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A new proof of the theorem: Harmonic manifolds with minimal horospheres are flat

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Abstract. In this note we reprove the known theorem: Harmonic manifolds with minimal horospheres are flat. It turns out that our proof is simpler and more direct than the original one. We also reprove the theorem: Ricci flat harmonic manifolds are flat, which is generally affirmed by appealing to Cheeger–Gromov splitting theorem. We also confirm that if a harmonic manifold M has same volume density function as $\mathbb{R}^n$, then M is flat.

Keywords. Harmonic manifold; Busemann function; minimal horospheres.

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1. Introduction

A complete Riemannian manifold (M, g) is called (globally) harmonic if for any $p \in M$ the volume density $\omega_p(q) = \sqrt{\det g_{ij}}(q)$ in normal co-ordinates, centered at any point $p \in M$ is a radial function. The known examples of harmonic manifolds include flat spaces and locally rank one symmetric spaces. In 1944, Lichnerowicz conjectured that any simply connected harmonic manifold is either flat or a rank one symmetric space. For the development of the conjecture, see the references in [6]. In 2002, the author in collaboration with A. Ranjan conjectured Theorem 1.1 stated below.

Theorem 1.1. *Harmonic manifolds with minimal horospheres are flat.*

Towards this, in [6] the following theorem was established.

Theorem 1.2. *Harmonic manifolds with polynomial volume growth are flat.*

In view of Theorem 1.2, it remained to prove that harmonic manifolds with minimal horospheres have polynomial volume growth. Thanks to the result of Nikolayevsky [3], who in 2005 demonstrated the result stated below.

Theorem 1.3. *The volume growth of a harmonic manifold is either polynomial or exponential.*

Using Theorems 1.2 and 1.3 it is easy to conclude the statement of Theorem 1.1.

The original proof of Theorem 1.2 is elegant but needs some complicated reasoning. Now we describe the proof of Theorem 1.2 in short.

In [6] we used the idea of Szabo's proof of the Lichnerowicz's conjecture, in compact case. By a result of P. Li and J. Wang it follows that span$\{b_v^2\}$, where b_v is a Busemann function for v (defined in §2), is a finite dimensional vector space. Then we average ${b_v}^2$ (idea which can be employed only for harmonic manifolds), and obtain a parallel displaced family g_γ, of real valued functions on $\mathbb{R}$, for every geodesic γ. span$\{g_\gamma\}$ is also a finite dimensional vector space. Hence, it follows that the generator function g is a trigonometric polynomial. By using properties of g, it can be written in the simpler form. Then we introduce another family of radial functions μ_γ. The generator function, μ is obtained by generalizing co-ordinate function $r\cos\theta$ on a harmonic manifold. Then we study properties of μ, and relate the two families. We observe that g and μ are almost periodic functions. Finally, using the characteristic property of an almost periodic function we prove that M is Ricci flat.

In view of this, it is natural to ask: Can one affirm that harmonic manifolds with minimal horospheres are Ricci flat, by a simpler method? The purpose of this note is to show that it is possible to assert this directly, by basic approach. In fact, we confirm the theorem that Ricci flat harmonic manifolds are flat, without appealing to the Cheeger–Gromov splitting theorem. We also confirm that harmonic manifolds with same volume density function as that of the corresponding Euclidean space is flat.

The paper is divided into three sections. Section 2 describes preliminaries required to prove the main theorem and in §3, the main theorem is proved.

2. Preliminaries

In this section, we describe important concepts about harmonic manifolds, useful in proving the main theorem. All the fundamentals about harmonic manifolds used in this paper are contained in Besse's book [1].

In the sequel M will denote a complete, non-compact, simply connected globally harmonic manifold. This implies that M is a manifold without conjugate points and thus by a theorem of Cartan–Hadamard the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism. Consequently, every geodesic of M is a line (see p. 170 of [1]).

Global harmonicity of M implies that volume density function in polar coordinates is a radial function. That is $\omega_m = \sqrt{\det(g_{ij})}$, the density in normal neighbourhood of m, is a radial function for all m in M. Correspondingly the volume density function of the geodesic sphere $S(m,r)$ is denoted by Θ_m, where $\Theta_m(n) = r_m^{d-1}(n)\omega_m(n)$ is also a radial function. Hence, $\Theta_m(n) = \Theta_m(r_m(n),\varphi) = \Theta_m(r(m,n)) = \Theta_m(r)$, where $r_m(n) = r(m,n)$ is the geodesic distance between m and n and $(r_m(n),\varphi)$ are the polar co-ordinates of n.

It can be shown that $\Theta_m(r) = \Theta_n(r), \forall m,n \in M$ (refer p. 157 of [1]), i.e. Θ is independent of a point in M. Thus, the function $\Theta : \mathbb{R}^+ \to \mathbb{R}$, associated naturally with a harmonic manifold is an important invariant.

Let SM be the unit tangent bundle of M. For $v \in SM$, let γ_v be the geodesic line with $\gamma_v'(0) = v$. Let ${b_v}^+$ and ${b_v}^-$ be two *Busemann functions* associated to γ_v, respectively towards $+\infty$ and $-\infty$, defined as follows:

$$
{b_v}^+(x) = \lim_{t\to\infty} d(x,\gamma_v(t)) - t,
$$

$$
b_v^-(x) = \lim_{t\to-\infty} d(x,\gamma_v(t)) + t.
$$

The level sets, ${b_v}^{\pm^{-1}}(t)$ are called *horospheres, geodesic spheres of infinite radius*, of M. Refer to [4] for details on Busemann function.

The following important results were proved in [5].

Theorem 2.1. *Let (M, g) be a harmonic manifold with volume density function Θ. Then, $\frac{\Theta'}{\Theta}(r)$, the mean curvature of geodesic sphere of radius r, is a monotonically decreasing positive function. Consequently, $\lim_{r\to\infty} \frac{\Theta'}{\Theta}(r)$ exists and is non negative. We denote this limit by h. It is the mean curvature of horospheres.*

Theorem 2.2. *Let (M, g) be a harmonic manifold with volume density function Θ. Let γ_v be a geodesic line in M. Then, $\Delta {b_v}^+ = \lim_{r\to\infty} \frac{\Theta'}{\Theta}(r) = h$. Consequently, b_v^+ is a real analytic function on M. Similarly, we can show that $\Delta {b_v}^- = h$ and b_v^- is a real analytic function on M. Thus, all the horospheres of M have (non negative) constant mean curvature h.*

Minimal horospheres

Let (M, g) be a harmonic manifold. From Theorem 2.2, any such manifold is naturally partitioned into two classes:

(1) having horospheres of constant positive mean curvature and
(2) having horospheres of zero mean curvature.

In the second case harmonic manifold is called as harmonic manifold with minimal horospheres. And thus, Theorem 1.1 is equivalent to the following statement.

Theorem 2.3. *Let (M, g) be a harmonic manifold. If $\Delta {b_v}^+ = h \equiv 0$ for any $v \in SM$, then M is flat.*

As ${b_v}^\pm$ is a real analytic function on M, in particular ${b_v}^\pm$ is C^2. Hence, we can define $(1, 1)$ tensor fields u^+ and u^- as follows:

For $v \in SM$ and $x \in v^\perp$, let

$$u^+(v)(x) = \nabla_x \nabla b_{-v}^+ \qquad \text{and} \qquad u^-(v)(x) = -\nabla_x \nabla b_v^+.$$

Thus $u^\pm(v) \in \text{End}\,(v^\perp)$. The endomorphism fields $u^\pm$ satisfy the Riccati equation along the orbits of the geodesic flow $\varphi^t : SM \to SM$. Thus if $u^\pm(t) := u^\pm(\varphi^t v)$ and $R(t) := R(\cdot, \gamma_v'(t))\gamma_v'(t) \in \text{End}(\gamma_v'(t)^\perp)$, then

$$(u^\pm)' + (u^\pm)^2 + R = 0.$$

u^+ and u^- are called as unstable and stable Riccati solutions respectively, and $R(t)$ is called the Jacobi operator.

Note that tr $u^+(v) = \Delta {b_{-v}}^+ = h$ and tr $u^-(v) = -\Delta {b_v}^+ = -h$.

The following theorem follows from standard techniques (see e.g. [5]).

Theorem 2.4. *Let (M, g) be a harmonic manifold. Then, the map $v \to u^\pm(v)$ is continuous on SM.*

Infinitesimal harmonicity

A complete Riemannian manifold (M, g) is said to be *infinitesimally harmonic* (p. 161 of [1]) if all the derivatives $\nabla^{(k)}_{\sigma_p \cdots \sigma_p}\omega_p$ with respect to the unit vectors $\sigma_p \in T_pM$ define constant functions on manifolds.

The derivatives $\nabla^{(k)}_{\sigma_p \cdots \sigma_p}\omega_p$ can be expressed in terms of the curvature tensor and its covariant derivatives. For example, we have for $v \in S_pM$,

$$\nabla_v \nabla_v \, \omega_p = -\frac{n}{3}\mathrm{Ricci}(v, v) = \text{constant}, \tag{1}$$

where Ricci (v, v) stands for the Ricci curvature. Equation (1) can be found in [2], where series expansion of the volume function was obtained.

As any globally harmonic manifold is real analytic it follows that global harmonicity and infinitesimal harmonicity are equivalent (p. 161 of [1]). Thus, we obtain Lemma 2.5 stated below.

Lemma 2.5. *Any harmonic manifold is an Einstein manifold.*

3. Proof of Theorem 1.1

In this section we prove our main theorem viz. Theorem 1.1.

Lemma 3.1. *Let (M, g) be a harmonic manifold with minimal horospheres. Let γ_v be a geodesic line, then $b_v^+ + b_v^- = 0$.*

Proof. Let γ_v be a geodesic line. As (M, g) is a harmonic manifold with minimal horospheres, $\Delta b_v^\pm = h = 0$. Also,

$$b_v{}^+(x) + b_v^-(x) = \lim_{t\to\infty} d(x, \gamma_v(t)) + d(x, \gamma_v(-t)) - 2t.$$

Hence $b_v{}^+(x) + b_v^-(x) \geq 0$ for all x, by triangle inequality, and $(b_v{}^+ + b_v^-)(\gamma_v(t)) = 0$, since γ_v is a line. Thus, the minimum principle shows that $b_v^+ + b_v^- = 0$. □

Remark 3.2. The above lemma shows that the stable and unstable horospheres of harmonic manifolds with minimal horospheres coincide like flat spaces.

COROLLARY 3.3

$u^+(v) = u^-(v)$ for all $v \in SM$. Consequently, $u^+(-v) = -u^+(v)$.

Proof. From the definition of $b_v^\pm$, the equation $b_v^+ + b_v^- = 0$ is equivalent to $b_v^+(x) = -b_{-v}^+(x)$. Hence $\nabla_x \nabla b_v^+ = -\nabla_x \nabla b_{-v}^+$. Thus, from the definition of $u^\pm$, we get

$$u^+(v) = u^-(v), \quad \forall v \in SM. \tag{2}$$

From definition of $u^\pm(v)$ we have $u^+(-v) = -u^-(v)$. Hence, eq. (2) shows that

$$u^+(-v) = -u^+(v). \tag{3}$$

□

Lemma 3.4. *For every point $p \in M$, there exists $v \in S_pM$ such that $u^+(v) = 0$. In particular*, $\mathrm{Ricci}(v, v) = 0$.

Proof. By Proposition 2.4, eigenvalues of $u^+(v)$ vary continuously with $v \in S_pM$. Fix $p \in M$. Note that as dimension of M is n, $u^+(v)$ is a $(n-1) \times (n-1)$ traceless symmetric matrix. Hence, there exists a basis of $v^\perp$ such that $u^+(v)$ is represented by a diagonal matrix. Let $\lambda_1^+(v), \lambda_2^+(v), \ldots, \lambda_{n-1}^+(v)$ be eigenvalues of $u^+(v)$ such that

$$\lambda_1^+(v) \leq \lambda_2^+(v) \leq \cdots \leq \lambda_{n-1}^+(v). \tag{4}$$

We may identify the tangent sphere S_pM with the standard $(n-1)$-sphere S^{n-1}. Now consider the continuous map $f : S^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$f(v) = (\lambda_1^+(v), \lambda_2^+(v), \ldots, \lambda_{n-1}^+(v)).$$

Then by Borsuk–Ulam theorem there exists $v \in S^{n-1}$ such that $f(v) = f(-v)$. Therefore,

$$\lambda_i^+(v) = \lambda_i{}^+(-v), \quad \forall i = 1, 2, \ldots, (n-1). \tag{5}$$

Hence

$$u^+(v) = u^+(-v). \tag{6}$$

Therefore, eqs (3) and (6) imply that $u^+(v) = 0$. Now $\mathrm{Ricci}(v, v) = 0$ follows from the Riccati equation. □

COROLLARY 3.5

Harmonic manifolds with minimal horospheres are Ricci flat.

Now we conclude that (M, g) is flat by two different methods.

First proof

COROLLARY 3.6

(M, g) *is flat.*

Proof. Since (M, g) is Ricci flat, taking trace of the Riccati equation, it follows that $\mathrm{tr}\,(u^+)^2 \equiv 0$ as $\mathrm{tr}\, u^+ = 0$. This implies that $u^+ \equiv 0$ and again the Riccati equation shows that the Jacobi operator is identically zero. Hence M is flat. □

Second proof

Lemma 3.7. *Let (M, g) be a Ricci flat harmonic manifold. Then, M has same volume density function as* $\mathbb{R}^n$.

Proof. By eq. (1), $\nabla^2\omega_p \equiv 0$. Hence, for any geodesic γ,

$$\omega_p(\gamma(t)) = \omega_p(\gamma(0)) + \alpha t. \tag{7}$$

Now consider any radial geodesic γ starting from $p \in M$. Then, as ω is a volume density function, $\omega_p(\gamma(0)) = \omega_p(0) = 1$ and $\nabla_v\omega_p(0) = 0$ [2]. Hence, $\omega_p \equiv 1$. But, by definition $\Theta(r) = r^{n-1}\omega(r)$. Thus, $\Theta(r) = r^{n-1}$. Therefore (M, g) has same volume density function as $\mathbb{R}^n$. □

COROLLARY 3.8

(M, g) is flat.

Proof. We will sketch the proof here. Let $D(p, r)$ denote the ball of radius r and S denotes the unit sphere in T_pM. Then

$$V(r) = \mathrm{Vol}((D(p,r)) = \left(\left(\int_0^r \Theta(t)\,\mathrm{d}t\right) \text{Area } S\right). \tag{8}$$

We use the series expansion of the volume function and the function Θ as obtained in [2]. Let $v \in S_pM$ and let $v = \sum_{i=1}^n a_i e_i$, where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of T_pM. Then

$$\Theta(r) = \Theta_p(\exp_p(rv)) = r^{n-1}\left\{1 - \frac{\sum_{i,j=1}^n \rho_{ij}a_i a_j}{6} r^2 + \cdots\right\}$$

and

$$V(r) = \frac{(\pi r^2)^{n/2}}{\frac{n}{2}!}\left\{1 - \frac{\tau}{6(n+2)} r^2 + \frac{1}{360(n+2)(n+4)}(-3||R||^2 + 8||\rho||^2 + 5\tau^2 - 18\Delta\tau)r^4 + \cdots\right\}.$$

In the above series expansion, τ denotes the scalar curvature, ρ denotes the Ricci tensor, ρ_{ij}'s denote the components of the Ricci tensor and R denotes the curvature tensor (see [2] for more details).

We have $\Theta(r) = r^{n-1}$. Hence, by comparing coefficients of Θ with the above series expansion of Θ and eq. (8) with the above series expansion of $V(r)$, it follows that $\tau = 0$, $\rho = 0$ and $3||R||^2 = 8||\rho||^2$. Hence, $||R|| = 0$ implying that $R \equiv 0$ and thus M is flat. □

Thus, the second proof above also proves Theorem 3.9 stated below.

Theorem 3.9. *If a harmonic manifold (M, g) has same volume density function as $\mathbb{R}^n$, then M is flat.*

Remark 3.10. Note that both the proofs above *reprove* the known theorem that *Ricci flat harmonic manifolds are flat*, which is generally affirmed by appealing to the *Cheeger–Gromov splitting theorem*.

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