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On the stability of the L^p-norm of the Riemannian curvature tensor

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Abstract. We consider the Riemannian functional $\mathcal{R}_p(g) = \int_M |R(g)|^p dv_g$ defined on the space of Riemannian metrics with unit volume on a closed smooth manifold M where $R(g)$ and dv_g denote the corresponding Riemannian curvature tensor and volume form and $p \in (0, \infty)$. First we prove that the Riemannian metrics with non-zero constant sectional curvature are strictly stable for $\mathcal{R}_p$ for certain values of p. Then we conclude that they are strict local minimizers for $\mathcal{R}_p$ for those values of p. Finally generalizing this result we prove that product of space forms of same type and dimension are strict local minimizer for $\mathcal{R}_p$ for certain values of p.

Keywords. Riemannian functional; critical point; stability; local minima.

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1. Introduction

Let M be a closed smooth manifold of dimension $n \geq 3$ and $\mathcal{M}$ denote the space of Riemannian metrics on M endowed with the $C^{2,\alpha}$-topology for any $\alpha \in (0, 1)$. In this paper we study the following Riemannian functional:

$$\mathcal{R}_p(g) = \int_M |R(g)|^p dv_g$$

where $R(g)$ and dv_g denote the corresponding Riemannian curvature and volume form. Since the functional is not scale-invariant, we restrict the functional to the subspace $\mathcal{M}_1 \subset \mathcal{M}$ consisting of metrics with unit volume. For $p < \frac{n}{2}$, it was pointed out by Gromov that $\inf_g \mathcal{R}_{p|\mathcal{M}_1} = 0$ [1]. Note that for $p = \frac{n}{2}$ the functional is scale-invariant. In dimension four, the Chern–Gauss–Bonnet theorem implies that Einstein metrics give an absolute minimum $8\pi^2\chi(M)$ for the functional $\mathcal{R}_2$, where $\chi(M)$ denotes the Euler characteristic of M. In [2], Anderson conjectured that if M be a closed hyperbolic 3-manifold then $\inf_g \mathcal{R}_{\frac{3}{2}}$ is realized by the hyperbolic metric. In this paper, we study the local minimizing property of $\mathcal{R}_p$ for $p \geq 2$ at some certain critical metrics.

Before stating our results we recall a canonical decomposition of tangent space of $\mathcal{M}$. From Lemma 4.57 in [3], if M is a compact Riemannian manifold, we have the orthogonal decomposition of the tangent space of $\mathcal{M}$ at g (which is the space $S^2(T^*M)$ of symmetric 2-tensors on M):

$$T_g\mathcal{M} = S^2(T^*M) = (\operatorname{Im}\delta_g^* + C^\infty(M).g) \oplus (\delta_g^{-1}(0) \cap \operatorname{Tr}_g^{-1}(0)). \qquad (1.1)$$

Here $\operatorname{Im}\delta_g^*$ is precisely the tangent space of the orbit of g under the action of the group of diffeomorphisms of M. Since $T_g\mathcal{M}_1 = \{h \in S^2(T^*M) | \int_M tr(h)dv_g = 0\}$, we have a corresponding decomposition

$$T_g\mathcal{M}_1 = (\operatorname{Im}\delta_g^* + C^\infty(M).g) \cap T_g\mathcal{M}_1 \oplus (\delta_g^{-1}(0) \cap \operatorname{Tr}_g^{-1}(0)). \tag{1.2}$$

$\mathcal{M}$ is an open convex subset of $S^2(T^*M)$ equipped with $C^{2,\alpha}$-topology. Since $S^2(T^*M)$ is a vector space we can differentiate $\mathcal{R}_p$ on $\mathcal{M}$ along any vector in $S^2(T^*M)$. $\nabla\mathcal{R}_p(g)$ in $S^2(T^*M)$ is called the *gradient* of $\mathcal{R}_p$ at g if for every $h \in S^2(T^*M)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{R}_p(g+th) = \mathcal{R}'_{p|g}.h = \langle \nabla\mathcal{R}_p(g), h\rangle.$$

g is called a *critical point* for $\mathcal{R}_p$ if the component of $\nabla\mathcal{R}_p(g)$ along $T_g\mathcal{M}_1$ is zero. By a standard technique one can prove that every compact irreducible locally symmetric space is a critical point of $\mathcal{R}_p$. Let g be a critical point of $\mathcal{R}_p$. The *Hessian* H of $\mathcal{R}_p$ is a symmetric bilinear map,

$$H : T_g\mathcal{M}_1 \times T_g\mathcal{M}_1 \to \mathbb{R}$$

defined by

$$H(h_1, h_2) = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathcal{R}_p(g(s,t))_{|t=0,s=0},$$

where $g(s,t)$ is a two-parameter family of metrics in $\mathcal{M}_1$ with $g(0,0) = g$ and $\frac{\partial}{\partial t}g(t,0)_{|t=0} = h_1$, $\frac{\partial}{\partial s}g(0,s)_{|s=0} = h_2$.

Let $\mathcal{W}$ denote the orthogonal complement of $\operatorname{Im}\delta_g^*$ in $T_g\mathcal{M}_1$.

DEFINITION 1.1

Let (M, g) be a critical point for $\mathcal{R}_{p|\mathcal{M}_1}$. The metric g is called *infinitesimally rigid* for $\mathcal{R}_p$ if the kernel of the bi-linear form H restricted to $\mathcal{W} \times \mathcal{W}$ is zero.

In [9], Muto proved that (S^n, can) is infinitesimally rigid for $\mathcal{R}_2$. For $p = 2$, the application of the differential Bianchi identity simplifies the expression for the gradient of $\mathcal{R}_2$. So it is easier to study the second variation of $\mathcal{R}_2$ than $\mathcal{R}_p$ for any arbitrary p, at a critical point. However it is not known that $\mathcal{R}_2$ is infinitesimally rigid even for any arbitrary irreducible symmetric space.

DEFINITION 1.2

Let (M, g) be a critical point for $\mathcal{R}_p$. (M, g) is *strictly stable* for $\mathcal{R}_p$ if there is an $\epsilon > 0$ such that for every element h in $\mathcal{W}$,

$$H(h,h) \geq \epsilon\|h\|^2, \tag{1.3}$$

where $\|.\|$ denotes the L^2-norm on $S^2(T^*M)$ defined by g.

For a metric with constant sectional curvature or product of metrics with constant sectional curvature we prove that $\mathcal{R}_p$ is infinitesimally rigid. In fact, we prove that $\mathcal{R}_p$ is *strictly stable* for these metrics.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold with dimension $n \geq 3$. If (M, g) is one of the following then g is strictly stable for $\mathcal{R}_p$ for the indicated values of p:*

(i) *A spherical space form and $p \in [2, \infty)$.*
(ii) *A hyperbolic manifold and $p \in [\frac{n}{2}, \infty)$.*
(iii) *A product of spherical space forms and $p \in [2, n]$.*
(iv) *A product of hyperbolic manifold and $p \in [\frac{n}{2}, n]$.*

Moreover, in all these cases, H is diagonalizable with respect to the decomposition (1.2), *for all $p \in [2, \infty)$.*

The product of a spherical space form and a compact hyperbolic manifold with the same dimension is a critical point of $\mathcal{R}_p$ but we are not able to prove that this is stable for $\mathcal{R}_p$. From the proof of the theorem we observe the following Proposition, which gives some information in the hyperbolic case when $p \leq \frac{n}{2}$.

PROPOSITION 1

Let (M, g) be a compact hyperbolic manifold with the sectional curvature c. If the first positive eigenvalue of the Laplacian λ_1 satisfies the inequality

$$\lambda_1 > \frac{|c|(n-2p)}{n+2p+4},$$

then g is strictly stable for $p \in [2, \frac{n}{2})$.

DEFINITION 1.3

Let (M, g) be a critical metric for $\mathcal{R}_{p|\mathcal{M}_1}$. Then g is called a *strict local minimizer* if there exists a $C^{2,\alpha}$-neighborhood $\mathcal{U}$ of g in $\mathcal{M}_1$, such that for all metrics $\tilde{g} \in \mathcal{U}$,

$$\mathcal{R}_p(\tilde{g}) \geq \mathcal{R}_p(g).$$

The equality holds if and only if $\tilde{g} = \phi^* g$ for some $C^{3,\alpha}$-diffeomorphism $\phi : M \to M$.

Since $\mathcal{M}$ and its sub-manifolds are Fréchet manifolds modeled on $S^2(T^*M)$, the usual inverse function theorem can not be applied. Using the Slicing Lemma 2.10 in [7], we observe that *if (M, g) is a closed Riemannian manifold such that g is strictly stable then it is a strict local minimizer for $\mathcal{R}_p$.* Applying Hölder inequality one can prove that if g is a strict local minimizer for $\mathcal{R}_2$ then it is also a strict local minimizer for $\mathcal{R}_p$ for $p \geq 2$. But it does not imply strict stability of $\mathcal{R}_p$ at g.

Similar results have been proved by Besson *et al.* in [4] for all irreducible locally symmetric spaces of non-compact type for the functional

$$\int_M |s|^{\frac{n}{2}} dv_g,$$

where s denotes the scalar curvature of g.

In §4, we study the second variation of $\mathcal{R}_p$ at metrics with constant curvature and prove parts (i) and (ii) of the theorem using the decomposition (1.2). We first prove that for any $h \in (\delta_g^{-1}(0) \cap \mathrm{Tr}_g^{-1}(0))$, there exists an $\epsilon_0 > 0$ such that $H(h, h) \geq \epsilon_0 \|h\|^2$ for all $p \geq 2$ in this case.

Next, we study the second variation of $\mathcal{R}_p$ along the conformal variations of the metric. A positive lower bound of the Ricci curvature gives a lower bound for the first eigenvalue of the Laplacian for compact manifolds. Using this estimate we prove that for any $f \in C^\infty(M)$, there exists an $\epsilon_1 > 0$ such that

$$H(fg, fg) \geq \epsilon_1 \|fg\|^2 \tag{1.4}$$

for metrics with constant positive sectional curvature for $p \geq 2$. When the sectional curvature is negative (1.4) follows immediately for $p \geq \frac{n}{2}$ from the expression of $H(h, h)$ we obtained in this section. For $p < \frac{n}{2}$, if the first eigenvalue of the Laplacian λ_1 satisfies the inequality $\lambda_1 > \frac{|c|(n-2p)}{n+2p+4}$ (c is the sectional curvature), then H satisfies (1.4).

Finally, proving that H is diagonalizable by the decomposition (1.2) for all $p \geq 2$, we get the desired result.

In §5, we prove parts (iii) and (iv) of the theorem. The main steps of the proof are similar to the proof of (i) and (ii). In §6, we study the local minimization property of $\mathcal{R}_p$.

2. Index of notations and definitions

The following notations and definitions will be used throughout this paper. Let (M, g) be a Riemannian manifold with dimension $n \geq 3$.

R, r, s: (4, 0) Riemannian curvature tensor, Ricci curvature, scalar curvature respectively.

$dv_g, V(g)$: The volume form and the volume of (M, g).

$(\, , \,), | \, . \, |$: The point-wise inner product and norm in the fibres of a various tensor bundle M defined by g.

$\langle \, , \, \rangle, \|.\|$: The global inner-product and norm defined on the space of sections of a tensor bundle on M induced by g.

D, D^* : The Riemannian connection and its formal adjoint.

$S^2(T^*M)$: The sections of symmetric 2-tensor bundle over M.

$d^D : S^2(T^*M) \to \Gamma(T^*M \otimes \Lambda^2 M)$ defined by $d^D\alpha(x, y, z) := (D_y\alpha)(x, z) - (D_z\alpha)(x, y)$, where $\Lambda^2 M$ denotes the space of alternating 2-tensors and $\Gamma(T^*M \otimes \Lambda^2 M)$ denotes the sections of $(T^*M \otimes \Lambda^2 M)$. Its formal adjoint δ^D is defined by, $\delta^D(A)(x, y) = \sum\{D_{e_i}A(x, y, e_i) + D_{e_i}A(y, x, e_i)\}$, where $\{e_i\}$ is an orthonormal basis at a point $x \in M$.

$\check{R}(x, y) := \sum R(x, e_i, e_j, e_k)R(y, e_i, e_j, e_k)$.

Next, consider a one-parameter family of metrics $g(t)$ with $g(0) = g$ and $h := \frac{\partial}{\partial t} g(t)_{|t=0}$.

Define, $\Pi_h(x, y) = \frac{\partial}{\partial t} D_x y_{|t=0}$ and $C_h(x, y, z) := (\Pi_h(x, y), z)$. A simple calculation shows that $C_h(x, y, z) = \frac{1}{2}[D_x h(y, z) + D_y h(x, z) - D_z h(x, y)]$, where x, y, z are fixed vector fields on M. The suffix h will be omitted when there is no ambiguity.

$\bar{R}_h := \frac{\partial}{\partial t} R_{|t=0}$ and $\bar{r}_h(x, y) := \bar{R}_h(x, e_i, y, e_i)$.

$\delta_g : S^2(T^*M) \to \Omega^1(M)$ defined by $\delta_g(h)(x) = -D_{e_i}h(e_i, x)$. Its formal adjoint δ_g^* is defined by $\delta_g^*\omega(x, y) := \frac{1}{2}(D_x y + D_y x)$.

L : A (0, 3)-tensor is defined by

$$\begin{aligned} L_h(w, y, z) : = & \sum [R(y, z, \Pi(e_i, e_i), w) + R(y, z, e_i, \Pi(e_i, w)) \\ & + R(z, e_i, \Pi(y, e_i), w) + R(z, e_i, e_i, \Pi(y, w)) \\ & + R(e_i, y, \Pi(z, e_i), w) + R(e_i, y, e_i, \Pi(z, w))]. \end{aligned}$$

$W_h := (D^*)'(h)(R) - L_h$.

d, δ : The exterior derivative acting on the space of deferential forms and its formal adjoint.

Δ : The Laplace operator acting on $C^\infty(M)$ defined by $\Delta f = \delta df = -tr D df$.

3. Gradient of $\mathcal{R}_p$

In this section, we compute the Euler–Lagrange equation of $\mathcal{R}_p$.

PROPOSITION 2

The functional $\mathcal{R}_p$ is differentiable with the gradient

$$\nabla \mathcal{R}_{p|\mathcal{M}} = -p\delta^D D^* |R|^{p-2} R - p|R|^{p-2} \check{R} + \frac{1}{2} |R|^p g$$

and

$$\nabla \mathcal{R}_{p|\mathcal{M}_1} = -p\delta^D D^* |R|^{p-2} R - p|R|^{p-2} \check{R} + \frac{1}{2} |R|^p g + \left(\frac{p}{n} - \frac{1}{2}\right) \|R\|^p g.$$

Proof.

$$(\mathcal{R}'_p)_g(h) = \int_M \frac{\partial}{\partial t} |R|^p dv_{g|t=0} + \frac{1}{2} \int_M |R|^p tr(h) dv_g,$$

$$(|R|^p)'_g(h) = \frac{\partial}{\partial t} (|R|^2)^{\frac{p}{2}}_{|t=0} = p|R|^{p-2}(R, R'_g.h) - 2p|R|^{p-2}(\check{R}, h).$$

From Proposition 4.70 in [3] we have

$$R'_g.h(x, y, z, t) = D_y C(h)(x, z, t) - D_x C(h)(y, z, t) + R(x, y, z, h^\sharp(t)).$$

Since R is skew-symmetric in the 1st and 2nd entries,

$$(|R|^{p-2} R, R'_g(h)) = -2(|R|^{p-2} R, DC(h)) + (|R|^{p-2} \check{R}, h).$$

Therefore,

$$\begin{aligned} \langle |R|^{p-2} R, R'_g(h) \rangle &= -2\langle |R|^{p-2} R, DC(h) \rangle + \langle |R|^{p-2} \check{R}, h \rangle \\ &= -2\langle D^* |R|^{p-2} R, C(h) \rangle + \langle |R|^{p-2} \check{R}, h \rangle. \end{aligned}$$

The skew-symmetry of $D^*(|R|^{p-2} R)$ in the last two entries yields

$$2\langle D^*(|R|^{p-2} R), C(h) \rangle = \langle D^*(|R|^{p-2} R), d^D(h) \rangle.$$

This implies that

$$\langle |R|^{p-2} R, R'_g.h \rangle = -\langle \delta^D D^* |R|^{p-2} R, h \rangle + \langle |R|^{p-2} \check{R}, h \rangle.$$

Hence

$$\mathcal{R}'_g.h = -p\langle \delta^D D^*|R|^{p-2}R, h\rangle - p\langle |R|^{p-2}\check{R}, h\rangle + \frac{1}{2}\langle |R|^p g, h\rangle.$$

Therefore,

$$\nabla \mathcal{R}_{p|\mathcal{M}} = -p\delta^D D^*|R|^{p-2}R - p|R|^{p-2}\check{R} + \frac{1}{2}|R|^p g.$$

Now,

$$\int_M tr(\nabla \mathcal{R}_p) dv_g = \left(\frac{n}{2} - p\right) \|R\|^p.$$

Therefore,

$$\nabla \mathcal{R}_{p|\mathcal{M}_1} = -p\delta^D D^*|R|^{p-2}R - p|R|^{p-2}\check{R} + \frac{1}{2}|R|^p g + \left(\frac{p}{n} - \frac{1}{2}\right) \|R\|^p g. \tag{3.1}$$

□

By a standard technique one can easily check that every compact isotropy irreducible homogeneous space, and in particular every irreducible symmetric space is a critical point for $\mathcal{R}_p$. Let (M_1, g_1) and (M_2, g_2) be two homogeneous critical points of $\mathcal{R}_p$ with $|R|_{g_1} = |R|_{g_2} \neq 0$. Then $(M_1 \times M_2, g_1 + g_2)$ is a critical metric for $\mathcal{R}_p$ if and only if the dimensions are the same.

4. Second variation at space forms

In this section, we study the second variation of $\mathcal{R}_p$. Let (M, g) be a closed locally symmetric space and $h_1, h_2 \in S^2(T^*M)$. Then

$$\begin{aligned} H(h_1, h_2) &= \langle (\nabla \mathcal{R}_{p|\mathcal{M}_1})'_g(h_1), h_2\rangle \\ &= -p\langle (\delta^D D^*(|R|^{p-2}R))'_g(h_1), h_2\rangle - p\langle (|R|^{p-2})'_g(h_1)\check{R}, h_2\rangle \\ &\quad -p\langle |R|^{p-2}(\check{R})'_g(h_1), h_2\rangle + \frac{1}{2}\langle (|R|^p)'_g(h_1) g, h_2\rangle \\ &\quad +\frac{1}{2}|R|^p\langle h_1, h_2\rangle + \left(\frac{p}{n} - \frac{1}{2}\right)\|R\|^p\langle h_1, h_2\rangle. \end{aligned}$$

Since g is homogeneous and R is parallel,

$$\begin{aligned} (\delta^D D^*(|R|^{p-2}R))'_g(h_1) &= (\delta^D)'_g(h_1) D^*(|R|^{p-2}R) + \delta^D (D^*)'_g(h_1)(|R|^{p-2}R) \\ &\quad + \delta^D D^*((|R|^{p-2})'_g(h_1)R) + \delta^D D^*(|R|^{p-2}R'_g(h_1)) \\ &= |R|^{p-2}(D^*)'_g(h_1)R + |R|^{p-2}\delta^D D^*\bar{R}_{h_1} \\ &\quad +\delta^D D^*((|R|^{p-2})'_g(h_1)R). \end{aligned}$$

Since g satisfies eq. (3.1), $\check{R} = \frac{1}{n}|R|^2 g$. Hence

$$\begin{aligned} H(h_1, h_2) &= -p|R|^{p-2}(\langle \delta^D (D^*)'_g(h_1)R, h_2\rangle + \langle D^*\bar{R}_{h_1}, d^D h_2\rangle) \\ &\quad -p|R|^{p-2}\langle \check{R}'_g(h_1), h_2\rangle - p\langle (|R|^{p-2})'_g(h_1)R, Dd^D h_2\rangle \end{aligned}$$

$$-\frac{p}{n}|R|^2\langle(|R|^{p-2})'_g(h_1)g, h_2\rangle + \frac{1}{2}\langle(|R|^p)'_g(h_1)g, h_2\rangle$$
$$+\frac{p}{n}\|R\|^p\langle h_1, h_2\rangle. \tag{4.1}$$

Next, we assume (M, g) to be a Riemannian manifold with non-zero constant sectional curvature throughout this section. We need the following lemma to prove parts (i) and (ii) of the theorem.

Lemma 4.1. *Let (M, g) be a Riemannian manifold with non-zero constant sectional curvature c. Then*

(i) $(\check{R})'_g.h = 2c^2(n+1)h - 4c^2tr(h)g + 2c[-2\delta^*_g\delta_g h - Ddtr(h) + D^*Dh]$.
(ii) $\delta^D W_h = c(n-2)\delta^D d^D h + 2cDdtr(h) + 2c\Delta tr(h)g$.
(iii) $D^*\bar{R}_h = -d^D\bar{r}_h - L_h$.
(iv) $\bar{r}_h = \frac{1}{2}[2(n-1)ch - 2\delta^*_g\delta_g h - Ddtr(h) + D^*Dh]$.
(v) $\delta^D d^D h = 2D^*Dh - 2\delta^*_g\delta_g h + 2nch - 2ctr(h)g$.
(vi) $(|R|^p)'_g.h = -2pc|R|^{p-2}(2tr\delta^*_g\delta_g h - \Delta tr(h) + (n-1)ctr(h))$.

4.1 *Proof of Lemma* 4.1

Let $\tilde{g}(t)$ be a one-parameter family of Riemannian metrics with $\tilde{g}(0) = g$ and $\tilde{g}'(0) = h$. Choose a normal co-ordinate $\{e_i\}$ with respect to g. Let D be the Riemannian connection corresponding to g.

Proof of (*i*) *and* (*iv*).

$$\check{R}_{pq} = \tilde{g}^{i_1i_2}\tilde{g}^{j_1j_2}\tilde{g}^{k_1k_2}R_{pi_1j_1k_1}R_{qi_2j_2k_2}.$$

Therefore,

$$\begin{aligned}(\check{R}_g.h)'_{pq} = {} & (\tilde{g}^{i_1i_2})'\tilde{g}^{j_1j_2}\tilde{g}^{k_1k_2}R_{pi_1j_1k_1}R_{qi_2j_2k_2} + \tilde{g}^{i_1i_2}(\tilde{g}^{j_1j_2})'\tilde{g}^{k_1k_2}R_{pi_1j_1k_1}R_{qi_2j_2k_2}\\ & +\tilde{g}^{i_1i_2}\tilde{g}^{j_1j_2}(\tilde{g}^{k_1k_2})'R_{pi_1j_1k_1}R_{qi_2j_2k_2} + \tilde{g}^{i_1i_2}\tilde{g}^{j_1j_2}\tilde{g}^{k_1k_2}(R_{pi_1j_1k_1})'R_{qi_2j_2k_2}\\ & +\tilde{g}^{i_1i_2}\tilde{g}^{j_1j_2}\tilde{g}^{k_1k_2}R_{pi_1j_1k_1}(R_{qi_2j_2k_2})'.\end{aligned}$$

Note that $(\tilde{g}^{ij})' = -\tilde{g}^{im}h_{mn}\tilde{g}^{nj}$. Therefore,

$$\begin{aligned}(\check{R}_g.h)'_{pq} &= -h_{mn}\left(R_{pmij}R_{qnij} + R_{pimj}R_{qinj} + R_{pijm}R_{qijn}\right)\\ &\quad +(R'_g.h)_{pijk}R_{qijk} + R_{pijk}(R'_g.h)_{qijk}.\end{aligned}$$

Since $R(0) = cI$, $R_{ijij} = -R_{ijji} = c$ for all $1 \le i, j \le n$, and $R_{ijkl} = 0$. This implies

$$\sum_{m,n,i,j}[h_{mn}(R_{pmij}R_{qnij} + R_{pimj}R_{qinj} + R_{pijm}R_{qijn})] = 2(n-3)c^2h_{pq} + 4c^2tr(h)g_{pq},$$

$$(R'_g(h))_{pijk}R_{qijk} = (R'_g(h))_{piqi}R_{qiqi} + (R'_g(h))_{piiq}R_{qiiq} = 2c(R'_g(h))_{piqi}$$

and

$$(R'_g(h))_{qijk}R_{pijk} = 2c(R'_g(h))_{qipi} = 2c(R'_g(h))_{piqi}.$$

From equation 1.174(c) in [3], we have

$$\begin{aligned} 2(R'_g(h))_{piqi} &= [(D^2_{iq}h)_{pi} + (D^2_{pi}h)_{qi} - (D^2_{pq}h)_{ii} - (D^2_{ii}h)_{pq} \\ &\quad +h_{ij}R_{piqj} - h_{qj}R_{piij}]. \end{aligned}$$

Using the Ricci identity we have

$$\begin{aligned} \Sigma_i[(D^2_{iq}h)_{pi}+(D^2_{pi}h)_{qi}] &= \Sigma_i[(D^2_{iq})h_{pi}-(D^2_{qi}h)_{pi}+(D^2_{qi}h)_{pi}+(D^2_{pi}h)_{qi}] \\ &= \Sigma_{i,j}[h_{ij}R_{iqpj}+h_{pj}R_{iqij}]-D\delta_g h_{pq}-D\delta_g h_{qp} \\ &= \Sigma_{i,j}[h_{ij}R_{iqpj}+h_{pj}R_{iqij}]-2\delta^*_g\delta_g h_{pq}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2(R'_g(h))_{piqi} &= h_{ij}R_{iqpj} + h_{pj}R_{iqij} - 2\delta^*_g\delta_g h_{pq} - Ddtr(h)_{pq} \\ &\quad +D^*Dh_{pq} + h_{ij}R_{piqj} - h_{qj}R_{piij}. \end{aligned}$$

Using $R = cI$ again we obtain,

$$h_{ij}R_{iqpj} + h_{pj}R_{iqij} + h_{ij}R_{piqj} - h_{qj}R_{piij} = 2(n-1)ch_{pq}.$$

Combining these two equations, the proof of Lemma 4.1(iv) follows. Next,

$$\begin{aligned} (\check{R}'_g(h))_{pq} &= -2(n-3)c^2h_{pq} - 4c^2tr(h)g_{pq} + 4c\Sigma_{i,j}(R'_g.h)_{piqi} \\ &= 2(n+1)c^2h_{pq} - 4c^2tr(h)g_{pq} + 2c[-2\delta^*_g\delta_g h_{pq} \\ &\quad -Ddtr(h)_{pq} + D^*Dh_{pq}]. \end{aligned}$$

This completes the proof of Lemma 4.1(i). □

Proof of (*ii*). Let T be a (0, 4) tensor independent of t. Then using the expression for D^* in a local co-ordinate chart and differentiating it with respect to t we obtain

$$\begin{aligned} (D^*)'_g(h)(T)(x,y,z) &= -(\tilde{g}^{kj})'(D_kT)_{jxyz} + \tilde{g}^{kj}[T_{\Pi_{kj}xyz} + T_{j\Pi_{kx}yz} \\ &\quad +T_{jx\Pi_{ky}z} + T_{jxy\Pi_{kz}}]. \end{aligned}$$

Note that, Π is a vector valued symmetric two form. Next,

$$(D^*)'_g(h)(R)_{jkl} = R_{\Pi_{ii}jkl} + R_{i\Pi_{ij}kl} + R_{ij\Pi_{ik}l} + R_{ijk\Pi_{il}}.$$

By the definition of L_h,

$$L_{hjkl} = \{R_{kl\Pi_{ii}j} + R_{kli\Pi_{ij}} + R_{li\Pi_{ik}j} + R_{ik\Pi_{il}j} + R_{lii\Pi_{kj}} + R_{iki\Pi_{lj}}\}.$$

Combining these two and using the symmetries of R we have

$$W_{hjkl} = [R_{ij\Pi_{ik}l} + R_{ijk\Pi_{il}} - R_{li\Pi_{ik}j} - R_{ik\Pi_{il}j} - R_{lii\Pi_{kj}} - R_{iki\Pi_{lj}}].$$

Pairing it with $d^D\alpha$ for any $\alpha \in S^2(T^*M)$ and using the symmetries of R and $d^D\alpha$ we have

$$\sum W_{hjkl} d^D\alpha_{jkl} = 2\sum \left(R_{ij\Pi_{ik}l} - R_{li\Pi_{ik}j} - R_{lii\Pi_{kj}}\right)(d^D\alpha)_{jkl}.$$

$R = cI$ gives

$$\begin{aligned} \sum R_{ij\Pi_{ki}l} d^D\alpha_{jkl} &= c\sum C_{kim} R_{ijml} d^D\alpha_{jkl} \\ &= c\sum C_{kii} d^D\alpha_{jkj} - c\sum C_{klj} d^D\alpha_{jkl}, \end{aligned}$$

$$\begin{aligned} \sum R_{li\Pi_{ik}j} d^D\alpha_{jkl} &= c\sum C_{ikm} R_{limj} d^D\alpha_{jkl} \\ &= c\sum C_{jkl} d^D\alpha_{jkl} - c\sum C_{iki} d^D\alpha_{lkl} \end{aligned}$$

and

$$\begin{aligned} \sum R_{lii\Pi_{kj}} d^D\alpha_{jkl} &= c\sum C_{kjm} R_{liim} d^D\alpha_{jkl} \\ &= -(n-1)c\sum C_{jkl} d^D\alpha_{jkl}. \end{aligned}$$

Since C is symmetric in the first two entries and $d^D\alpha$ is skew-symmetric in the last two entries,

$$\sum C_{klj} d^D\alpha_{jkl} = 0.$$

Next a simple calculation gives $\sum_i C_{kii} = \frac{1}{2} dtr(h)_k$ and $\sum_j d^D\alpha_{jkj} = dtr\alpha_k + \delta_g\alpha_k$. Then

$$\sum C_{jkl} d^D\alpha_{jkl} = \frac{1}{2}\sum (C_{jkl} - C_{jlk}) d^D\alpha_{jkl} = \frac{1}{2}\sum d^D h_{jkl} d^D\alpha_{jkl}.$$

Combining all these equations we have

$$\delta^D W_h = (n-2)c\delta^D d^D h + 2cDdtr(h) + 2c\Delta tr(h)g.$$

□

Proof of (*iii*). Let x, y, z, u, w be fixed vector fields. Then

$$\begin{aligned} (D_x R)'(y,z,u,w) &= (x.R(y,z,u,w))' - \{\bar{R}_h(D_x y, z, u, w) \\ &\quad + \bar{R}_h(y, D_x z, u, w) + \bar{R}_h(y, z, D_x u, w) \\ &\quad + \bar{R}_h(y, z, u, D_x w) + R(\Pi(x,y), z, u, w) \\ &\quad + R(y, \Pi(x,z), u, w) + R(y, z, \Pi(x,u), w) \\ &\quad + R(y, z, u, \Pi(x,w))\} \\ &= D_x\bar{R}_h(y,z,u,w) - \{R(\Pi(x,y), z, u, w) \\ &\quad + R(y, \Pi(x,z), u, w) + R(y, z, \Pi(x,u), w) \\ &\quad + R(y, z, u, \Pi(x,w)\}. \end{aligned}$$

Applying the differential Bianchi identity we get

$$(D_x R)'(y,z,u,w) + (D_y R)'(z,x,u,w) + (D_z R)'(x,y,u,w) = 0.$$

This gives

$$\begin{aligned}
&D_x\bar{R}_h(y,z,u,w)+D_y\bar{R}_h(z,x,u,w)+D_z\bar{R}_h(x,y,u,w)\\
&\quad=R(\Pi(x,y),z,u,w)+R(y,\Pi(x,z),u,w)+R(y,z,\Pi(x,u),w)\\
&\qquad+R(y,z,u,\Pi(x,w))+R(\Pi(y,z),x,u,w)+R(z,\Pi(y,x),u,w)\\
&\qquad+R(z,x,\Pi(y,u),w)+R(z,x,u,\Pi(y,w))+R(\Pi(z,x),y,u,w)\\
&\qquad+R(x,\Pi(z,y),u,w)+R(x,y,\Pi(z,u),w)+R(x,y,u,\Pi(z,w))\\
&\quad=R(y,z,\Pi(x,u),w)+R(y,z,u,\Pi(x,w))+R(z,x,\Pi(y,u),w)\\
&\qquad+R(z,x,u,\Pi(y,w))+R(x,y,\Pi(z,u),w)+R(x,y,u,\Pi(z,w)).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum(D_{e_i}\bar{R}_h)(e_i,w,y,z) &= \sum(D_{e_i})\bar{R}_h(y,z,e_i,w)\\
&= -\sum\{(D_y\bar{R}_h)(z,e_i,e_i,w)+(D_z\bar{R}_h)(e_i,y,e_i,w)\}\\
&\quad+L_h(w,y,z)\\
&= \sum\{(D_y\bar{R}_h)(z,e_i,w,e_i)-(D_z\bar{R}_h)(e_i,y,e_i,w)\}\\
&\quad+L_h(w,y,z)\\
&= d^D\bar{r}_h(w,y,z)+L_h(w,y,z).
\end{aligned}$$

Therefore,

$$D^*\bar{R}_h=-d^D\bar{r}_h-L_h.$$

□

Proof of (*v*). From the identity (2.8) of [6], we have

$$\delta^D d^D h_{pq}=2D^*Dh_{pq}-2\delta_g^*\delta_g h_{pq}+\sum_i(r_{pi}h_{iq}+r_{qi}h_{ip})-2\sum_{i,j}R_{piqj}h_{ij}. \tag{4.2}$$

A straightforward computation using $R=cI$ gives the required result. □

Proof of (*vi*). From the proof of Proposition 2,

$$\begin{aligned}
(|R|^p)'_g.h &= p|R|^{p-2}(R,R'_g.h)-2p|R|^{p-2}(\check{R},h)\\
&= 2cp|R|^{p-2}\sum(R'_g.h)_{ijij}-2\frac{p}{n}|R|^p tr(h).
\end{aligned}$$

Using (iv) we have

$$\begin{aligned}
\sum(R'_g.h)_{ijij} &= tr(\bar{r}_h)\\
&= c(n-1)tr(h)-tr\delta_g^*\delta_g h+\frac{1}{2}(trD^*Dh-trDdtr(h))\\
&= c(n-1)tr(h)-tr\delta_g^*\delta_g h+\Delta tr(h).
\end{aligned}$$

Since $|R|^2=2c^2n(n-1)$ we have

$$(|R|^p)'_g(h)=-2cp|R|^{p-2}(tr\delta_g^*\delta_g h-\Delta tr(h)+(n-1)ctr(h)).$$

□

Next, we study the stability of $\mathcal{R}_p$ which is a space form. A symmetric covariant 2-tensor h is called Transverse-Traceless tensor (TT-tensor) if $\delta_g h = 0$ and $tr(h) = 0$. First, we study H on TT-variations.

4.2 *Transverse-traceless variations*

Let (M, g) be a Riemannian manifold with constant sectional curvature $c \neq 0$ and $h \in \delta_g^{-1}(0) \cap \mathrm{Tr}^{-1}(0)$. In this case the expression for $H(h, h)$ reduces to

$$\begin{aligned} H(h,h) &= -p|R|^{p-2}[\langle \delta^D (D^*)'_g.h(R), h\rangle + \langle D^*\bar{R}_h, d^D h\rangle + \langle \check{R}'_g(h), h\rangle] \\ &\quad + \frac{p}{n}\|R\|^p\langle h, h\rangle. \end{aligned}$$

Using Lemma 4.1(iii) we have

$$\begin{aligned} H(h,h) &= -p|R|^{p-2}[\langle \delta^D W_h, h\rangle - \langle \bar{r}_h, \delta^D d^D h\rangle + \langle \check{R}'_g(h), h\rangle] \\ &\quad + \frac{p}{n}\|R\|^p\langle h, h\rangle. \end{aligned}$$

Then from Lemma 4.1(i) we have

$$\begin{aligned} \frac{p}{n}\|R\|^p\langle h,h\rangle - p\|R\|^{p-2}\langle (\check{R})'_g.h, h\rangle &= 2pc^2(n-1)\|R\|^{p-2}\|h\|^2 \\ &\quad - p\|R\|^{p-2}\{(n-1)c^2\langle h,h\rangle \\ &\quad + 2c\langle Dh, Dh\rangle\} \\ &= -2pc\|R\|^{p-2}\|Dh\|^2. \end{aligned}$$

Using Lemma 4.1(ii) and (v) we have

$$\begin{aligned} \langle \delta^D W_h, h\rangle &= c(n-2)\langle \delta^D d^D h, h\rangle \\ &= 2c(n-2)\langle D^*Dh, h\rangle + 2c^2 n(n-2)\langle h, h\rangle \\ &= 2c(n-1)\|Dh\|^2 + 2c^2 n(n-2)\|h\|^2. \end{aligned}$$

Next using Lemma 4.1(iv) and (v) we have

$$\begin{aligned} \langle \bar{r}_h, \delta^D d^D h\rangle &= -\langle 2(n-1)ch + D^*Dh, D^*Dh + nch\rangle \\ &= -[\|D^*Dh\|^2 + (3n-2)c\|Dh\|^2 + 2c^2 n(n-1)\|h\|^2]. \end{aligned}$$

Combining all these results we have

$$H(h,h) = p\|R\|^{p-2}\{\|D^*Dh\|^2 + nc\|Dh\|^2 + 2nc^2\|h\|^2\}.$$

It is clear from the above expression that if $c > 0$, then $H(h,h) > 2nc^2\|h\|^2$. Suppose $c < 0$. Since $\|d^D h\|^2 \geq 0$, using Lemma 4.1(v) we have that the least eigenvalue of the rough Laplacian is bounded below by $-nc$. Now,

$$\begin{aligned} \|D^*Dh\|^2 + nc\|Dh\|^2 &= \|D^*Dh + nch\|^2 - nc\langle D^*Dh + nch, h\rangle \\ &\geq -nc\langle D^*Dh, h\rangle - n^2c^2\|h\|^2. \end{aligned}$$

Hence $H(h,h) > 2nc^2\|h\|^2$.

4.3 *Conformal variations*

Next we study H on the space of conformal variations of g. Consider any f in $C^\infty(M)$ with $\int f dv_g = 0$. In this section we prove that there exists $\epsilon_1 > 0$ such that

$$H(fg, fg) \geq \epsilon_1 \|fg\|^2 = n\epsilon_1 \|f\|^2.$$

First we compute each term appearing in the expression of H in (4.1).

$$\frac{p}{n}\|R\|^p\|fg\|^2 = 2n(n-1)pc^2\|R\|^{p-2}\int_M f^2 dv_g. \tag{4.3}$$

Applying Lemma 4.1(vi) we have

$$\begin{aligned}(|R|^p)'_g(fg) &= -2pc|R|^{p-2}(trD\delta_g fg - \Delta tr fg + (n-1)ctr fg)\\ &= -2pc|R|^{p-2}(\Delta f - n\Delta f + n(n-1)cf)\\ &= -2p|R|^{p-2}(n-1)c(ncf - \Delta f).\end{aligned}$$

Consequently,

$$\begin{aligned}tr((|R|^{p-2})'(fg)g) &= -2cn(n-1)(p-2)|R|^{p-4}(ncf - \Delta f)\\ &= \frac{(p-2)}{c}|R|^{p-2}(\Delta f - ncf).\end{aligned}$$

Hence

$$\begin{aligned}-\frac{p}{n}\|R\|^2\langle(|R|^{p-2})'(fg)g, fg\rangle &= -2pc(p-2)(n-1)\|R\|^{p-2}\Big[\|df\|^2\\ &\quad -nc\int_M f^2 dv_g\Big]\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\frac{1}{2}\langle(|R|^p)'g, fg\rangle &= -pnc(n-1)\|R\|^{p-2}\int_M(-f\Delta f + ncf^2)dv_g\\ &= npc(n-1)\|R\|^{p-2}\left[\|df\|^2 - nc\int_M f^2 dv_g\right].\end{aligned} \tag{4.5}$$

From Lemma 4.1(i),

$$tr(\check{R})'(fg) = -2c^2n(n-1)f + 4c(n-1)\Delta f.$$

Therefore,

$$-p\|R\|^{p-2}\langle(\check{R})'(fg), fg\rangle = -2cp(n-1)\|R\|^{p-2}\left[2\|df\|^2 - cn\int_M f^2 dv_g\right]. \tag{4.6}$$

Next, we compute the 4th term in the expression of H in (4.1). By a straightforward computation, we have the following identity:

$$Dd^D h(x, y, z, w) = D^2_{x,z}h(y, w) - D^2_{x,w}h(y, z).$$

This yields

$$\begin{aligned}(R, Dd^D fg) &= 2\sum R_{ijkl} Dd^D fg_{ijkl} \\ &= 2\sum R_{ijij}((D^2_{ii} fg)_{jj} - (D^2_{ij} fg)_{ij}) \\ &= 2c\sum (tr Ddtr fg + tr D\delta_g fg) \\ &= -2c(n-1)\Delta f.\end{aligned}$$

Therefore,

$$\begin{aligned}-p\langle (|R|^{p-2})' R, Dd^D fg\rangle &= -p\int_M (|R|^{p-2})'_{fg}(fg)(R, Dd^D fg) dv_g \\ &= 4p(n-1)^2(p-2)c^2\|R\|^{p-4}[\|\Delta f\|^2 \\ &\quad -nc\|df\|^2].\end{aligned} \tag{4.7}$$

Next using Lemma 4.1(v) we have

$$\begin{aligned}tr\delta^D d^D fg &= 2tr D^* D(fg) - 2tr D\delta_g(fg) \\ &= 2\left(\Delta(tr(fg)) + tr Ddf\right) \\ &= 2(n-1)\Delta f.\end{aligned}$$

This identity combining with Lemma 4.1(ii) implies that

$$\begin{aligned}\langle \delta^D W_{(fg)}, fg\rangle &= c(n-2)\int_M (tr\delta^D d^D fg) f dv_g \\ &\quad +2nc\int_M (tr Ddf) f dv_g + 2n^2 c\int_M f\Delta f dv_g \\ &= 4c(n-1)^2\|df\|^2.\end{aligned}$$

Therefore,

$$-p\|R\|^{p-2}\langle \delta^D W_{(fg)}, fg\rangle = -4pc(n-1)^2\|R\|^{p-2}\|df\|^2. \tag{4.8}$$

Next, we compute the remaining term appearing in the expression of the Hessian. From Lemma 4.1(iv) we obtain

$$\begin{aligned}\bar{r} &= \frac{1}{2}\{2(n-1)cfg - 2\delta_g^*\delta_g fg - Ddtr fg + D^* Dfg\} \\ &= \frac{1}{2}\{2c(n-1)fg + 2Ddf - nDdf + \Delta fg\} \\ &= \frac{1}{2}\{2c(n-1)fg - (n-2)Ddf + \Delta fg\}.\end{aligned}$$

By a simple calculation using Lemma 4.1(v) we have

$$\delta^D d^D fg = 2(\Delta fg + Ddf).$$

Therefore,

$$\begin{aligned}\langle \bar{r}, \delta^D d^D fg\rangle &= (2n-3)\langle \Delta f, \Delta f\rangle - (n-2)\langle Ddf, Ddf\rangle + 2c(n-1)^2\langle df, df\rangle \\ &= (n-1)\langle Ddf, Ddf\rangle + (n-1)(4n-5)c\langle df, df\rangle.\end{aligned}$$

Using Bochner–Weitzenbök formula on the space of one forms we have

$$\Delta df = D^* Ddf + (n-1)cdf.$$

This implies

$$\|\Delta f\|^2 = \langle \delta df, \delta df\rangle = \langle \Delta df, df\rangle = \|Ddf\|^2 + (n-1)c\|df\|^2.$$

Therefore,

$$\langle \bar{r}, \delta^D d^D fg\rangle = (n-1)\|\Delta f\|^2 + c(n-1)(3n-4)\|df\|^2. \tag{4.9}$$

Hence combining all the equations from (4.3) to (4.9) we have

$$H(fg, fg) = p\|R\|^{p-2}(a\|\Delta f\|^2 - bc\langle \Delta f, f\rangle + dc^2\|f\|^2),$$

where

$$a = (n-1) + 2(p-2)\left(1 - \frac{1}{n}\right),$$
$$b = 4(n-1)(p-1),$$
$$d = n(n-1)(2p-n).$$

Consider the polynomial, $q(x) = ax^2 - bx + d$. Suppose f be an eigenfunction of the Laplacian corresponding to the eigenvalue λc. Then

$$H(fg, fg) = q(\lambda)c^2\|f\|^2.$$

To prove our claim it is sufficient to prove that $q(\lambda) > 0$. Notice that

$$q(x) = (x-n)\left(ax - \frac{d}{n}\right).$$

Let $c > 0$. Since $\frac{d}{an} < n$ and the first eigenvalue $c\lambda_1$ of Δ satisfies $\lambda_1 \geq n$ we have that $q(\lambda) \geq 0$. $q(\lambda) = 0$ if and only if $\lambda = \lambda_1 = n$. This implies that (M, g) is a sphere with the standard metric. In this case, the eigenfunctions are the first-order spherical harmonics. These functions satisfy $\delta_g^* df = Ddf = -fg$. Hence the proof follows.

If $c < 0$, then the proof immediately follows from the expression of $H(fg, fg)$. □

Next to obtain the stability of $\mathcal{R}_p$ for space forms it is sufficient to prove that $H(h, fg) = 0$ be a TT-tensor for any h and $f \in C^\infty M$. From [5], the decomposition (1.1) is preserved by the rough Laplacian. Hence, it is easy to see from Lemma 4.1 that

$$tr((\check{R})'(h)) = tr(\delta^D d^D h) = tr(\delta^D W_h) = tr(\bar{r}_h) = 0$$

and

$$\delta_g(\bar{r}_h) = 0.$$

This implies that $tr(\delta^D d^D \bar{r}_h) = 0$. Lemma 4.1(vi) implies that $(|R|^p)'(h)$ is also zero. Hence,

$$H(h, fg) = 0.$$

5. Second variation at product of space forms

In this section we prove the stability of $\mathcal{R}_p$ for product of space forms of same type for certain values of p. Let (M_1^m, g_1) and (M_2^m, g_2) be two closed Riemannian manifolds with dimension $m \geq 3$ and constant sectional curvature $c \neq 0$. Let $(M, g) = (M_1 \times M_2, g_1 + g_2)$.

From Lemma 4.57(ii) in [3], we have the following orthogonal decomposition of $T_g\mathcal{M}_1$:

$$T_g\mathcal{M}_1 = \mathrm{Im}\delta_g^* \oplus C^\infty(M) \oplus (\delta_g^{-1}(0) \cap tr_g^{-1}(0)) \tag{5.1}$$

Let $E_1 = \{e_1, e_2, \ldots, e_m\}$ and $E_2 = \{e_{m+1}, \ldots, e_{2m}\}$ denote normal basis at some points p_1 and p_2 corresponding to (M_1^m, g_1) and (M_2^m, g_2) respectively. The curvature R satisfies the following properties:

(R1) $R(e_i, e_j, e_i, e_j) = -R(e_i, e_j, e_j, e_i) = c$, when $\{e_i, e_j\} \subset E_k$, $k = 1, 2$.
(R2) $R(e_m, e_n, e_i, e_j) = 0$, otherwise.

A traceless symmetric tensor splits as

$$h = h_1 + fg_1 + \tilde{h} + h_2 - fg_2, \tag{5.2}$$

where h_1 is tangent to the first factor, h_2 is tangent to the second factor and $\tilde{h}$ is non-zero only for the mixed set of vectors and $f \in C^\infty(M_1 \times M_1)$. This decomposition is preserved by the rough Laplacian and

$$tr(h_1) = tr(h_2) = tr(\tilde{h}) = 0.$$

Let $h \in C^\infty(M) \cdot g \oplus (\delta_g^{-1}(0) \cap tr^{-1}(0))$. Then we have that $\delta_g h = -\frac{1}{n} dtrh$. Moreover, if h is a TT-tensor, then

$$\delta_g^* \delta_g h_1 = \delta_g^* \delta_g h_2 = \delta_g^* \delta_g \tilde{h} = 0.$$

To prove the theorem, we need the following lemma.

Lemma 5.1.

$$\begin{aligned}
\check{R}'(\tilde{h}) &= 4\delta_g^* \delta_g \tilde{h} + D^* D\tilde{h},\\
\check{R}'(h_1) &= 2(m+1)c^2 h_1 + 2cD^* Dh_1 - 4c\delta_g^* \delta_g h_1,\\
\check{R}'(fg_1) &= -2(m-1)c^2 fg_1 + 2c[\Delta_1 fg_1 - (m-2)\delta_g^* df_1],
\end{aligned}$$

where df_1 is the component of df along the first factor.

Proof. From the proof of Lemma 4.1(i),

$$\begin{aligned}
\check{R}'(h)_{pq} = &- \sum_{m,n,i,j} h_{mn} \left(R_{pmij} R_{qnij} + R_{pimj} R_{qinj} + R_{pijm} R_{qijn}\right)\\
&+ \sum_{i,j,k} R'(h)_{pijk} R_{qijk} + \sum_{i,j,k} R_{pijk} R'(h)_{qijk}.
\end{aligned}$$

Using (R1) and (R2) we have that

$$\sum_{m,n,i,j} \tilde{h}_{mn} \left(R_{pmij}R_{qnij} + R_{pimj}R_{qinj} + R_{pijm}R_{qijn}\right) = 0$$

and $\sum_{i,j,k} R'(\tilde{h})_{pijk}R_{qijk}$ is non-zero only if $\{e_p, e_i, e_j, e_k\} \subset E_k$, $k = 1, 2$. Now,

$$\begin{aligned} 2\sum_i R'(\tilde{h})_{piqi} &= [(D^2_{iq}\tilde{h})_{pi} + (D^2_{pi}\tilde{h})_{qi} - (D^2_{pq}\tilde{h})_{ii} - (D^2_{ii}\tilde{h})_{pq} \\ &\quad + \tilde{h}_{ij}R_{piqj} - \tilde{h}_{qj}R_{piij}]. \end{aligned}$$

It is clear from the above expression that $\check{R}'(\tilde{h})_1 = \check{R}'(\tilde{h})_2 = 0$. Hence

$$\sum R'(\tilde{h})_{pijk}R_{qijk} = c^2\delta_g^*\delta_g\tilde{h} + \frac{1}{2}D^*D\tilde{h}.$$

Therefore,

$$\check{R}'(\tilde{h}) = 4\delta_g^*\delta_g\tilde{h} + D^*D\tilde{h}.$$

Next using (R1) and (R2) again we have

$$\sum_{m,n,i,j} h_{1mn}(R_{pmij}R_{qnij} + R_{pimj}R_{qinj} + R_{pijm}R_{qijn}) = 2(m-3)c^2h_{1pq}.$$

If $e_p, e_q \in E_2$, a simple computation shows that $\sum_{i\in E_2} R'(h_1)_{piqi} = 0$. If $e_p, e_q \in E_1$, then

$$\begin{aligned} \sum R'(h_1)_{pijk}R_{qijk} &= 2c\sum_{i\in E_1} R'(h_1)_{piqi} = c[D^*Dh_1 \\ &\quad +2(m-1)ch_1 - 2\delta_g^*\delta_g h_1]. \end{aligned}$$

Hence

$$\check{R}'(h_1)_{pq} = 2(m+1)c^2h_{1pq} + 2cD^*Dh_{1pq} - 4\delta_g^*\delta_g h_1.$$

Similarly,

$$\check{R}'(fg_1) = 2(m+1)c^2fg_1 - 4mc^2fg_1 + 2c[-mDdf_1 + 2\delta_g^*df_1 + \Delta_1 fg_1].$$

□

Next the following lemmas follow from the proof of Lemma 5.1 and Lemma 4.1.

Lemma 5.2.

$$\begin{aligned} \bar{r}_{\tilde{h}} &= \frac{1}{2}[D^*D\tilde{h} - 2\delta_g^*\delta_g\tilde{h}], \\ \bar{r}_{h_1} &= \frac{1}{2}[2c(m-1)h_1 + D^*Dh_1 - 2\delta_g^*\delta_g h_1], \\ \bar{r}_{fg_1} &= \frac{1}{2}[2c(m-1)fg_1 + 2\delta_g^*df_1 - mDdf + \Delta fg_1]. \end{aligned}$$

Lemma 5.3.

$$(|R|^p)'\tilde{h} = 0,$$
$$(|R|^p)'h_1 = -4pc|R|^{p-2}tr(\delta_g^*\delta_g h_1),$$
$$(|R|^p)'(fg_1) = 2cp(m-1)|R|^{p-2}(\Delta_1 f - mcf).$$

Lemma 5.4.

$$\delta^D d^D \tilde{h} = 2D^*D\tilde{h} + 2c(m-1)\tilde{h} - 2\delta_g^*\delta_g\tilde{h},$$
$$\delta^D d^D h_1 = 2D^*Dh_1 + 2mch_1 - 2\delta_g^*\delta_g h_1,$$
$$\delta^D d^D fg_1 = 2\Delta fg_1 + 2\delta_g^* df_1.$$

The proof easily follows from the proof of Lemma 4.1(v).

Lemma 5.5.

$$(\delta^D W_{\tilde{h}})_k = 0, \qquad for\ k = 1, 2,$$
$$\langle W_{\tilde{h}}, d^D\tilde{h}\rangle = (m-1)c\|d^D\tilde{h}\|^2 + \frac{c}{2}K, \quad where\ \ 0 \le K \le \|d^D\tilde{h}\|^2,$$
$$\delta^D W_{h_1} = c(m-2)\delta^D d^D h_1,$$
$$\delta^D W_{fg_1} = (m-1)c\delta^D d^D(fg_1) + 2cm\Delta_1 fg_1 + 2cm\delta_g^* df_1.$$

Proof. From the proof of Lemma 4.1(ii) we have that for any $h, \alpha \in S^2(T^*M)$,

$$\sum W_{hjkl}d^D\alpha_{jkl} = 2\sum\left(R_{ij\Pi_{ikl}} - R_{li\Pi_{ik}j} - R_{lii\Pi_{kj}}\right)(d^D\alpha)_{jkl}.$$

Now consider $\tilde{h}$.

$$\begin{aligned}\sum R_{ij\Pi_{ki}l}d^D\alpha_{jkl} &= \sum C_{kim}R_{ijml}d^D\alpha_{jkl}\\ &= c\sum_{i,j\in E_1} C_{kii}d^D\alpha_{jkj} - c\sum_{j,l\in E_1} C_{klj}d^D\alpha_{jkl}\\ &\quad +c\sum_{i,j\in E_2} C_{kii}d^D\alpha_{jkj} - c\sum_{j,l\in E_2} C_{klj}d^D\alpha_{jkl},\end{aligned} \tag{5.3}$$

$$\sum_{i\in E_1} C_{kii} = dtr_{g_1}(\tilde{h})_k = 0,$$

As we have seen in Lemma 4.1(ii), $\sum_{j,l\in E_1} C_{klj}d^D\alpha_{jkl} = 0$. Similarly, the last two terms of (5.4) are also zero. Next,

$$\begin{aligned}\sum R_{lii\Pi_{kj}}(d^D\alpha)_{jkl} &= C_{\tilde{h}kjl}R_{liil}d^D\alpha_{jkl}\\ &= -(m-1)c\sum C_{\tilde{h}kjl}d^D\alpha_{jkl}\\ &= -\frac{c(m-1)}{2}d^D\tilde{h}_{jkl}d^D\alpha_{jkl},\end{aligned}$$

$$\begin{aligned}\sum R_{li\Pi_{ik}j}d^D\alpha_{jkl} &= \sum C_{\tilde{h}ikm}R_{limj}d^D\alpha_{jkl}\\ &= \sum C_{\tilde{h}ikl}R_{lili}d^D\alpha_{ikl} + \sum C_{\tilde{h}iki}R_{liil}d^D\alpha_{lkl}\end{aligned}$$

$$= c \sum_{l,i \in E_1} C_{\tilde{h}ikl} d^D \alpha_{ikl} + c \sum_{l,i \in E_2} C_{\tilde{h}ikl} d^D \alpha_{ikl}.$$

Clearly for $\alpha = h_1$ or $\alpha = h_2$, the above expression is zero. Let $\alpha = \tilde{h}$. Then by a simple calculation we have

$$\sum_{l,i \in E_1} C_{\tilde{h}ikl} d^D \tilde{h}_{ikl} = -\frac{1}{4} \sum_{i,l \in E_1} |d^D \tilde{h}_{ikl}|^2$$

and

$$\sum_{l,i \in E_1} C_{\tilde{h}ikl} d^D \tilde{h}_{ikl} = -\frac{1}{4} \sum_{i,l \in E_1} |d^D \tilde{h}_{ikl}|^2.$$

Suppose

$$K = \frac{1}{4} \int_M \left(\sum_{i,l \in E_1} |d^D \tilde{h}_{ikl}|^2 + \sum_{i,l \in E_1} |d^D \tilde{h}_{ikl}|^2 \right) dv_g.$$

Then $0 \leq K \leq \frac{1}{4} \|d^D \tilde{h}\|^2$. Hence the result follows.

Next, consider h_1. It is easy to see using the formula for C_{h_1} that $C_{h_1 ijk}$ is zero if $\{e_i, e_j, e_k\}$ intersects E_2. Using this and following a similar computation as in Lemma 4.1(ii), we get the result.

Now, consider $h = f g_1$. In this case, a straightforward calculation gives

$$\sum (R_{ij\Pi_{ik}l} - R_{li\Pi_{ik}j}) d^D \alpha_{jkl} = 2 \sum C_{kii} R_{ijij} d^D \alpha_{jkj} + \sum C_{kij} R_{ijij} d^D \alpha_{ikj}.$$

Since $C_{kii} = 0$ when $e_i \in E_2$,

$$\begin{aligned} 2 \sum C_{kii} R_{ijij} d^D \alpha_{jkj} &= 2c \sum_{i,j \in E_1} C_{kii} d^D \alpha_{jkj} \\ &= c(m-1) \sum df_k (dtr\alpha_{1k} + \delta_g \alpha_{1k}). \end{aligned}$$

Since $C_{kij} = \frac{1}{2}(df_k g_{ij} + df_i g_{kj} - df_j g_{ik})$,

$$\sum C_{kij} R_{ijij} d^D \alpha_{ikj} = c \sum_{i,j \in E_1} df_j (dtr\alpha_{1j} + \delta_g \alpha_{1j}).$$

Therefore,

$$\sum (R_{ij\Pi_{ik}l} - R_{li\Pi_{ik}j}) d^D \alpha_{jkl} = cm \sum df_k (dtr\alpha_{1k} - \delta_g \alpha_{1k})$$

$$\sum R_{lii\Pi_{kj}} (d^D \alpha)_{jkl} = -\frac{c}{2}(m-1) d^D (f g_1)_{jkl} d^D \alpha_{jkl}.$$

Hence

$$\delta^D W_{fg_1} = (m-1) c \delta^D d^D (f g_1) + 2cm \Delta_1 f g_1 + 2cm \delta_g^* df_1.$$

□

Next we study the stability of $\mathcal{R}_p$ for product of space forms. First we study the action of H on TT-tensors.

5.1 *Transverse-traceless variations*

Consider $h \in \delta_g^{-1}(0) \cap tr^{-1}(0)$. Suppose $h = h_1 + \tilde{h} + h_2 + fg_1 - fg_2$. It is easy to see using the above lemma that

$$H(h_1, h_2) = H(h_1, \tilde{h}) = H(h_2, \tilde{h}) = 0$$

and

$$\begin{aligned} H(h_1, h_1) &= p|R|^{p-2}[\|D^* Dh_1\|^2 + mc\|Dh_1\|^2 + 2(m-2)c^2\|h_1\|^2], \\ H(h_2, h_2) &= p|R|^{p-2}[\|D^* Dh_2\|^2 + mc\|Dh_2\|^2 + 2(m-2)c^2\|h_2\|^2], \\ H(\tilde{h}, \tilde{h}) &= p|R|^{p-2}[\|D^* D\tilde{h}\|^2 + c(m-1)\|D\tilde{h}\|^2 + 2c^2(m-1)\|\tilde{h}\|^2 - \frac{c}{2}K]. \end{aligned}$$

Using similar arguments as in §4.1, we have ϵ_1 and ϵ_2 such that $H(h_1, h_1) \geq \epsilon_1\|h_1\|^2$ and $H(h_2, h_2) \geq \epsilon_2\|h_2\|^2$. Now, using the estimate for K given in Lemma 5.1(v), we have

$$H(\tilde{h}, \tilde{h}) \geq p|R|^{p-2}[\|D^* D\tilde{h}\|^2 + c\left(m - \frac{5}{4}\right)\|D\tilde{h}\|^2 + \frac{7}{4}c^2(m-1)\|\tilde{h}\|^2].$$

If $c > 0$, then it is clear from the above expression that

$$H(\tilde{h}, \tilde{h}) \geq \epsilon_3\|\tilde{h}\|^2.$$

Suppose $c < 0$, then $c(m - \frac{5}{4}) \geq c(m-1)$. Now, $\|d^D\tilde{h}\|^2 \geq 0$ implies that

$$\|D^* D\tilde{h}\|^2 + c(m-1)\|D\tilde{h}\|^2 \geq 0.$$

Hence

$$H(\tilde{h}, \tilde{h}) \geq \epsilon_3\|\tilde{h}\|^2.$$

Using bi-linearity of H, we have

$$\begin{aligned} H(h, h) \;=\; & H(h_1, h_1) + H(h_2, h_2) + H(\tilde{h}, \tilde{h}) + H(fg_1, fg_1) \\ & + H(fg_2, fg_2) + H(fg_1, fg_2). \end{aligned} \tag{5.4}$$

Next we shall compute the remaining terms of (5.4). From Lemma 5.1 we have

$$\begin{aligned} \langle(\check{R})'(fg_1), fg_1\rangle \;&=\; -2(m-1)c^2\|fg_1\|^2 + 2c[\langle\Delta_1 fg_1, fg_1\rangle \\ &\qquad - (m-2)\langle\delta_g^* df_1, fg_1\rangle] \\ &=\; -2c^2 m(m-1)\|f\|^2 + 4c(m-1)\|df_1\|^2, \end{aligned}$$

where df_1 is the component of df along the tangent space of M_1.

$$\begin{aligned} \langle\bar{r}_{fg_1}, \delta^D d^D fg_1\rangle &= \langle 2c(m-1)fg_1 + 2\delta_g^* df_1 - mDdf + \Delta fg_1, \Delta fg_1 + \delta_g^* df_1\rangle \\ &= 2cm(m-1)\|df\|^2 + (m-3)\langle\Delta_1 f, \Delta f\rangle + m\|\Delta f\|^2 \\ &\quad -(m-2)\|\delta_g^* df_1\|^2 - 2c(m-1)\|df_1\|^2 \\ &= 2cm(m-1)\|df\|^2 + (2m-3)\|\Delta_1 f\|^2 \end{aligned}$$

$$+3(m-1)\langle \Delta_1 f, \Delta_2 f\rangle + m\|\Delta_2 f\|^2$$
$$-(m-2)\|\delta_g^* df_1\|^2 - 2c(m-1)\|df_1\|^2.$$

Using Bochner–Weitzenbök formula on the space of one forms we have

$$\Delta df_1 = D^* Ddf_1 + (m-1)cdf_1.$$

Next, a simple calculation yields the following identity for a one-form ω,

$$2\delta_g\delta_g^*\omega + \delta d\omega = 2D^* D\omega. \tag{5.5}$$

Using this identity we have

$$\|\delta_g^* df_1\|^2 = \langle \delta_g\delta_g^*(df_1), df_1\rangle = \|\Delta_1 f\|^2 - c(m-1)\|df_1\|^2.$$

Therefore,

$$\begin{aligned}\langle \bar{r}_{fg_1}, \delta^D d^D fg_1\rangle &= 2cm(m-1)\|df\|^2 + (m-1)\|\Delta_1 f\|^2\\ &\quad + c(m-1)(m-4)\|df_1\|^2 + 3(m-1)\langle \Delta_1 f, \Delta_2 f\rangle\\ &\quad + m\|\Delta_2 f\|^2.\end{aligned}$$

Next,

$$\begin{aligned}\langle \delta^D W_{fg_1}, fg_1\rangle &= 2c(m-1)[\langle \Delta fg_1 + \delta_g^* df_1, fg_1\rangle + 2cm\langle \Delta_1 fg_1, fg_1\rangle\\ &\quad + 2cm\langle \delta_g^* df_1, fg_1\rangle\\ &= 2cm(m-1)\|df\|^2 + 2c(m-1)^2\|df_1\|^2,\end{aligned}$$

$$\begin{aligned}(R, Dd^D fg_1) &= 2c\sum_{i,j\in E_1}(Dd^D fg_1)_{ijij} + 2c\sum_{i,j\in E_2}(Dd^D fg_1)_{ijij}\\ &= 2c\sum_{i,j\in E_1}((D^2_{ii} fg_1)_{jj} - (D^2_{ij} fg_1)_{ij})\\ &= -2c(m-1)\Delta_1 f.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle (|R|^{p-2})'(fg_1)R, Dd^D fg_1\rangle &= -4c^2(p-2)(m-1)^2|R|^{p-4}\langle \Delta_1 f - mcf, \Delta_1 f\rangle\\ &= -(p-2)\left(1-\frac{1}{m}\right)|R|^{p-2}[\|\Delta_1 f\|^2 - mc\|df_1\|^2],\end{aligned}$$

$$\begin{aligned}&\frac{1}{n}|R|^2\langle (|R|^{p-2})'(fg_1).(g_1+g_2), fg_1\rangle\\ &\quad = c(p-2)\left(1-\frac{1}{m}\right)|R|^{p-2}\langle (\Delta_1 f - mcf)g_1, fg_1\rangle\\ &\quad = c(p-2)(m-1)|R|^{p-2}[\|df_1\|^2 - mc\|f\|^2],\end{aligned}$$

$$\frac{1}{2}\langle (|R|^p)'(fg_1)g_1, fg_1\rangle = mpc(m-1)|R|^{p-2}[\|df_1\|^2 - mc\|f\|^2].$$

Combining all these results, we have

$$\begin{aligned} H(fg_1, fg_1) &= p(m-1)|R|^{p-2}[a\|\Delta_1 f\|^2 - bc\|df_1\|^2 + dc^2\|f\|^2] \\ &\quad + p|R|^{p-2}[3(m-1)\langle\Delta_1 f, \Delta_2 f\rangle + m\|\Delta_2 f\|^2], \end{aligned}$$

where $a = \frac{1}{m}(m+p-2)$, $b = 2(p+1)$, $d = m(p-m+2)$.

Performing similar computation we have

$$\begin{aligned} H(fg_1, fg_2) = {} & p|R|^{p-2}[2\langle\Delta_1 f, \Delta_2 f\rangle + m(m-1)c\|df\|^2 - m^2(m-1)c^2\|f\|^2] \\ & + p(p-2)(m-1)|R|^{p-2}\left[\frac{1}{m}\langle\Delta_1 f, \Delta_2 f\rangle - c\|df\|^2 + mc^2\|f\|^2\right] \end{aligned}$$

and

$$\begin{aligned} H(fg_2, fg_2) &= p(m-1)|R|^{p-2}[a\|\Delta_2 f\|^2 - bc\|df_2\|^2 + dc^2\|f\|^2] \\ &\quad + p|R|^{p-2}[3(m-1)\langle\Delta_1 f, \Delta_2 f\rangle + m\|\Delta_1 f\|^2]. \end{aligned}$$

Therefore,

$$\begin{aligned} H(fg_1 - fg_2, fg_1 - fg_2) &= H(fg_1, fg_1) - 2H(fg_1, fg_2) + H(fg_2, fg_2) \\ &= p|R|^{p-2}[a_1\|\Delta_1 f\|^2 + a_1\|\Delta_2 f\|^2 \\ &\quad + b_1 c\|df\|^2 + 2d_1c^2\|f\|^2] + p|R|^{p-2}u_1\langle\Delta_1 f, \Delta_2 f\rangle, \end{aligned}$$

where

$$\begin{aligned} a_1 &= (m-1)a + m, \\ u_1 &= \frac{2}{m}\{3m^2 - 3m - 2 - p(m-1)\}, \\ b_1 &= -2(m-1)(m+3), \\ d_1 &= 4m(m-1). \end{aligned}$$

Case 1. $c > 0$. We know that the first eigenvalue of the Laplacian is greater than mc. Suppose, f be an eigenfunction corresponding to the eigenvalue $c\lambda$ of the Laplacian of $(M_1 \times M_2, g_1 + g_2)$. Then $f = f_1 f_2$ and $\lambda = \mu_1 + \mu_2$ where f_1 and f_2 are eigenfunctions of the Laplacian for (M_1, g_1) and (M_2, g_2) corresponding to the eigenvalues $c\mu_1$ and $c\mu_2$ respectively. Therefore,

$$\langle\Delta_1 f, \Delta_2 f\rangle = c^2\mu_1\mu_2|f|^2.$$

Since $u_1 \geq 0$ for $p \leq 2m$, we have

$$\begin{aligned} H(fg_1 - fg_2, fg_1 - fg)_2 &\geq p|R|^{p-2}[a_1\|\Delta_1 f\|^2 + a_1\|\Delta_2 f\|^2 \\ &\quad + b_1 c\|df\|^2 + d_1c^2\|f\|^2] \\ &\geq p|R|^{p-2}[a_1\|\Delta_1 f\|^2 + b_1 c\|df_1\|^2 + d_1c^2\|f\|^2] \\ &\quad + p|R|^{p-2}[a_1\|\Delta_2 f\|^2 + b_1 c\|df_1\|^2 + d_1c^2\|f\|^2]. \end{aligned}$$

Now consider the polynomial

$$q_1(x) = a_1x^2 + b_1x + d_1.$$

Note that

$$H(fg_1 - fg_2, fg_1 - fg_2) \geq pc^2|R|^{p-2}(q_1(\mu_1) + q_1(\mu_2))\|f\|^2.$$

So, it is sufficient to prove that $q_1(x) > 0$ for $x \geq m$.

$$q_1'(x) = 2a_1 x + b_1.$$

By a simple computation we have that $q_1'(x) > 0$ for $x \geq m$ and $q_1(m) > 0$.

This completes the proof. □

Case 2. $c < 0$. Since $b_1 < 0$ and $u_1\langle \Delta_1 f, \Delta_2 f\rangle > 0$ we have that

$$H(fg_1 - fg_2, fg_1 - fg_2) \geq 2p|R|^{p-2} d_1 c^2 \|f\|^2.$$

It is easy to see from Lemma 5.1 that H is diagonalizable by the decomposition (5.1). Therefore to complete the proof it is sufficient to show that there exists an $\epsilon_3 > 0$ such that $H(fg, fg) \geq \epsilon_3 \|fg\|^2$.

5.2 *Conformal variations*

Consider f in $C^\infty(M_1 \times M_2)$. Using the computations in 5.1 we have

$$\begin{aligned} H(fg_1 + fg_2, fg_1 + fg_2) &= H(fg_1, fg_1) + 2H(fg_1, fg_2) + H(fg_2, fg_2) \\ &= p|R|^{p-2}[a_2\|\Delta f\|^2 + u_2\langle \Delta_1 f, \Delta_2 f\rangle \\ &\quad + b_2 c\|df\|^2 + d_2 c^2 \|f\|^2], \end{aligned}$$

where

$$\begin{aligned} &a_2 = a_1, u_2 = 2m, \\ &b_2 = -2(m-1)(2p - m - 1), \\ &d_2 = 4m(m-1)(p-m). \end{aligned}$$

Since $u_2 > 0$,

$$H(fg_1 + fg_2, fg_1 + fg_2) \geq p|R|^{p-2}[a_2\|\Delta f\|^2 + b_2 c\|df\|^2 + d_2 c^2 \|f\|^2].$$

Case 1. $c > 0$. Consider the polynomial

$$q_2(\lambda) = a_2 \lambda^2 + b_2 \lambda + d_2.$$

A simple computation gives if $p \leq 2m$, then $2a_2 m + b_2 > 0$ and $q_2(m) > 0$. Using the argument as in 5.1, the proof follows.

Case 2. $c < 0$. When $p \geq m$ it is easy to see that $b_2 < 0$ and $d_2 > 0$. Therefore, $q_2(\lambda) > 0$. This completes the proof. □

6. Local minimization

To obtain local minimization property for $\mathcal{R}_p$, we follow the techniques used in [7]. First we consider the scale-invariant functional defined by

$$\tilde{\mathcal{R}}_p(g) = (V(g))^{\frac{2p}{n} - 1} . \mathcal{R}_p(g).$$

A simple calculation shows that

$$\nabla\tilde{\mathcal{R}}_p(g) = V^{\frac{2p}{n}-1}\nabla\mathcal{R}_p(g) + \left(\frac{p}{n} - \frac{1}{2}\right) V^{\frac{2p}{n}-2}\mathcal{R}_p(g)g.$$

It is easy to see that g is a critical metric for $\mathcal{R}_{p|\mathcal{M}_1}$ if and only if it is critical for $\tilde{\mathcal{R}}_p$. Let $\tilde{H}_{\tilde{g}}$ denote the second derivative of $\tilde{\mathcal{R}}_p$ at $\tilde{g}$. Recall that

$$\mathcal{W} = (\mathrm{Im}\delta_g^*)^{\perp} \cap T_g\mathcal{M}_1.$$

Let (M, g) be a critical point for $\tilde{\mathcal{R}}_p$. (M, g) is $L^{2,2}$- *stable* for $\tilde{\mathcal{R}}_p$, if there exists $\epsilon > 0$ such that for any $h \in \mathcal{W}$,

$$\tilde{H}_g(h, h) \geq \epsilon\|h\|^2_{L^{2,2}},$$

where

$$\|h\|^2_{L^{2,2}} = \|D^2h\|^2 + \|Dh\|^2 + \|h\|^2.$$

PROPOSITION 3

Let (M, g) be a closed Riemannian manifold. If (M, g) is $L^{2,2}$-stable for $\tilde{\mathcal{R}}_p$ then it is a strict local minimizer for $\tilde{\mathcal{R}}_p$.

We need the following lemma to prove the proposition.

Lemma 6.1. *For each metric $\tilde{g} = g + \theta_1$ in a sufficiently small $C^{l+1,\alpha}$-neighborhood of g $(l \geq 1)$, there is a $C^{l+2,\alpha}$-diffeomorphism $\phi : M \to M$ and a constant c such that*

$$\tilde{\theta} = e^c\phi^*\tilde{g} - g$$

satisfies

$$\delta_g\tilde{\theta} = 0 \ \text{ and } \ \int tr(\tilde{\theta})dv_g = 0.$$

Moreover, we have the estimate

$$\|\tilde{\theta}\|_{C^{l+1,\alpha}} \leq C\|\theta_1\|_{C^{l+1,\alpha}}.$$

Proof. Consider the operator

$$\delta_g\delta_g^* : T^*M \to T^*M.$$

Since this is an elliptic operator, the lemma follows from the proof of Lemma 2.10 in [7]. □

We denote by $A * B$ any tensor field which is a real linear combination of tensor fields, each formed by starting with the tensor field $A \otimes B$, using the metric to switch the type of any number of T^*M components to TM components, or vice versa taking any number of contractions, and switching any number of components in the product. For any two tensor A and B we have $|A * B| \leq C|A||B|$ for some constant C which will depend neither on A nor B.

Lemma 6.2. *There exists a neighborhood V of g and a positive constant C_1 such that for any $\tilde{g} \in V$,*

$$|\tilde{\mathcal{R}}_p(\tilde{g}) - \tilde{\mathcal{R}}_p(g)| \leq C_1 \|\tilde{g} - g\|^2_{C^{2,\alpha}}. \tag{6.1}$$

Proof. Let $\tilde{g} = g + \theta$ and T be a tensor. We have the following relation between the connection of g and $\tilde{g}$:

$$D_{g+\theta} T = D_g T + (g+\theta)^{-1} * D_g \theta * T. \tag{6.2}$$

The curvature of g and $\tilde{g}$ related by

$$R(g+\theta) = R(g) + (g+\theta)^{-1} * D^2\theta + (g+\theta)^{-2} * (D\theta * D\theta). \tag{6.3}$$

We also have the following formula:

$$(g+\theta)^{-1} - g^{-1} = -g^{-1}(g+\theta)^{-1}\theta. \tag{6.4}$$

The lemma follows by using some standard techniques and the above equations. □

Lemma 6.3. *Let g be a Riemannian metric on M with unit volume. There exists a neighborhood U of g in $\mathcal{M}_1$ such that for any $\tilde{g} \in U$ and $h \in \mathcal{W}$,*

$$|\tilde{H}_{\tilde{g}}(h,h) - \tilde{H}_g(h,h)| \leq C\|\tilde{g} - g\|^4_{C^{2,\alpha}} \|h\|_{L^{2,2}}.$$

Proof. By a straightforward computation we have

$$\begin{aligned} \tilde{H}_g &= -2\langle \nabla\tilde{\mathcal{R}}_p, h \circ h\rangle_g + \langle (\nabla\tilde{\mathcal{R}}_p)'(h), h\rangle_g \\ &= 2[p\langle |R|^{p-2}R, Dd^D(h \circ h)\rangle + p\langle |R|^{p-2}\tilde{\mathcal{R}}_p, h \circ h\rangle - \frac{1}{2}\langle |R|^p, |h|^2\rangle] \\ &\quad +\langle (\nabla\mathcal{R}_p)'(h), h\rangle - \left(\frac{p}{n} - \frac{1}{2}\right)\mathcal{R}_p(g)\|h\|^2. \end{aligned}$$

We observe from the expression of $\tilde{H}$ that $\tilde{H}(g) = \int_M f|R|^{p-2} dv_g$, where $f \in C^\infty(M)$ and $\int_M f dv_g$ is the second derivative of $\tilde{\mathcal{R}}_2$. Using the previous lemma it is sufficient to prove the lemma for the second derivative of $\tilde{\mathcal{R}}_2$.

Suppose $\tilde{H}$ denotes the second derivative of $\tilde{\mathcal{R}}_2$. We have

$$\begin{aligned} &(R, Dd^D(h \circ h)) = g^{-1} * g^{-1} * g^{-1} * g^{-1} * R * (D^2h + Dh * Dh), \\ &(\check{R}, h \circ h) = g^{-1} * g^{-1} * g^{-1} * g^{-1} * R * R, \\ &(\bar{R}_h, Dd^D h) = g^{-1} * g^{-1} * g^{-1} * g^{-1}(D^2h * D^2h + h * R), \\ &\langle W_h, d^D h\rangle = \int_M (g^{-1} * g^{-1} * g^{-1} * g^{-1} * R * Dh * Dh) dv_g, \\ &((\check{R})'(h), h) = g^{-1} * g^{-1} * g^{-1} * g^{-1} * R * h * (R * h + D^2h), \\ &(|R|^p)'(h) = |R|^{p-2} * g^{-1} * g^{-1} * g^{-1} * g^{-1} * (R * D^2h + R * R * h), \\ &\langle (\delta^D)'(h) D^*(R) = g^{-1} * g^{-1} * g^{-1} * g^{-1} * d^2h * h * R. \end{aligned}$$

Combining the above equations we obtain the required result. □

Proof of Proposition 3. Choose a neighborhood U of g in $C^{2,\alpha}$-topology such that the following conditions hold:

(i) Lemmas 6.1 and 6.3 hold on U.

(ii) Let $\tilde{g} = g + \theta_1 \in U$. Then using Lemma 6.1 we have $\tilde{\theta}$ satisfying the conditions given in Lemma 6.1. We can assume $g + t\tilde{\theta} \in U$ for all $t \in [0, 1]$.
(iii) Since g is $L^{2,2}$-stable, we can assume that for any $\tilde{g} \in U$ with $V(\tilde{g}) = V(g)$, $\tilde{H}_g(h, h) > 0$ for all $h \in \mathcal{W}$.

We have

$$\tilde{\mathcal{R}}_p(g + \tilde{\theta}) = \tilde{\mathcal{R}}_p(e^c \phi^* \tilde{g}) = \tilde{\mathcal{R}}_p(\phi^* \tilde{g}) = \tilde{\mathcal{R}}_p(\tilde{g}) = \tilde{\mathcal{R}}_p(g + \theta_1).$$

Define

$$\gamma(t) = g + t\tilde{\theta},$$

$\gamma(t) \in U$ for $t \in [0, 1]$. Let

$$a(t) = \tilde{\mathcal{R}}_p(\gamma(t)).$$

Then $a(0) = \tilde{\mathcal{R}}_p(g)$, $a(1) = \tilde{\mathcal{R}}_p(g + \tilde{\theta})$ and $a'(0) = 0$. Since $\tilde{\theta} \in \mathcal{W}$,

$$a''(t) = \tilde{H}_{\gamma(t)}(\tilde{\theta}, \tilde{\theta}) > 0.$$

Therefore,

$$a(1) - a(0) = \int_0^1 \int_0^1 a''(st) ds dt > 0.$$

If $\tilde{\mathcal{R}}_p(\tilde{g}) = \tilde{\mathcal{R}}_p(g)$, then $\tilde{\theta} = 0$. Hence $\tilde{g}$ is isometric to g. This completes the proof. □

The following corollary is an immediate consequence of this proposition.

COROLLARY 6.1

Let (M, g) be a closed Riemannian manifold with dimension $n \geq 3$. If (M, g) is one of the following then g is a strict local minimizer for $\mathcal{R}_p$ for the indicated values of p:

(i) *A spherical space form and $p \in [2, \infty)$.*
(ii) *A hyperbolic manifold and $p \in [\frac{n}{2}, \infty)$.*
(iii) *A product of spherical space forms and $p \in [2, n]$.*
(iv) *A product of hyperbolic manifolds and $p \in [\frac{n}{2}, n]$.*

Proof. In light of Proposition 3, it is sufficient to prove that (M, g) is $L^{2,2}$-stable. Define

$$\|h\|_1^2 = \|D^* Dh\|^2 + \|Dh\|^2 + \|h\|^2.$$

From the proof of Theorem 1.1, we have that there exists a positive constant k such that $H(h, h) \geq k\|h\|_1^2$ for all $h \in \mathcal{W}$. When (M, g) has unit volume one can easily check that $\tilde{H}(h, h) = H(h, h)$. Hence to prove the corollary it is sufficient to prove that $\|.\|_{L^{2,2}}$-norm and $\|.\|_1$-norm are equivalent.

Since M is compact and $D^* D$ is an elliptic operator using elliptic estimate, we have $C > 0$ such that

$$\|h\|_{L^{2,2}}^2 \leq C[\|D^* Dh\|^2 + \|h\|^2].$$

Therefore, $\|h\|^2_{L^{2,2}} \leq C\|h\|^2_1$. Since at every point $|D^2h| > |D^*Dh|$ we have $\|h\|^2_1 \leq \|h\|^2_{L^{2,2}}$. Hence, the proof follows. □

As a consequence, we have the following:

COROLLARY 6.2

Let (M, g) be a spherical space form or product of spherical space forms. There exists a neighborhood $\mathcal{U}$ of g in $\mathcal{M}$ such that for every $g_0 \in \mathcal{U}$,

(i) *If $\mathcal{R}_p(g_0) < \mathcal{R}_p(g)$ for any $p > \frac{n}{2}$, then $V(g_0) > V(g)$.*
(ii) *If $\mathcal{R}_p(g_0) < \mathcal{R}_p(g)$ for any $p \in [2, \frac{n}{2})$, then $V(g_0) < V(g)$.*
(iii) *If $\mathcal{R}_p(g_0) \geq \mathcal{R}_p(g)$ for any $p \in [2, \infty)$ and $V(g_0) = V(g)$, then g_0 is isometric to g.*

COROLLARY 6.3

Let (M, g) be a compact hyperbolic manifold or product of compact hyperbolic manifolds. There exists a neighborhood $\mathcal{V}$ of g in $\mathcal{M}$ such that for every $g_1 \in \mathcal{V}$,

(i) *If $\mathcal{R}_p(g_1) < \mathcal{R}_p(g)$ for any $p \in (\frac{n}{2}, n)$, then $V(g_1) > V(g)$.*
(ii) *If $\mathcal{R}_p(g_1) \geq \mathcal{R}_p(g)$ for any $p \in [\frac{n}{2}, n]$ and $V(g_1) = V(g)$, then g_1 is isometric to g.*

Remark 6.2. Consider the Lie group $SU(2)$ with bi-invariant metric g which is isometric to the standard sphere S^3. Let $\tilde{g}(t)$, $t > 0$ denote the volume normalized Berger's collapsing metrics on $SU(2)$. Suppose $\tilde{\mathcal{R}}_p(t)$ is the restriction of $\tilde{\mathcal{R}}_p$ on $\tilde{g}(t)$. Since $\tilde{\mathcal{R}}_p(t) \to 0$ as $t \to 0$ and $\tilde{\mathcal{R}}_p(t)$ has a minima at $\tilde{g}(1)$, $\tilde{\mathcal{R}}_p(t)$ has a maxima $\tilde{g}(t_o)$ for some t_o between 0 and 1. $\tilde{g}(t_o)$ is precisely the critical metric for $\tilde{\mathcal{R}}_p$ which is exhibited by Lamontagne in [8] for $p = 2$.

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References

[1] Anderson M T, Degeneration of metrics with bounded curvature and applications to critical metrics of Riemannian functional, *Proc. Symposia in Pure Math.* **54(Part 3)** (1993) 53–79
[2] Anderson M T, Extrema of curvature functionals on the space of metrics on 3-manifolds, *Calc. Var. Partial Differential Equations* **5(3)** (1997) 199–269
[3] Besse A L, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)] (1987) (Berlin: Springer-Verlag) vol. 10
[4] Besson G, Courtois G and Gallot S, Volume et entropie minimale des espaces localement symétriques, *Invent. Math.* **103** (1991) 417–445
[5] Berger M and Ebin D G, Some decompositions of the space of symmetric tensors on a riemannian manifold, *J. Differential Geometry* **3** (1969) 379–392

[6] Berger M, Quelques formules de variation pour une structure riemannienne, *Ann. Sci. Ecole Norm. Sup. 4^e série* **3** (1970) 285–294

[7] Gursky M J and Viaclovsky J A, Rigidity and stability of Einstein metrics for quadratic functionals, arXiv:1105.4648v1 [math.DG] 23 May 2011

[8] Lamontagne F, A critical metric for the L^2-norms of the curvature tensor on S^3, *Proc. Amer. Math. Soc.* **126(2)** (1998) 589–593

[9] Muto Y, Curvature and critical Riemannian metrics, *J. Math. Sci. Japan* **26** (1974) 686–697