 2.1 [6]. *Let μ_0 be defined as above and $\nu \in i\Omega^+$. For a sufficiently large $R \in (0, +\infty)$, there exists a real number $\Lambda \in (0, +\infty)$ such that for $\lambda \geq \Lambda$, $\mathcal{F}_\lambda^{-1}(\mu_0 + \nu)$ does not intersect with the sphere of radius R in $U_\lambda \oplus V_\lambda$.*

Let $W_\lambda = U'_\lambda \oplus V'_\lambda$ and $\nu_0 = \mu_0 + \nu$. Then we have the following S^1-equivariant map:

$$\mathcal{F}_\lambda : (B_\lambda, \partial B_\lambda \cup (0 \times V_\lambda) \cap B_\lambda) \to (W_\lambda, W_\lambda - \nu_0),$$

where B_λ is the ball of radius R at the origin in $U_\lambda \oplus V_\lambda$. Note that the quotient of $\mathcal{F}_\lambda$ gives the following S^1-equivariant map:

$$f : S^{V_\lambda \oplus R} \wedge S(U_\lambda) \to S^{W_\lambda},$$

where $S^{V_\lambda \oplus R}$ and $S(U_\lambda)$ denote the Thom spaces.

Let $\Phi \in H^*_{S^1}(S^{W_\lambda}, \mathrm{Z})$ denote the equivariant Thom class. From Thom isomorphism theorem, there exists a $\theta \in H^{2(m-1-d)}_{S^1}(S(U_\lambda)) \approx H^{2(m-1-d)}(\mathbf{C}P^{m-1})$ such that $f^*(\Phi) = \sigma(\theta)$. In the above two equations, $m = \dim_{\mathbf{C}} U_\lambda$, $2d = \dim \mathcal{M}_c = \frac{1}{4}(c_1(L)^2 - (2\chi(X) - 3\sigma(X)))$ and σ is the suspension isomorphism. Note that $H^{2(m-1-d)}(\mathbf{C}P^{m-1}) \cong \mathbf{Z}$ with a generator x^{m-1-d}, where $x \in H^2(\mathbf{C}P^{m-1-d})$ is a generator. Thus we can regard θ as an integer given by the coefficient of x^{m-1-d}.

In [8], Kim summarized the results of Furuta and Fang as follows.

Theorem 2.2 [5, 6]. *Let X be a smooth 4-manifold with $b_1(X) = 0$ and $b_2^+(X) \geq 2$. Let c denote spin^c structure on X.*
(1) *For a sufficiently large $\lambda \geq \Lambda$, the Seiberg–Witten invariants $SW(X, c) = \theta$. Furthermore, if X has an action $\mathbf{Z}_p$ preserving the spin^c structure c and $H^2_+(X/\mathbf{Z}_p, \mathbf{R}) \neq 0$ then there exists an $S^1 \times \hat{\mathbf{Z}}_p$-equivariant map*

$$f : S^{V_\lambda \oplus R} \wedge S(U_\lambda) \to S^{W_\lambda}$$

and $SW(X, c) = \theta$.
(2) *Let t be the standard 1-dimensional complex representation of S^1, and let $T = 1 - t$. Then $\beta = \beta_f$ satisfies the following identity*:

$$\beta_f(t) = (-1)^n \theta \left(\frac{log(1+T)}{T} \right)^l T^{m-1-d}, T^{m-d} = 0,$$

where $n = \dim_{\mathbf{C}} U'_\lambda$ and $l = \frac{b_+(X)-1}{2}$. In particular, we have $\beta = \pm SW(X, c) T^{m-1-d}$.

2.2 *Spin number*

Let X be a homotopy $E(4)$ surface and τ be a periodic diffeomorphism of odd prime order p. Then for the lifting $\hat{\tau}$, the spin number is defined to be

$$\mathrm{Spin}(\hat{\tau}, X) = \mathrm{ind}_{\hat{\tau}} D = \mathrm{tr}(\hat{\tau}|_{\ker D}) - \mathrm{tr}(\hat{\tau}|_{\mathrm{coker} D}).$$

In our study, we also use the following results about spin number. For details, we can refer to [10].

Lemma 2.3. *Let X be a homotopy $E(4)$ surface which admits a periodic diffeomorphism τ of odd prime order p and satisfies $b_2^+(X/\tau) = 7$. Then $k_0 \leq 6$.*

Lemma 2.4. *Let X be a homotopy $E(4)$ surface and admit a periodic diffeomorphism τ of odd prime order p satisfying $b_2^+(X/\tau) = 7$. Suppose that the spin number* Spin$(\hat{\tau}, X)$ *is both rational and negative. Then we have*

$$k_1 = k_2 = \cdots = k_{p-1} = \frac{4 - k_0}{p-1} \geq 1 \quad and \quad k_0 < \frac{4}{p}.$$

2.3 *Adams ψ-operation*

In this section, we will recall some results about Adams ψ-operation. For details, we can refer to [5].

Let $\beta \in K_{S^1\times Z_3}(S(U_\lambda))$ be the K-theoretic degree. Applying the Adams ψ-operation to β, Fang [5] proved that

$$\psi^q(\beta) = q^l\beta(1+t+\cdots+t^{q-1})^{n_0}(1+t\xi+\cdots+t^{q-1}\xi^{q-1})^{n_1}$$
$$(1+t\xi^2+\cdots+t^{q-1}\xi^{2(q-1)})^{n_2}. \tag{2.1}$$

Note that t is the standard 1-dimensional complex representation of S^1 and ξ is a one-dimensional representation such that $\xi^p = 1$. Besides, $\psi^2(\xi) = \xi^2$ and $\psi^2(t) = t^2$.

We also need the following lemma in [5].

Lemma 2.5.

$$K_{S^1\times Z_3}(S(U_\lambda)) \cong \frac{R(S^1)\otimes R(Z_3)}{(1-t)^{m_0}(1-t\xi)^{m_1}(1-t\xi^2)^{m_2}},$$

where $R(S^1)\otimes R(Z_3)$ denotes the representation ring of $S^1 \otimes Z_3$.

3. Main results

We can obtain the following proposition which plays a key role in the proof of our main result.

PROPOSITION 3.1

Let X be a homotopy $E(4)$ surface that admits a periodic diffeomorphism τ of odd prime order p satisfying $b_1(X) = 0$, $b_+(X) \geq 2$ and $b_+(X/\tau) = 7$. Suppose $k_0 > \frac{-4}{p-2}$ and the spin number is rational and negative. Then we have

(1) $\beta_f = a(1+\xi+\xi^2+\cdots+\xi^{p-1})(1-t)^{m_0}(1-t\xi)^{m_1}\cdots(1-t\xi^{p-1})^{m_{p-1}-1}, \quad a \in Z$
(2) *The Seiberg–Witten invariant for the trivial spin^c structure vanishes identically.*

Proof.

(1) For simplicity, we only prove the case of $p = 3$. For other cases, the proof is completely similar to this one.

To prove the first part of Proposition 3.1, we need the following lemmas. □

Lemma 3.2. *There is a class $\beta^{(1)} \in R(S^1)\otimes R(Z_3)/(1-t\xi)^{m_1}(1-t\xi^2)^{m_2}$ such that $\beta_f = \beta^{(1)}(1-t)^{m_0}$.*

Proof. Let $\beta_f = \Sigma_i(\Sigma_{j=0}^2 a_j^i\xi^j)T^i$, where $T = 1-t$. Applying operation (2.1) for $q = 2$ and noticing that $\beta_f = \beta(\xi, t)$, $\psi^2(\beta) = \beta(\xi^2, t^2)$, we have

$$\begin{aligned}\psi^2(\beta) &= \Sigma_i(\Sigma_{j=0}^2 a_j^i\xi^{2j})(2T-T^2)^i \\ &= 2^l\Sigma_i(\Sigma_{j=0}^2 a_j^i\xi^j)T^i(1+t)^{n_0}(1+t\xi)^{n_1}(1+t\xi^2)^{n_2}.\end{aligned}$$

Substituting $t = 1-T$ to the above equation and comparing the coefficients of T^i, we get

$$2^i(\Sigma_{j=0}^2 a_j^i\xi^{2j}) = 2^{l+n_0}(\Sigma_{j=0}^2 a_j^i\xi^j)(1+\xi)^{n_1}(1+\xi^2)^{n_2}. \tag{3.1}$$

Replacing ξ with 1, we have

$$2^i(\Sigma_{j=0}^2 a_j^i) = 2^{l+n}(\Sigma_{j=0}^2 a_j^i).$$

Thus if $i < l+n$, then $\Sigma_{j=0}^2 a_j^i = 0$.

Next, substituting ν and ν^2 for ξ in (3.1) and using the relation $(1+\nu)(1+\nu^2) = 1$, we have

$$2^{2i}\Pi_{k=1}^2(\Sigma_{j=0}^2 a_j^i \nu^{2kj}) = 2^{2(l+n_0)}\Pi_{k=1}^2(\Sigma_{j=0}^2 a_j^i \nu^{kj}).$$

Note that $\Pi_{k=1}^2(\Sigma_{j=0}^2 a_j^i \nu^{2kj}) = \Pi_{k=1}^2(\Sigma_{j=0}^2 a_j^i \nu^{kj})$. If $i < l+n_0 \leq l+n$ then there must be some $k=1$ or $k=2$ satisfying $\Sigma_{j=0}^2 a_j^i \nu^{kj} = 0$. Thus $a_0^i = a_1^i = a_2^i$. Then we have $a_j^i = 0,\ (j = 0, 1, 2)$ for $\Sigma_{j=0}^2 a_j^i = 0$.

Since the virtual dimension of the Seiberg–Witten moduli space for the trivial spin^c structure is zero, we have $l+n = m-1$. Thus

$$l + n_0 = l + n - n_1 - n_2 = m - 1 - n_1 - n_2 = k_1 + k_2 + m_0 - 1. \quad (3.2)$$

By the assumption of the spin number for $E(4)$, we have $k_1 + k_2 \geq 2$. Then we have $l + n_0 > m_0 - 1$. Hence $a_j^i = 0$ for $j = 0, 1, 2,\ i \leq m_0 - 1$. In this case, the image of β_f in $\frac{R(S^1)\otimes R(\mathbb{Z}_3)}{(1-t)^{m_0}}$ is zero. From Lemma 2.5, there exists a $\beta^{(1)} \in \frac{R(S^1)\otimes R(\mathbb{Z}_3)}{(1-t\xi)^{m_1}(1-t\xi^2)^{m_2}}$. □

Lemma 3.3. *There is a class $\beta^{(2)} \in R(S^1)\otimes R(\mathbb{Z}_3)/(1-t\xi^2)^{m_2}$ such that $\beta^{(1)} = \beta^{(2)}(1-t\xi)^{m-1}$.*

Proof. Let $\beta^{(1)} = \Sigma_i(\Sigma_{j=0}^2 b_j^i \xi^j)S^i$, where $S = 1 - t\xi$. Using Adams operation for $q = 2$, we have

$$\begin{aligned}\Sigma_i(\Sigma_{j=0}^2 b_j^i \xi^{2j})(1-t^2\xi^2)^i = \psi^2(\beta^{(1)}) \;&=\; 2^l\,\Sigma_i(\Sigma_{j=0}^2 b_j^i \xi^j)S^i(1+t)^{n_0}\\ &\quad\times(1+t\xi)^{n_1}(1+t\xi^2)^{n_2}.\end{aligned}$$

Substituting $t = \xi^2(1-S)$ to the above equation and comparing the coefficients of S^i, we get

$$2^i(\Sigma_{j=0}^2 b_j^i \xi^{2j}) = 2^{l+n_1}(\Sigma_{j=0}^2 b_j^i \xi^j)(1+\xi^2)^{n_0}(1+\xi)^{n_2}. \quad (3.3)$$

Replacing ξ with 1, we have

$$2^i(\Sigma_{j=0}^2 b_j^i) = 2^{l+n}(\Sigma_{j=0}^2 b_j^i).$$

If $i < l+n$ then $\Sigma_{j=0}^2 b_j^i = 0$. Using the same methods as in Lemma 3.2, we get

$$2^{2i}\Pi_{k=1}^2(\Sigma_{j=0}^2 b_j^i \nu^{2kj}) = 2^{2(l+n_1)}\Pi_{k=1}^2(\Sigma_{j=0}^2 b_j^i \nu^{kj}).$$

If $i < l+n_1$, then there must be some $k=1$ or $k=2$ such that $\Sigma_{j=0}^2 b_j^i \nu^{kj} = 0$ and

$$l + n \geq l + n_0 = m - 1 - n_0 - n_2 = k_0 + k_2 + m_1 - 1. \quad (3.4)$$

Under the assumption of $\frac{4}{3} > k_0 > -4$ for $p = 3$, we have $k_0 + k_2 > 0$. Thus $b_j^i = 0$ for $j = 0, 1, 2$ and $i \leq m_1 - 1$. Then the image of $\beta^{(1)}$ in $\frac{R(S^1)\otimes R(\mathbb{Z}_3)}{(1-t\xi)^{m_1}}$ is zero. Applying

Lemma 2.5 again, we obtain that there must be some $\beta^{(2)}$ in $\frac{R(S^1)\otimes R(Z_3)}{(1-t\xi^2)^{m_2}}$ such that $\beta^{(1)} = \beta^{(2)}(1-t\xi)^{m_1}$. Then the lemma is proved. □

Let $\beta^{(2)} = \Sigma_i(\Sigma_{j=0}^2 c_j^i \xi^j)Y^i \in R(S^1)\otimes R(Z_3)/(1-t\xi^2)^{m_2}$, where $Y = 1-t\xi^2$. Next, we repeat the above procedure for $\beta_f = \beta^{(2)}(1-t)^{m_0}(1-t\xi)^{m_1}$.

Applying the Adams operation for $q = 2$ to β, we have

$$\begin{aligned}&\Sigma_i(\Sigma_{j=0}^2 c_j^i \xi^{2j})(1-t^2\xi)^i(1-t^2)^{m_0}(1-t^2\xi^2)^{m_1} = \psi^2(\beta)\\ &\quad = 2^l\Sigma_i(\Sigma_{j=0}^2 c_j^i\xi^j)Y^i(1-t)^{m_0}(1-t\xi)^{m_1}(1+t)^{n_0}(1+t\xi)^{n_1}(1+t\xi^2)^{n_2}.\end{aligned}$$

Putting $Y = 1-t\xi^2$ in the above equation, we have

$$\begin{aligned}&\Sigma_i(\Sigma_{j=0}^2 c_j^i\xi^{2j})(2Y-Y^2)^i(1+\xi-\xi Y)^{k_0}(1+\xi^2-\xi^2Y)^{k_1}\\ &\quad = 2^l\Sigma_i(\Sigma_{j=0}^2 c_j^i\xi^j)Y^i(2-Y)^{n^2}.\end{aligned} \tag{3.5}$$

Comparing the coefficients of Y^i in (3.5), we have

$$2^i(\Sigma_{j=0}^2 c_j^i\xi^{2j})(1+\xi)^{k_0}(1+\xi^2)^{k_1} = 2^{l+n_2}(\Sigma_{j=0}^2 c_j^i\xi^j).$$

Replacing ξ with 1, we have

$$2^{i+k_0+k_1}(\Sigma_{j=0}^2 c_j^i) = 2^{l+n_2}(\Sigma_{j=0}^2 c_j^i). \tag{3.6}$$

Thus $\Sigma_{j=0}^2 c_j^i = 0$ for $i < m_2 - 1 = l + n_2 - (k_0+k_1)$.

Besides, substituting ν and ν^2 for ξ in (3.6) and using the same methods as in Lemma 3.2, we get

$$2^{2i}\Pi_{k=1}^2(\Sigma_{j=0}^2 c_j^i\nu^{2kj}) = 2^{2(l+n_2)}\Pi_{k=1}^2(\Sigma_{j=0}^2 c_j^i\nu^{kj}).$$

If $i < l+n_2$ then there must be some $k = 1$ or $k = 2$ such that $\Sigma_{j=0}^2 c_j^i\nu^{kj} = 0$. Since $k_0 > \frac{-4}{p-2}$, we have $k_0+k_1 > 0$. Then $m_2 - 1 < l + n_2$. Thus $c_j^i = 0$ for $j = 0, 1, 2$ and $i \le m_2 - 2$. Hence $\beta^{(2)} = (\Sigma_{j=0}^2 c_j^{m_2-1})Y^{(m_2-1)}$.

Comparing the coefficient of Y^{m_2-1} in (3.5), we have

$$2^{m_2-1}(\Sigma_{j=0}^2 c_j^{m_2-1}\xi^{2j})(1+\xi)^{k_0}(1+\xi^2)^{k_1} = 2^{l+n_2}(\Sigma_{j=0}^2 c_j^{m_2-1}\xi^j).$$

From $(1+\nu)(1+\nu^2) = 1$ and $l+n_2-m_2+1 = k_0+k_1$, we have

$$\Pi_{k=1}^2(\Sigma_{j=0}^2 c_j^{m_2-1}\nu^{2kj}) = 2^{2(k_0+k_1)}\Pi_{k=1}^2(\Sigma_{j=0}^2 c_j^{m_2-1}\nu^{kj}).$$

Since $k_0+k_1 \neq 0$, there must be some k such that $\Sigma_{j=0}^2 c_j^{m_2-1}\nu^{kj} = 0$. Thus $c_0^{m_2-1} = c_1^{m_2-1} = c_2^{m_2-1} = c$ which means

$$\beta_f = c(1+\xi+\xi^2)(1-t)^{m_0}(1-t\xi)^{m_1}(1-t\xi^2)^{m_2-1}, \quad c \in Z.$$

Hence the result (1) in Proposition 3.1 is proved.

(2) Since $\beta_f = c(1+\xi+\xi^2)(1-t)^{m_0}(1-t\xi)^{m_1}(1-t\xi^2)^{m_2-1}$, $c \in Z$, we only need to prove $c = 0$ for trivial spin^c structure.

Applying the Adams operation for $q = 3$ to β_f, we have

$$3c(1-t^3)^{m-1} = \beta_f(1-t)^{n_0}(1-t\xi)^{n_1}(1-t\xi^2)^{n_2}.$$

Substituting 1 for ξ, we have $c(1-t^3)^{m-1} = c(1-t)^{m+n-1}$. If $c \neq 0$, then

$$(1-t^3)^{m-1} = (1-t)^{m+n-1}.$$

Comparing the coefficient of t in the above equation, we have $m-1 = -n$. Thus $l+n = -n$ which means $n = -\frac{l}{2} = -\frac{3}{2}$. This is a contradiction for $n \in Z$. Then part (2) of Proposition 3.1 is proved.

Using Proposition 3.1, we prove the following main theorem.

Theorem 3.4. *Let X be a homotopy $E(4)$ surface and admit a periodic diffeomorphism τ of odd prime order p. If $k_0 > \frac{-4}{p-2}$ and the spin number is rational and negative then τ can not act trivially on $H^2_+(X; \mathbf{R})$.*

Proof. Assume the action of τ on $H^2_+(X; \mathbf{R})$ is trivial, then $b_+(X/\tau) = 7$. By the assumption of this theorem and Proposition 3.1, the Seiberg–Witten invariants for $E(4)$ is zero. This contradicts the result of Morgan and Szabó [12] that the Seiberg–Witten invariants for $E(4)$ is 2. Thus Theorem 3.4 is proved. □

4. Applications of Theorem 3.4

In [11], the locally linear pseudofree Z_3 actions on elliptic surfaces $E(4)$ is classified. Specifically, they get a classification table (table 1), where $\#X^G$ denotes the number of fixed points, m_+ denotes the number of fixed points with local representations (1, 2) or (2, 1), m_- denotes the number of fixed points with local representations (1, 1) or (2, 2).

At first, by using Theorem 3.4, we can prove that the action of type C_1 can not be a diffeomorphism on elliptic surfaces $E(4)$.

Table 1. The classification of actions.

Type	$\#X^G$	m_+	m_-	b_2^G	b_+^G	b_-^G	Sign(X/G)
A_0	3	3	0	16	3	13	−10
A_1	6	0	6	18	3	15	−12
B_0	15	0	15	24	5	19	−14
B_1	12	3	9	22	5	17	−12
B_2	9	6	3	20	5	15	−10
C_0	24	0	24	30	7	23	−16
C_1	21	3	18	28	7	21	−14
C_2	18	6	12	26	7	19	−12
C_3	15	9	6	24	7	17	−10
C_4	12	12	0	22	7	15	−8

On the one hand, from the G-index formula for Dirac operator, we have

$$\begin{aligned} \mathrm{ind}_g D_X &= k_0 + \zeta k_1 + \zeta^2 k_2 = \frac{1}{3}(m_+ - m_-), \\ \mathrm{ind}_{g^2} D_X &= k_0 + \zeta^2 k_1 + \zeta k_2 = \frac{1}{3}(m_+ - m_-), \\ \mathrm{ind}_1 D_X &= k_0 + k_1 + k_2 = 4. \end{aligned}$$

Solving these equations, we obtain

$$\begin{aligned} k_0 &= \frac{1}{9}\{12 + 2(m_+ - m_-)\}, \\ k_1 = k_2 &= \frac{1}{9}\{12 - (m_+ - m_-)\}. \end{aligned}$$

In the case of type C_1, $m_+ = 3$ and $m_- = 18$. Hence, we have $k_0 = -2 > \frac{-4}{3-2} = -4$ and $k_1 = k_2 = 3$. Furthermore,

$$\mathrm{Spin}(\hat{\tau}, X) = k_0 + k_1\nu + k_2\nu^2 = k_0 - k_1 = -5.$$

On the other hand, $b_+^G = 7$ which means the action of τ on $H_+^2(X; \mathbf{R})$ is trivial. Then by Theorem 3.4, we can conclude that the action of type C_1 can not be a diffeomorphism on elliptic surface $E(4)$.

By the same method, we can conclude that the action of type C_2 can not be a diffeomorphism either.

Secondly, we cannot judge whether the action of type B_0 is a diffeomorphism by Theorem 3.4.

In fact, by computation as before we can get $k_0 = -2 > \frac{-4}{3-2} = -4$, $k_1 = k_2 = 3$ and $\mathrm{Spin}(\hat{\tau}, X) = -5$. Besides, $b_+^G = 5$ which means the action of τ on $H_+^2(X; \mathbf{R})$ is nontrivial. Since the classification in [11] is a topological one, we are not sure whether the action of type B_0 is a diffeomorphism.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (No. 11301334).

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