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Geometry of the cotangent bundle with Sasakian metrics and its applications

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Abstract. The main aim of this paper is to study paraholomorpic Sasakian metric and Killing vector field with respect to the Sasakian metric in the cotangent bundle.

Keywords. Sasakian metric; cotangent bundle; vertical and horizontal lift; almost complex structure; Killing vector field.

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1. Introduction

Sasakian metrics on tangent bundle were introduce in 1958 by the Japanese geometer Sasaki [15]. Sasakian metrics (diagonal lifts of metrics) on tangent bundles were also studied in [4,8,16]. In a more general case of tensor bundles of type $(1, q)$, $(0, q)$ and (p, q), Sasakian metrics and their geodesics are considered in [3, 10, 11]. Curvature properties for the Sasakian metric of the tangent bundle are given in [1,4,6,9,16]. The geometry of Sasaki metric is studied for the cotangent bundle in [13, 14, 16]. This paper is concerned with paraholomorphic Sasakian metric and Killing vector field property of Sasakian metrics in the cotangent bundle.

2. Preliminaries

Let M^n be an n-dimensional Riemannian manifold of class C^∞ and with metric g, T^*M^n its cotangent bundle and π the natural projection $T^*M^n \to M^n$. A system of local coordinates (U, x^i), $i = 1, \ldots, n$ in M^n induces on T^*M^n a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$, $\bar{i} := n + i = n + 1, \ldots, 2n$, where $x^{\bar{i}} = p_i$ is the component of covectors p in each cotangent space $T^*_x M^n$, $x \in U$ with respect to the natural coframe $\{\mathrm{d}x^i\}$.

We denote by $\Im^r_s(M^n)(\Im^r_s(T^*M^n))$ the module over $F(M^n)(F(T^*M^n))$ of C^∞ tensor fields of type (r, s), where $F(M^n)(F(T^*M^n))$ is the ring of real-valued C^∞ functions on $M^n(T^*M^n)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i \mathrm{d}x^i$ be the local expressions in $U \subset M^n$ of a vector and a covector (1-form) fields $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$, respectively. Then the complete and horizontal lifts ${}^{\mathrm{C}}X, {}^{\mathrm{H}}X \in \Im_0^1(T^*M^n)$ of $X \in \Im_0^1(M^n)$ and the vertical lift ${}^{V}\omega \in \Im_0^1(T^*M^n)$ of $\omega \in \Im_1^0(M^n)$ are given, respectively by

$$ {}^{\mathrm{C}}X = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}}, \tag{1} $$

$$ {}^{\mathrm{H}}X = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma_{ij}^h X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{2} $$

$$ {}^{\mathrm{V}}\omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}}, \tag{3} $$

with respect to the natural frame $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}} \right\}$, where Γ_{ij}^h are the components of the Levi-Civita connection ∇_g on M^n (see [16] for more details).

For each $x \in M^n$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T_x^* M^n$ by

$$ g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j $$

for all $\omega, \theta \in \Im_1^0(M^n)$.

A Sasakian metric ${}^{\mathrm{S}}g$ is defined on T^*M^n by the following three equations:

$$ {}^{\mathrm{S}}g \left({}^{\mathrm{V}}\omega, {}^{\mathrm{V}}\theta \right) = {}^{\mathrm{V}}(g^{-1}(\omega, \theta)) = g^{-1}(\omega, \theta) \circ \pi, \tag{4} $$

$$ {}^{\mathrm{S}}g \left({}^{\mathrm{V}}\omega, {}^{\mathrm{H}}Y \right) = 0, \tag{5} $$

$$ {}^{\mathrm{S}}g \left({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y \right) = {}^{\mathrm{V}}(g(X, Y)) = g(X, Y) \circ \pi, \tag{6} $$

for any $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$. Since any tensor field of type (0,2) on T^*M^n is completely determined by its action on vector fields of type ${}^{\mathrm{H}}X$ and ${}^{\mathrm{V}}\omega$ (see p. 280 of [16]), it follows that ${}^{\mathrm{S}}g$ is completely determined by its eqs (4), (5) and (6).

3. Levi-Civita connection of ${}^{\mathrm{S}}g$

In $U \subset M^n$, we put

$$ X_{(i)} = \frac{\partial}{\partial x^i}, \quad \theta^{(i)} = \mathrm{d}x^i, i = 1, \ldots, n. $$

Then from (2) and (3) we see that ${}^{\mathrm{H}}X_{(i)}$ and ${}^{\mathrm{V}}\theta^{(i)}$ have respectively local expressions of the form

$$ \tilde{e}_{(i)} = {}^{\mathrm{H}}X_{(i)} = \frac{\partial}{\partial x^i} + \sum_h p_a \Gamma_{hi}^a \frac{\partial}{\partial x^{\bar{h}}}, \tag{7} $$

$$ \tilde{e}_{(\bar{i})} = {}^{\mathrm{V}}\theta^{(i)} = \frac{\partial}{\partial x^{\bar{i}}}. \tag{8} $$

We call the set $\{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(i)}, \tilde{e}_{(\bar{i})}\} = \{{}^{\mathrm{H}}X_{(i)}, {}^{\mathrm{V}}\theta^{(i)}\}$ the frame adapted to Levi-Civita connection ∇_g. The indices $\alpha, \beta, \ldots = 1, \ldots, 2n$ indicate the indices with respect to the adapted frame.

Now from the equations (1), (2), (3), (7) and (8), we see that ${}^{\mathrm{H}}X$ and ${}^{\mathrm{V}}\omega$ have components

$$ {}^{\mathrm{C}}X = X^i \tilde{e}_{(i)} - \sum_i p_h \nabla_i X^h \tilde{e}_{(\bar{i})}, \quad {}^{\mathrm{C}}X = ({}^{\mathrm{C}}X^\alpha) = \begin{pmatrix} X^i \\ -p_h \nabla_i X^h \end{pmatrix}, \tag{9} $$

$$ {}^{\mathrm{H}}X = X^i \tilde{e}_{(i)}, \quad {}^{\mathrm{H}}X = ({}^{\mathrm{H}}X^\alpha) = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{10} $$

$$ {}^{\mathrm{V}}\omega = \sum_i \omega_i \tilde{e}_{(\bar{i})}, \quad {}^{\mathrm{V}}\omega = ({}^{\mathrm{V}}\omega^\alpha) = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}, \tag{11} $$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, where X^i and ω_i are the local components of $X \in \Im^1_0(M^n)$ and $\omega \in \Im^0_1(M^n)$, respectively.

From (4), (5) and (6) we see that

$$ \begin{aligned} {}^{\mathrm{S}}g_{\bar{i}\bar{j}} &= {}^{\mathrm{S}}g(\tilde{e}_{(\bar{i})}, \tilde{e}_{(\bar{j})}) = {}^{\mathrm{V}}(g^{-1}(\mathrm{d}x^i, \mathrm{d}x^j)) = g^{ij}, \\ {}^{\mathrm{S}}g_{\bar{i}j} &= {}^{\mathrm{S}}g(\tilde{e}_{(\bar{i})}, \tilde{e}_{(j)}) = 0, \\ {}^{\mathrm{S}}g_{ij} &= {}^{\mathrm{S}}g(\tilde{e}_{(i)}, \tilde{e}_{(j)}) = {}^{\mathrm{V}}(g(\partial_i, \partial_j)) = g_{ij}, \end{aligned} $$

i.e. ${}^{\mathrm{S}}g$ has components

$$ {}^{\mathrm{S}}g = \begin{pmatrix} g_{ij} & 0 \\ 0 & g^{ij} \end{pmatrix} \tag{12} $$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$.

We now consider local 1-forms $\tilde{\omega}^\alpha$ in $\pi^{-1}(U)$ defined by

$$ \tilde{\omega}^\alpha = \bar{A}^\alpha{}_B \mathrm{d}x^B, $$

where

$$ A^{-1} = (\bar{A}^\alpha{}_B) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{\bar{j}} \\ \bar{A}^{\bar{i}}{}_j & \bar{A}^{\bar{i}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ -p_a \Gamma^a_{ij} & \delta^j_i \end{pmatrix}. \tag{13} $$

The matrix (13) is the inverse of the matrix

$$ A = (A_\beta{}^A) = \begin{pmatrix} A_j{}^i & A_{\bar{j}}{}^i \\ A_j{}^{\bar{i}} & A_{\bar{j}}{}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ p_a \Gamma^a_{ij} & \delta^j_i \end{pmatrix} \tag{14} $$

of the transformation $\tilde{e}_\beta = A_\beta{}^A \partial_A$ (see (7) and (8)). We easily see that the set $\{\tilde{\omega}^\alpha\}$ is the coframe dual to the adapted frame $\{\tilde{e}_{(\beta)}\}$, i.e., $\tilde{\omega}^\alpha(\tilde{e}_{(\beta)}) = \bar{A}^\alpha{}_B A_\beta{}^B = \delta^\alpha_\beta$.

Since the adapted frame $\{\tilde{e}_{(\beta)}\}$ is non-holonomic, we put

$$ [\tilde{e}_\gamma, \tilde{e}_\beta] = \Omega_{\gamma\beta}{}^\alpha \tilde{e}_\alpha $$

from which we have

$$\Omega_{\gamma\beta}{}^{\alpha} = (\tilde{e}_{\gamma} A_{\beta}{}^{A} - \tilde{e}_{\beta} A_{\gamma}{}^{A}) \bar{A}^{\alpha}{}_{A}.$$

According to (7), (8), (13) and (14), the components of non-holonomic object $\Omega_{\gamma\beta}{}^{\alpha}$ are given by

$$\begin{cases} \Omega_{l\bar{j}}{}^{\bar{i}} = -\Omega_{\bar{j}l}{}^{\bar{i}} = -\Gamma^{j}_{li}, \\ \Omega_{lj}{}^{\bar{i}} = p_a R_{lji}{}^{a}, \end{cases} \tag{15}$$

all the others being zero, where $R_{lji}{}^{a}$ is the local component of the curvature tensor R of ∇_g.

Let ${}^{\mathrm{S}}\nabla$ be the Levi-Civita connection determined by the Sasakian metric ${}^{\mathrm{S}}g$. We put

$${}^{\mathrm{S}}\nabla_{\tilde{e}_{\gamma}} \tilde{e}_{\beta} = {}^{\mathrm{S}}\Gamma^{\alpha}_{\gamma\beta} \tilde{e}_{\alpha}.$$

From the equation ${}^{\mathrm{S}}\nabla_X Y - {}^{\mathrm{S}}\nabla_Y X = [X, Y], \forall X, Y \in \Im^1_0(T^*M^n)$ we have

$${}^{\mathrm{S}}\Gamma^{\alpha}_{\gamma\beta} - {}^{\mathrm{S}}\Gamma^{\alpha}_{\beta\gamma} = \Omega_{\gamma\beta}{}^{\alpha}. \tag{16}$$

The equation $({}^{\mathrm{S}}\nabla_X {}^{\mathrm{S}}g)(Y, Z) = 0$ has the form

$$\tilde{e}_{\delta} {}^{\mathrm{S}}g_{\gamma\beta} - {}^{\mathrm{S}}\Gamma^{\varepsilon}_{\delta\gamma} {}^{\mathrm{S}}g_{\varepsilon\beta} - {}^{\mathrm{S}}\Gamma^{\varepsilon}_{\delta\beta} {}^{\mathrm{S}}g_{\gamma\varepsilon} = 0. \tag{17}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$. We have from (15) and (17)

$${}^{\mathrm{S}}\Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2} {}^{\mathrm{S}}g^{\alpha\epsilon} (\tilde{e}_{\gamma} {}^{\mathrm{S}}g_{\varepsilon\beta} + \tilde{e}_{\beta} {}^{\mathrm{S}}g_{\gamma\varepsilon} - \tilde{e}_{\varepsilon} {}^{\mathrm{S}}g_{\gamma\beta}) + \frac{1}{2} (\Omega_{\gamma\beta}{}^{\alpha} + \Omega^{\alpha}_{\cdot\ \gamma\beta} + \Omega^{\alpha}_{\cdot\ \beta\gamma}), \tag{18}$$

where $\Omega^{\alpha}{}_{\gamma\beta} = {}^{\mathrm{S}}g^{\alpha\varepsilon} {}^{\mathrm{S}}g_{\delta\beta} \Omega_{\varepsilon\gamma}{}^{\delta}$ and $({}^{\mathrm{S}}g)^{-1} = ({}^{\mathrm{S}}g^{\alpha\beta}) = \begin{pmatrix} g^{ij} & 0 \\ 0 & g_{ij} \end{pmatrix}$.

Taking account of (7), (8), (12), (15), we obtain from (18),

$$\begin{cases} {}^{\mathrm{S}}\Gamma^{h}_{ji} = \Gamma^{h}_{ji}, & {}^{\mathrm{S}}\Gamma^{h}_{\bar{j}\bar{i}} = {}^{\mathrm{S}}\Gamma^{\bar{h}}_{\bar{j}i} = {}^{\mathrm{S}}\Gamma^{\bar{h}}_{j\bar{i}} = 0, \\ {}^{\mathrm{S}}\Gamma^{h}_{j\bar{i}} = \frac{1}{2} p_m R^{h\ im}_{\ .j.}, & {}^{\mathrm{S}}\Gamma^{h}_{\bar{j}i} = \frac{1}{2} p_m R^{h\ jm}_{\ .i.}, \\ {}^{\mathrm{S}}\Gamma^{\bar{h}}_{ji} = \frac{1}{2} p_m R_{jih}{}^{m}, & {}^{\mathrm{S}}\Gamma^{\bar{h}}_{j\bar{i}} = -\Gamma^{i}_{jh}. \end{cases} \tag{19}$$

Let $\tilde{X}, \tilde{Y} \in \Im^1_0(T^*M^n)$ and $\tilde{X} = \tilde{X}^{\alpha} \tilde{e}_{\alpha}$, $\tilde{Y} = \tilde{Y}^{\beta} \tilde{e}_{\beta}$. Then the covariant derivative ${}^{\mathrm{S}}\nabla_{\tilde{Y}} \tilde{X}$ along $\tilde{Y}$ has components

$${}^{\mathrm{S}}\nabla_{\tilde{Y}} \tilde{X}^{\alpha} = \tilde{Y}^{\gamma} \tilde{e}_{\gamma} \tilde{X}^{\alpha} + {}^{\mathrm{S}}\Gamma^{\alpha}_{\gamma\beta} \tilde{X}^{\beta} \tilde{Y}^{\gamma}, \tag{20}$$

with respect to adapted frame $\{\tilde{e}_{(\alpha)}\}$.

Using (7), (8), (10), (11), (19) and (20) we have as follows.

Theorem 1 [14]. *Let M^n be a Riemannian manifold with metric g and ${}^{\mathrm{S}}\nabla$ be the Levi–Civita connection of the cotangent bundle T^*M^n equipped with the Sasakian metric ${}^{\mathrm{S}}g$. Then ${}^{\mathrm{S}}\nabla$ satisfies*

(i) ${}^{\mathrm{S}}\nabla_{{}^{\mathrm{V}}\omega} {}^{\mathrm{V}}\theta = 0$,

(ii) ${}^{S}\nabla_{{}^{V}\omega}{}^{H}Y = \frac{1}{2}{}^{H}(R(\tilde{p},\tilde{\omega})Y)$,

(iii) ${}^{S}\nabla_{{}^{H}X}{}^{V}\theta = {}^{V}(\nabla_X\theta) + \frac{1}{2}{}^{H}(R(\tilde{p},\tilde{\theta})X)$

(iv) ${}^{S}\nabla_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) + \frac{1}{2}{}^{V}(pR(X,Y))$

for all $X, Y \in \Im^1_0(M^n)$ $\omega, \theta \in \Im^0_1(M^n)$, *where* $\tilde{\omega} = g^{-1} \circ \omega \in \Im^1_0(M^n)$, $\tilde{\omega} = g^{-1} \circ \theta \in \Im^1_0(M^n)$, $\tilde{p} = g^{-1} \circ p \in \Im^1_0(M^n)$.

4. Killing vector fields

A vector field $X \in \Im^1_0(M^n)$ is called Killing vector field (or infinitesimal isometry) if $L_X g = 0$, and X is called the infinitesimal affine transformation if $L_X\nabla_g = 0$. A Killing vector field is necessarily an infinitesimal affine transformation, i.e. we have $L_X\nabla_g = 0$ as a consequence of $L_X g = 0$ [16].

Theorem 2 [2]. *Necessary and sufficient conditions in order that the*

(a) *complete* ${}^{C}X \in \Im^1_0(T^*M^n)$,

(b) *horizontal* ${}^{H}X \in \Im^1_0(T^*M^n)$ *and*

(c) *vertical* ${}^{V}\omega \in \Im^0_1(T^*M^n)$

lifts to T^*M^n *with the metric* ${}^{S}g$, *of a vector field* X *and covector field* ω *in* M^n *be a Killing vector field in* T^*M^n *such that*

(a) X *is a Killing vector field in* M^n,
(b) X *is a Killing vector field with vanishing second covariant derivative in* M^n *and*
(c) ω *is parallel* M^n.

Let X and Y be vector fields in M^n. If X and Y are Killing vector fields in M^n, from the definition of the Killing vector field, we have

$$L_{[X,Y]}g = L_X(L_Y g) - L_Y(L_X g) = 0,$$

i.e. $[X, Y]$ is an infinitesimal isometry in M^n [5].

It is well known that (p. 238, p. 277 of [16])

$$\begin{aligned}
&[{}^{C}X, {}^{H}X] = {}^{H}[X,Y] + {}^{V}(p(L_X\nabla)Y),\\
&[{}^{C}X, {}^{V}\omega] = {}^{V}(L_X\omega),\\
&[{}^{C}X, {}^{C}Y] = {}^{C}[X,Y],
\end{aligned}\tag{21}$$

where $(L_X\nabla)Y = \nabla_Y\nabla X + R(X,Y)$ and $(L_X\nabla)(Y,Z) = L_X(\nabla_Y X) - \nabla_Y(L_X Z) - \nabla_{[X,Y]}Z$.

Using (21), we compute the Lie derivatives of the metric ${}^{S}g$ with respect to ${}^{V}(L_X\omega)$ and ${}^{C}[X,Y]$:

$$\begin{aligned}
&L_{{}^{C}[X,Y]}{}^{S}g = L_{[{}^{C}X,{}^{C}Y]}{}^{S}g = L_{{}^{C}X}(L_{{}^{C}Y}{}^{S}g) - L_{{}^{C}Y}(L_{{}^{C}X}{}^{S}g),\\
&L_{{}^{V}(L_X\omega)}{}^{S}g = L_{[{}^{C}X,{}^{V}\omega]}{}^{S}g = L_{{}^{C}X}(L_{{}^{V}\omega}{}^{S}g) - L_{{}^{V}\omega}(L_{{}^{C}X}{}^{S}g).
\end{aligned}\tag{22}$$

Using Theorem 2 and (22), we get as follows.

Theorem 3. *Sufficient conditions in order that the vertical lift of a covector field* $L_X\omega$ *and complete lift of a vector field* $[X, Y]$ *in* M^n *to* T^*M^n *be a Killing vector with metric* ${}^{\mathrm{S}}g$ *are that* X *and* Y *are Killing vector fields with vanishing second covariant derivative and* ω *is parallel in* M^n.

Let X be infinitesimal affine transformation in M^n $(L_X\nabla) = 0$. Then $[{}^{\mathrm{C}}X, {}^{\mathrm{H}}X] = {}^{\mathrm{H}}[X, Y]$ and

$$L_{{}^{\mathrm{H}}[X,Y]}{}^{\mathrm{S}}g = L_{[{}^{\mathrm{C}}X,{}^{\mathrm{H}}Y]}{}^{\mathrm{S}}g = L_{{}^{\mathrm{C}}X}(L_{{}^{\mathrm{H}}Y}{}^{\mathrm{S}}g) - L_{{}^{\mathrm{H}}Y}(L_{{}^{\mathrm{C}}X}{}^{\mathrm{S}}g).$$

Using Theorem 2 and the last equation, we have as follows.

Theorem 4. *Sufficient conditions in order that the horizontal lift of a vector field* $[X, Y]$ *in* M^n *to* T^*M^n *be a Killing vector with metric* ${}^{\mathrm{S}}g$ *are that* X *and* Y *are a Killing vector fields with vanishing second covariant derivative in* M^n.

5. On almost paracomplex structures of special type in the cotangent bundle

An almost paracomplex manifold is an almost product manifold (M^n, φ), $\varphi^2 = I$, such that the two eigenbundles T^+M^n and T^-M^n associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarly even. Considering the paracomplex structure φ, we obtain the set $\{I, \varphi\}$ on M^n, which is an isomorphic representation algebra of order 2, which is called the algebra of paracomplex (or double) numbers and is denoted by $R(j)$, $j^2 = 1$.

A tensor field $\omega \in \Im_q^0(M^{2n})$ is said to be a pure with respect to the paracomplex structure φ , if

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \omega(X_1, X_2, \ldots, \varphi X_q)$$

for any $X_1, X_2, \ldots, X_q \in \Im_0^1(M^{2n})$.

We define the operator ϕ_φ associated with φ and apply to the pure tensor field ω :

$$\begin{aligned}(\phi_\varphi\omega)(Y, X_1, X_2, \ldots, X_q) = {} & (\varphi Y)(\omega(X_1, X_2, \ldots, X_q)) \\ & -Y(\omega(\varphi X_1, X_2, \ldots, X_q)) \\ & +\omega((L_{X_1}\varphi)Y, X_2, \ldots, X_q) \\ & + \cdots + \omega(X_1, X_2, \ldots, (L_{X_q}\varphi)Y). \qquad (23)\end{aligned}$$

where L_X denotes the Lie derivative with respect to X. We note that $\phi_\varphi\omega \in \Im_{q+1}^0(M^{2n})$. If $\phi_\varphi\omega = 0$, then ω is said to be almost paraholomorphic with respect to the paracomplex algebra $R(j)$ (see [7,9,12,14]).

DEFINITION 1

In a manifold with almost paracomplex structure φ, a vector field X is called an almost paraholomorphic vector field if $L_X\varphi = 0$.

Let $F \in \Im_1^1(M^n)$. We define a tensor field ${}^{\mathrm{D}}F$ of type (1,1) in T^*M^n by

$${}^{\mathrm{D}}F\,{}^{\mathrm{H}}X = {}^{\mathrm{H}}(FX), \qquad {}^{\mathrm{D}}F\,{}^{\mathrm{V}}\omega = -\,{}^{\mathrm{V}}(\omega \circ F) = -\,{}^{\mathrm{V}}(\omega F) \qquad (24)$$

for any $X \in \Im_0^1(M^n)$ and ${}^{\mathrm{V}}\omega \in \Im_1^0(M^n)$. We call ${}^{\mathrm{D}}F$ the diagonal lift of the tensor field F. ${}^{\mathrm{D}}F$ has components

$$ {}^{\mathrm{D}}F = \begin{pmatrix} F_i^h & 0 \\ 0 & -F_h^i \end{pmatrix} \tag{25} $$

with respect to the adapted frame $\tilde{e}_{(\alpha)}$. The diagonal lift ${}^{\mathrm{D}}I$ of identity tensor field I of type (1,1) has the components

$$ {}^{\mathrm{D}}I = \begin{pmatrix} \delta_j^i & 0 \\ 2p_a\Gamma_{ij}^a & -\delta_j^i \end{pmatrix} \tag{26} $$

with respect to the induced coordinates and satisfies ${}^{\mathrm{D}}I^2 = I$ .Thus ${}^{\mathrm{D}}I$ is an almost product structure determining the horizontal distribution and the distribution consisting of the tangent planes to fibres [16].

We put

$$ S(\tilde{X},\tilde{Y}) = {}^{\mathrm{S}}g({}^{\mathrm{D}}I\tilde{X},\tilde{Y}) - {}^{\mathrm{S}}g(\tilde{X},{}^{\mathrm{D}}I\tilde{Y}) $$

If $S(\tilde{X},\tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ${}^{\mathrm{V}}\omega$, ${}^{\mathrm{V}}\theta$ or ${}^{\mathrm{H}}X$, ${}^{\mathrm{H}}Y$, then $S = 0$. By virtue of ${}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega = -{}^{\mathrm{V}}\omega$, ${}^{\mathrm{D}}I\,{}^{\mathrm{H}}X = {}^{\mathrm{H}}X$ and (4), (5) and (6), we have

$$ \begin{aligned} S({}^{\mathrm{V}}\omega,{}^{\mathrm{V}}\theta) &= {}^{\mathrm{S}}g(-{}^{\mathrm{V}}\omega,{}^{\mathrm{V}}\theta) - {}^{\mathrm{S}}g({}^{\mathrm{V}}\omega,-{}^{\mathrm{V}}\theta) = 0,\\ S({}^{\mathrm{V}}\omega,{}^{\mathrm{H}}X) &= {}^{\mathrm{S}}g(-{}^{\mathrm{V}}\omega,{}^{\mathrm{H}}X) - {}^{\mathrm{S}}g({}^{\mathrm{V}}\omega,{}^{\mathrm{H}}X) = 0,\\ S({}^{\mathrm{H}}X,{}^{\mathrm{V}}\theta) &= {}^{\mathrm{S}}g({}^{\mathrm{H}}X,{}^{\mathrm{V}}\theta) - {}^{\mathrm{S}}g({}^{\mathrm{H}}X,-{}^{\mathrm{V}}\theta) = 0,\\ S({}^{\mathrm{H}}X,{}^{\mathrm{H}}Y) &= {}^{\mathrm{S}}g({}^{\mathrm{H}}X,{}^{\mathrm{H}}Y) - {}^{\mathrm{S}}g({}^{\mathrm{H}}X,{}^{\mathrm{H}}Y) = 0, \end{aligned} $$

i.e. ${}^{\mathrm{S}}g$ is pure metric with respect to ${}^{\mathrm{D}}I$.

Theorem 5. *$(T^*M^n, {}^{\mathrm{D}}I, {}^{\mathrm{S}}g)$ is an almost paracomplex Riemannian manifold.*

It is well known that (p. 238, p. 277 of [16])

$$ \begin{aligned} &{}^{\mathrm{V}}\omega\,{}^{\mathrm{V}}f = 0, \qquad {}^{\mathrm{H}}X\,{}^{\mathrm{V}}f = {}^{\mathrm{V}}(Xf),\\ &[{}^{\mathrm{H}}X,{}^{\mathrm{V}}\omega] = {}^{\mathrm{V}}(\nabla_X\omega), \qquad [{}^{\mathrm{V}}\omega,{}^{\mathrm{V}}\theta] = 0,\\ &[{}^{\mathrm{H}}X,{}^{\mathrm{H}}Y] = {}^{\mathrm{H}}[X,Y] + \gamma R(X,Y) = {}^{\mathrm{H}}[X,Y] + {}^{\mathrm{V}}(pR(X,Y)), \end{aligned} \tag{27} $$

$pR(X,Y) = (p_i(R(X,Y)_j^i))$ and

$$ \begin{aligned} &{}^{\mathrm{D}}I\,{}^{\mathrm{V}}(pR(X,Y)) = -{}^{\mathrm{V}}(pR(X,Y)),\\ &{}^{\mathrm{S}}g({}^{\mathrm{V}}(pR(X,Y)),{}^{\mathrm{H}}Z) = 0. \end{aligned} \tag{28} $$

From (23) we have

(i) $$ \begin{aligned} (\phi_{{}^{\mathrm{D}}I}\,{}^{\mathrm{S}}g)({}^{\mathrm{V}}\omega,{}^{\mathrm{V}}\theta,{}^{\mathrm{V}}\xi) &= ({}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega)({}^{\mathrm{S}}g({}^{\mathrm{V}}\theta,{}^{\mathrm{V}}\xi)) - {}^{\mathrm{V}}\omega({}^{\mathrm{S}}g({}^{\mathrm{D}}I\,{}^{\mathrm{V}}\theta,{}^{\mathrm{V}}\xi))\\ &\quad + {}^{\mathrm{S}}g((L_{{}^{\mathrm{V}}\theta}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\omega,{}^{\mathrm{V}}\xi) + {}^{\mathrm{S}}g({}^{\mathrm{V}}\theta,((L_{{}^{\mathrm{V}}\xi}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\omega) = 0, \end{aligned} $$

(ii) $$ \begin{aligned} (\phi_{{}^{\mathrm{D}}I}\,{}^{\mathrm{S}}g)({}^{\mathrm{V}}\omega,{}^{\mathrm{V}}\theta,{}^{\mathrm{H}}X) &= ({}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega)({}^{\mathrm{S}}g({}^{\mathrm{V}}\theta,{}^{\mathrm{H}}X)) - {}^{\mathrm{V}}\omega({}^{\mathrm{S}}g({}^{\mathrm{D}}I\,{}^{\mathrm{V}}\theta,{}^{\mathrm{H}}X))\\ &\quad + {}^{\mathrm{S}}g((L_{{}^{\mathrm{V}}\theta}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\omega,{}^{\mathrm{H}}X) + {}^{\mathrm{S}}g({}^{\mathrm{V}}\theta,((L_{{}^{\mathrm{H}}X}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\omega) = 0, \end{aligned} $$

(iii) $$(\phi_{{}^{\mathrm{D}}I}{}^{\mathrm{S}}g)({}^{\mathrm{V}}\omega, {}^{\mathrm{H}}X, {}^{\mathrm{V}}\xi) = ({}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega)({}^{\mathrm{S}}g({}^{\mathrm{H}}X, {}^{\mathrm{V}}\xi)) - {}^{\mathrm{V}}\omega({}^{\mathrm{S}}g({}^{\mathrm{D}}I\,{}^{\mathrm{H}}X, {}^{\mathrm{V}}\xi)) + {}^{\mathrm{S}}g((L_{{}^{\mathrm{H}}X}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\omega, {}^{\mathrm{V}}\xi) + {}^{\mathrm{S}}g({}^{\mathrm{H}}X, ((L_{{}^{\mathrm{V}}\xi}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\omega) = 0,$$

(iv) $$(\phi_{{}^{\mathrm{D}}I}{}^{\mathrm{S}}g)({}^{\mathrm{H}}X, {}^{\mathrm{V}}\omega, {}^{\mathrm{V}}\theta) = ({}^{\mathrm{D}}I\,{}^{\mathrm{H}}X)({}^{\mathrm{S}}g({}^{\mathrm{V}}\omega, {}^{\mathrm{V}}\theta)) - {}^{\mathrm{H}}X({}^{\mathrm{S}}g({}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega, {}^{\mathrm{V}}\theta)) + {}^{\mathrm{S}}g((L_{{}^{\mathrm{V}}\omega}{}^{\mathrm{D}}I)\,{}^{\mathrm{H}}X, {}^{\mathrm{V}}\theta) + {}^{\mathrm{S}}g({}^{\mathrm{V}}\omega, ((L_{{}^{\mathrm{V}}\theta}{}^{\mathrm{D}}I)\,{}^{\mathrm{H}}X) = 0,$$

(v) $$(\phi_{{}^{\mathrm{D}}I}{}^{\mathrm{S}}g)({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y, {}^{\mathrm{H}}Z) = ({}^{\mathrm{D}}I\,{}^{\mathrm{H}}X)({}^{\mathrm{S}}g({}^{\mathrm{H}}Y, {}^{\mathrm{H}}Z)) - {}^{\mathrm{H}}X({}^{\mathrm{S}}g({}^{\mathrm{D}}I\,{}^{\mathrm{H}}Y, {}^{\mathrm{H}}Z)) + {}^{\mathrm{S}}g((L_{{}^{\mathrm{H}}Y}{}^{\mathrm{D}}I)\,{}^{\mathrm{H}}X, {}^{\mathrm{H}}Z) + {}^{\mathrm{S}}g({}^{\mathrm{H}}Y, ((L_{{}^{\mathrm{H}}Z}{}^{\mathrm{D}}I)\,{}^{\mathrm{H}}X) = 0,$$

(vi) $$(\phi_{{}^{\mathrm{D}}I}{}^{\mathrm{S}}g)({}^{\mathrm{V}}\omega, {}^{\mathrm{H}}X, {}^{\mathrm{H}}Y) = ({}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega)({}^{\mathrm{S}}g({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y)) - {}^{\mathrm{V}}\omega({}^{\mathrm{S}}g({}^{\mathrm{D}}I\,{}^{\mathrm{H}}X, {}^{\mathrm{H}}Y)) + {}^{\mathrm{S}}g((L_{{}^{\mathrm{H}}X}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\omega, {}^{\mathrm{H}}Y) + {}^{\mathrm{S}}g({}^{\mathrm{H}}X, ((L_{{}^{\mathrm{H}}Y}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\omega) = 0,$$

(vii) $$\begin{aligned}(\phi_{{}^{\mathrm{D}}I}{}^{\mathrm{S}}g)({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y, {}^{\mathrm{V}}\omega) &= ({}^{\mathrm{D}}I\,{}^{\mathrm{H}}X)({}^{\mathrm{S}}g({}^{\mathrm{H}}Y, {}^{\mathrm{V}}\omega)) - {}^{\mathrm{H}}X({}^{\mathrm{S}}g({}^{\mathrm{D}}I\,{}^{\mathrm{H}}Y, {}^{\mathrm{V}}\omega)) \\ &\quad + {}^{\mathrm{S}}g((L_{{}^{\mathrm{H}}Y}{}^{\mathrm{D}}I)\,{}^{\mathrm{H}}X, {}^{\mathrm{V}}\omega) + {}^{\mathrm{S}}g({}^{\mathrm{H}}Y, ((L_{{}^{\mathrm{V}}\omega}{}^{\mathrm{D}}I)\,{}^{\mathrm{H}}X) \\ &= 2\,{}^{\mathrm{S}}g({}^{\mathrm{V}}\omega, {}^{\mathrm{V}}(pR(X,Y))) = 2g^{-1}(\omega, (pR(X,Y))),\end{aligned}$$

(viii) $$(\phi_{{}^{\mathrm{D}}I}{}^{\mathrm{S}}g)({}^{\mathrm{H}}X, {}^{\mathrm{V}}\omega, {}^{\mathrm{H}}Y) = 2\,{}^{\mathrm{S}}g({}^{\mathrm{V}}\omega, {}^{\mathrm{V}}(pR(X,Y))) = 2g^{-1}(\omega, (pR(X,Y))),$$

is analogous to (vii).

Therefore we have as follows.

Theorem 6. *The almost paracomplex Riemannian manifold $(T^*M^n, {}^{\mathrm{D}}I, {}^{\mathrm{S}}g)$ is paraholomorphic if and only if M^n is flat.*

We now consider the Lie derivative of ${}^{\mathrm{D}}I$ with respect to the complete lift ${}^{\mathrm{C}}X$, we obtain

$$\begin{aligned}(L_{{}^{\mathrm{C}}X}{}^{\mathrm{D}}I)\,{}^{\mathrm{V}}\theta &= L_{{}^{\mathrm{C}}X}({}^{\mathrm{D}}I\,{}^{\mathrm{V}}\theta) - {}^{\mathrm{D}}I(L_{{}^{\mathrm{C}}X}{}^{\mathrm{V}}\theta) = -L_{{}^{\mathrm{C}}X}{}^{\mathrm{V}}\theta - {}^{\mathrm{D}}I(L_{{}^{\mathrm{C}}X}{}^{\mathrm{V}}\theta) \\ &= -{}^{\mathrm{V}}(L_X\theta) - {}^{\mathrm{D}}I\,{}^{\mathrm{V}}(L_X\theta) = 0, \\ (L_{{}^{\mathrm{C}}X}{}^{\mathrm{D}}I)\,{}^{\mathrm{H}}Y &= L_{{}^{\mathrm{C}}X}({}^{\mathrm{D}}I\,{}^{\mathrm{H}}Y) - {}^{\mathrm{D}}I(L_{{}^{\mathrm{C}}X}{}^{\mathrm{H}}Y) = L_{{}^{\mathrm{C}}X}{}^{\mathrm{H}}Y - {}^{\mathrm{D}}I(L_{{}^{\mathrm{C}}X}{}^{\mathrm{H}}Y) \\ &= {}^{\mathrm{H}}[X,Y] + {}^{\mathrm{V}}(p(L_X\nabla)Y) - {}^{\mathrm{H}}[X,Y] + {}^{\mathrm{V}}(p(L_X\nabla)Y), \\ &= 2\,{}^{\mathrm{V}}(p(L_X\nabla)Y).\end{aligned}$$

Thus using Definition 1, we have as follows.

Theorem 7. *If X is an infinitesimal affine transformation of a Riemannian manifold M^n, i.e., if $L_X\nabla = 0$, then its complete lift ${}^{\mathrm{C}}X$ to the cotangent bundle T^*M^n is an almost paraholomorphic vector field with respect to the almost paracomplex structure $({}^{\mathrm{D}}I, {}^{\mathrm{S}}g)$.*

For almost paracomplex structure the integrability is equivalent to the vanishing of the Nijenhuis tensor [12],

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y].$$

Then using (27), we have

$$\begin{aligned}N_{{}^{\mathrm{D}}I}({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y) &= [{}^{\mathrm{D}}I\,{}^{\mathrm{H}}X, {}^{\mathrm{D}}I\,{}^{\mathrm{H}}Y] - {}^{\mathrm{D}}I[{}^{\mathrm{D}}I\,{}^{\mathrm{H}}X, {}^{\mathrm{H}}Y] - {}^{\mathrm{D}}I[{}^{\mathrm{H}}X, {}^{\mathrm{D}}I\,{}^{\mathrm{H}}Y] \\ &\quad + {}^{\mathrm{D}}I^2[{}^{\mathrm{H}}X, {}^{\mathrm{H}}Y] \\ &= [{}^{\mathrm{H}}X, {}^{\mathrm{H}}Y] - {}^{\mathrm{D}}I[{}^{\mathrm{H}}X, {}^{\mathrm{H}}Y] - {}^{\mathrm{D}}I[{}^{\mathrm{H}}X, {}^{\mathrm{H}}Y] + [{}^{\mathrm{H}}X, {}^{\mathrm{H}}Y]\end{aligned}$$

$$
\begin{aligned}
&= 2({}^{\mathrm{H}}[X,Y] + {}^{\mathrm{V}}(pR(X,Y))) - 2({}^{\mathrm{H}}[X,Y] - {}^{\mathrm{V}}(pR(X,Y)))\\
&= 4\,{}^{\mathrm{V}}(pR(X,Y)),\\
N_{{}^{\mathrm{D}}I}({}^{\mathrm{H}}X, {}^{\mathrm{V}}\omega) &= [{}^{\mathrm{D}}I\,{}^{\mathrm{H}}X, {}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega] - {}^{\mathrm{D}}I[{}^{\mathrm{D}}I\,{}^{\mathrm{H}}X, {}^{\mathrm{V}}\omega] - {}^{\mathrm{D}}I[{}^{\mathrm{H}}X, {}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega]\\
&\quad + {}^{\mathrm{D}}I^2[{}^{\mathrm{H}}X, {}^{\mathrm{V}}\omega]\\
&= 0,\\
N_{{}^{\mathrm{D}}I}({}^{\mathrm{V}}\omega, {}^{\mathrm{H}}X) &= [{}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega, {}^{\mathrm{D}}I\,{}^{\mathrm{H}}X] - {}^{\mathrm{D}}I[{}^{\mathrm{D}}I\,{}^{\mathrm{V}}\omega, {}^{\mathrm{H}}X] - {}^{\mathrm{D}}I[{}^{\mathrm{V}}\omega, {}^{\mathrm{D}}I\,{}^{\mathrm{H}}X]\\
&\quad + {}^{\mathrm{D}}I^2[{}^{\mathrm{V}}\omega, {}^{\mathrm{H}}X]\\
&= 0,\\
N_{{}^{D}I}({}^{V}\omega, {}^{V}\theta) &= [{}^{D}I\,{}^{V}\omega, {}^{D}I\,{}^{V}\theta] - {}^{D}I[{}^{D}I\,{}^{V}\omega, {}^{V}\theta] - {}^{D}I[{}^{V}\omega, {}^{D}I\,{}^{V}\theta]\\
&\quad + {}^{D}I^2[{}^{V}\omega, {}^{V}\theta]\\
&= 0.
\end{aligned}
$$

$N_{{}^{\mathrm{D}}I}(\tilde{X}, \tilde{Y}) = 0$ if and only if M^n is locally flat. Thus we have as follows.

Theorem 8. *Let (M^n, g) be a Riemannian manifold and T^*M^n be its cotangent bundle equipped with the Sasakian metric ${}^{\mathrm{S}}g$. Then the almost paracomplex structure ${}^{\mathrm{D}}I$ is integrable if and only if M^n is locally flat.*

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