Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 3, August 2014, pp. 349–364.

Time-periodic solution of a 2D fourth-order nonlinear parabolic equation

XIAOPENG ZHAO[1] and CHANGCHUN LIU[2]

[1]School of Science, Jiangnan University, Wuxi 214122, People's Republic of China
[2]Department of Mathematics, Jilin University, Changchun 130012, People's Republic of China
E-mail: zxp032@gmail.com; liucc@jlu.edu.cn

MS received 31 August 2012; revised 10 December 2013

Abstract. By using the Galerkin method, we study the the existence and uniqueness of time-periodic generalized solutions and time-periodic classical solutions to a fourth-order nonlinear parabolic equation in 2D case.

Keywords. Time-periodic solution; 2D fourth-order nonlinear parabolic equation; Galerkin method.

Mathematics Subject Classification. 35K55, 35B10, 35K35.

1. Introduction

In the study of epitaxial thin film growth, there arises the following diffusion equation

$$u_t = -\nabla \cdot j + f(x,t),$$

where $u(x,t)$ denotes the height of a film in epitaxial growth, $j = j(x,t)$ comprises all processes which move atoms along the surface, and $f(x,t)$ is some Gaussian noise. Taking $j = \gamma\nabla\Delta u - |\nabla u|^2\nabla u$, we obtain the well-known nonlinear parabolic equation (see [5, 11, 16, 19, 20])

$$\frac{\partial u}{\partial t} + \gamma\Delta^2 u - \operatorname{div}(|\nabla u|^2\nabla u) = f(x,t), \quad \text{in } \Omega \times \mathbf{R}, \tag{1.1}$$

where $\gamma > 0$ is a constant and $\Omega = (0,L)\times(0,L)$ is a two-dimensional bounded domain. Here, $u(x,t)$ denotes the height from the surface of the film in epitaxial growth, $\Delta^2 u$ denotes the capillarity-driven surface diffusion, and $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ corresponds to the hopping of atoms.

In [5], King *et al.* studied the following equation:

$$u_t + \Delta^2 u - \operatorname{div}[f(\nabla u)] = h(x).$$

In their paper, they proved the existence, uniqueness and regularity of solutions in an appropriate function space for the initial boundary value problem.

Kohn and Yan [6] considered the following fourth-order parabolic equation

$$u_t + \Delta^2 u + \nabla\cdot(2(1-|\nabla u|^2)\nabla u) = 0,$$

which is a specific example of energy-driven coarsening in two-space dimensions. Numerical simulations and heuristic arguments indicate that the standard deviation of u grows as $t^{-\frac{1}{3}}$, and the energy per unit area decays like $t^{-\frac{1}{3}}$. The authors proved that the time-averaged energy per unit area decays as faster than $t^{-\frac{1}{3}}$.

Recently, Liu [10, 11] studied the following equation:

$$u_t + \operatorname{div}\,[m(u)k\nabla\Delta u - |\nabla u|^{p-2}\nabla u] = 0,$$

for 1-D and 2-D case. On the basis of uniform Schauder type estimate and Campanato spaces, he proved the global existence of classical solutions.

In [20], Zhao and Liu considered the existence of global attractor for eq. (1.1) in a bounded domain $\Omega \subset \mathbf{R}^2$. Based on the regularity estimates for the semigroups and the classical existence theorem of global attractors, the authors proved that the equation possesses a global attractor in H^k $(0 \le k < 5)$ space, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

On the other hand, by using spectral method, Xu and Tang [16] studied eq. (1.1) with periodic boundary condition. In their paper, they constructed and analysed highly stable time discretizations which allow much larger time steps than those of a standard implicit–explicit approach. They also added an extra term (which is consistent with the order of time discretization) to stabilize the numerical schemes.

Furthermore, several authors have paid attention to the time-periodic problems [3, 7, 12, 13]. But, to the best of our knowledge, only a few papers deal with time periodic solutions of fourth-order diffusion equations. In [9, 17], the existence of time periodic solutions for the Cahn–Hilliard type equation and viscous Cahn–Hilliard equation with periodic concentration-dependent potentials and sources have been investigated. In [14, 15], Wang *et al.* considered the existence and uniqueness of time-periodic generalized solutions and time-periodic classical solutions to the generalized Ginzburg–Landau model equation in 1D and 2D cases. In [4], by using the Galerkin method and the Leray–Schauder fixed point theorem, Fu and Guo studied the existence and uniqueness of a time periodic solution for the viscous Camassa–Holm equation. There are also many papers devoted to the periodic problems, for example [8, 18] and so on.

Here, we investigate eq. (1.1) together with

$$u(\cdot, t) \text{ is } L \text{ periodic}, \quad t \in \mathbf{R}, \tag{1.2}$$

and the time-periodic condition

$$u(x, t + \omega) = u(x, t), \quad x \in \Omega,\ t \in \mathbf{R}, \tag{1.3}$$

where $\omega > 0$ is a constant and $f(x, t)$ is a ω-periodic function with respect to time t, which also satisfies

$$\int_\Omega f(x, t)\mathrm{d}x = 0.$$

Our main purpose is to obtain the existence and uniqueness of time-periodic generalized solutions and time-periodic classical solutions to problems (1.1)–(1.3). The proof is at its core based on a Leray–Schauder fixed point argument which is first applied to a spatially discretized version of problems (1.1)–(1.3) through a standard Galerkin procedure.

Throughout this paper, we use the following notations.

Let X be a Banach space, $C^k_\omega(\mathbf{R}; X)$ denotes the set of X-valued ω-periodic functions on $\mathbf{R}$ with continuous derivatives up to order k. The norm in $C^k_\omega(\mathbf{R}; X)$ is defined as

$$\|u\|_{C^k_\omega(\mathbf{R};X)} = \sup_{0\le t\le\omega}\left\{\sum_{i=0}^{k}\|D^i_t u\|_X\right\},$$

where $D_t = \frac{\partial}{\partial t}$, $\|\cdot\|_X$ is the norm in X. We also define $L^p_\omega(\mathbf{R}; X)$ $(1 \le p \le \infty)$ as the set of ω-periodic X-valued measurable functions on $\mathbf{R}$ such that

$$\|u\|_{L^p_\omega(\mathbf{R};X)} = \left(\int_0^\omega \|u\|^p_X \mathrm{d}t\right)^{\frac{1}{p}} < \infty, \quad \text{where } 1 \le p < \infty,$$
$$\|u\|_{L^p_\omega(\mathbf{R};X)} = \operatorname{ess}\sup_{0\le t\le\omega} \|u\|_X < \infty, \quad \text{where } p = \infty.$$

Let $W^{k,p}_\omega(\mathbf{R}; X)$ denote the set of functions which belong to $L^p_\omega(\mathbf{R}; X)$ together with their partial derivatives with respect to t up to order k.

In the following, we frequently use the Poincaré inequality (see [2]):

$$\|u\|^2 \le C^*\|\nabla u\|^2, \quad \text{where } \int_\Omega u(x,t)\mathrm{d}x = 0,$$

where C^* is a positive constant which depends only on the domain. For simplicity, denote $\|\cdot\|_{L^2(0,1)}$ by $\|\cdot\|$, $\|\cdot\|_{L^\infty(0,1)}$ by $\|\cdot\|_\infty$, $\|\cdot\|_{L^p(0,1)}$ by $\|\cdot\|_p$ and $\|\cdot\|_{H^m(0,1)}$ by $\|\cdot\|_{H^m}$, respectively.

2. Integration estimations and existence of the approximate solutions for problems (1.1)–(1.3)

Let $\{y_j(x)\}(j = 1, 2, \ldots)$ be the orthonormal base in $L^2(\Omega)$ being composed of the eigenfunctions of the eigenvalue problem

$$\Delta y + \lambda y = 0, \quad y \text{ is } L \text{ periodic},$$

corresponding to eigenvalues λ_j $(j = 1, 2, \ldots)$.

Suppose $u_N(x,t) = \sum_{j=1}^N u_{Nj}(t)y_j(x)$ is the Galerkin approximate solution to problems (1.1)–(1.3), where $u_{Nj}(t)(j = 1, 2, \ldots, N)$ is the undermined function and N is a natural number.

Performing the Galerkin procedure for eq. (1.1), we obtain

$$\begin{cases} (u_{Nt} + \gamma\Delta^2 u_N - \operatorname{div}[|\nabla u_N|^2\nabla u_N], y_j) = (f, y_j), \\ (u_N(\cdot, t+\omega), y_j) = (u_N(\cdot, t), y_j), \quad j = 1, 2, \ldots, N. \end{cases} \tag{2.1}$$

In order to use Leray–Schauder fixed-point theorem to prove the existence of the solutions $u_{Nj}(t)$ to the problem (2.1), we consider the following time-periodic problem of the system of ordinary differential equations with the parameter $\theta \in (0, 1)$,

$$\begin{cases} (u_{Nt} + \gamma\Delta^2 u_N, y_j) = \theta(\operatorname{div}[|\nabla u_N|^2\nabla u_N] + f, y_j), \\ (u_N(\cdot, t+\omega), y_j) = (u_N(\cdot, t), y_j), \quad j = 1, 2, \ldots, N. \end{cases} \tag{2.2}$$

Lemma 2.1. *Suppose that* $f \in C_\omega(\mathbf{R}; H^1_{per}(\Omega))$, $M_1 = \sup_{0\le t\le\omega}\|f(\cdot,t)\|_{H^1}$. *Then, there exists a solution* $(u_{Nj}(t))_{j=1}^N \in C^1_\omega[0,\omega]$ *for problem* (2.2), *where*

$$C^1_\omega[0,\omega] = \{r(t)|r(t+\omega) = r(t),\ \forall t\in\mathbf{R},\ \ r(t)\in C^1[0,\omega]\}.$$

In the meantime, the approximate solution $u_N(x,t)$ *has the estimation*

$$\sup_{0\le t\le\omega}\|u_N(\cdot,t)\|^2_{H^1} \le c_0(M_1),$$

where c_0 *is a positive constant independent of* N.

Proof. Multiplying both sides of eq. (2.2) by $u_{Nj}(t)$, and summing up the products over $j = 1,2,\dots,N$, we derive that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|^2 + \gamma\|\Delta u_N\|^2 + \theta\int_\Omega |\nabla u_N|^4\mathrm{d}x = \theta\int_\Omega f u_N\mathrm{d}x.$$

We also have

$$\|u_N\|^2 \le C^*\|\nabla u_N\|^2 = -C^*(u_N,\Delta u_N) \le \frac{1}{2}\|u_N\|^2 + \frac{C^*}{2}\|\Delta u_N\|^2.$$

Then

$$\|u_N\|^2 \le (C^*)^2\|\Delta u_N\|^2, \qquad \|\nabla u_N\|^2 \le C^*\|\Delta u_N\|^2. \tag{2.3}$$

By Hölder's inequality and (2.3), we get

$$\theta\int_\Omega f u_N\mathrm{d}x \le \frac{\gamma}{2(C^*)^2}\|u_N\|^2 + \frac{\theta^2(C^*)^2}{2\gamma}\|f\|^2 \le \frac{\gamma}{2}\|\Delta u_N\|^2 + \frac{(C^*)^2}{2\gamma}M_1^2.$$

Summing up, we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|^2 + \gamma\|\Delta u_N\|^2 \le \frac{(C^*)^2}{\gamma}M_1^2. \tag{2.4}$$

Integrating (2.4) over $[0,\omega]$, we obtain

$$\int_0^\omega \|\Delta u_N(\cdot,t)\|^2\mathrm{d}t \le \frac{(C^*)^2}{\gamma^2}M_1^2\omega. \tag{2.5}$$

It then follows from (2.5) that there exists a time $t_1 \in (0,\omega)$ such that

$$\|\Delta u_N(\cdot,t_1)\|^2 \le \frac{(C^*)^2}{\gamma^2}M_1^2. \tag{2.6}$$

Applying (2.3), it follows from (2.6) that

$$\|u_N(\cdot,t_1)\|^2 \le \frac{(C^*)^4}{\gamma^2}M_1^2. \tag{2.7}$$

Integrating (2.4) again over $[t_1, t+\omega](\forall t\in[0,\omega])$ and using (2.7), we deduce that

$$\sup_{0\le t\le\omega}\|u_N(\cdot,t)\|^2 \ \le\ \|u_N(\cdot,t_1)\|^2 + \frac{2\omega}{\gamma}(C^*)^2M_1^2$$

$$\leq \frac{1}{\gamma}\left[\frac{(C^*)^4}{\gamma} + 2\omega(C^*)^2\right] M_1^2. \tag{2.8}$$

Multiplying both sides of eq. (2.2) by $\lambda_j u_{Nj}(t)$, and summing up the products over $j = 1, 2, \ldots, N$, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_N\|^2 + \gamma\|\nabla\Delta u_N\|^2 \\
&\quad = -\theta\int_\Omega \nabla\cdot(|\nabla u_N|^2\nabla u_N)\Delta u_N \mathrm{d}x - \theta\int_\Omega \nabla f\cdot\nabla u_N \mathrm{d}x.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\int_\Omega \nabla\cdot[|\nabla u_N|^2\nabla u_N]\Delta u_N \mathrm{d}x \\
&\quad = \int_\Omega |\nabla u_N|^2|\Delta u_N|^2 \mathrm{d}x + 2\sum_{i=1}^{2}\int_\Omega u_{Ni}^2 u_{Nii}^2 \mathrm{d}x \\
&\qquad + 2\int_\Omega (u_{N1}^2 + u_{N2}^2)u_{N11}u_{N22}\mathrm{d}x + 4\int_\Omega \Delta u_N u_{N12}u_{N1}u_{N2}\mathrm{d}x \\
&\quad = \int_\Omega |\nabla u_N|^2|\Delta u_N|^2 \mathrm{d}x + 2\sum_{i=1}^{2}\int_\Omega u_{Ni}^2 u_{Nii}^2 \mathrm{d}x \\
&\qquad + 2\int_\Omega |\nabla u_N|^2 u_{N11}u_{N22}\mathrm{d}x + 4\int_\Omega \Delta u_N (u_{N12}u_{N1}u_{N2})\mathrm{d}x \\
&\quad \geq \int_\Omega |\nabla u_N|^{p-2}|\Delta u_N|^2 \mathrm{d}x + \sum_{i=1}^{2}\int_\Omega u_{Ni}^2 u_{Nii}^2 \mathrm{d}x \\
&\qquad - \int_\Omega |\nabla u_N|^{p-2}|\Delta u_N|^2 \mathrm{d}x \geq 0.
\end{aligned}$$

By Nirenberg's inequality, we deduce that

$$\|\nabla u_N\|^2 \leq (c_1'\|\nabla\Delta u_N\|^{\frac{1}{3}}\|u_N\|^{\frac{2}{3}} + c_2'\|u_N\|)^2 \leq \frac{\gamma}{2}\|\nabla\Delta u_N\|^2 + c_1(M_2). \tag{2.9}$$

Summing up the above three inequalities, using Hölder's inequality and noticing that $\theta \in (0, 1)$, we immediately conclude that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_N\|^2 + \gamma\|\nabla\Delta u_N\|^2 &\leq \frac{1}{4}\|\nabla f\|^2 + \|\nabla u_N\|^2 \\
&\leq \frac{M_2^2}{4} + \frac{\gamma}{2}\|\nabla\Delta u_N\|^2 + c_1(M_2),
\end{aligned}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_N\|^2 + \gamma\|\nabla\Delta u_N\|^2 \leq \frac{M_2^2}{2} + 2c_1(M_2). \tag{2.10}$$

Integrating (2.10) over $[0, \omega]$, we have

$$\int_0^\omega \|\nabla\Delta u_N\|^2 \mathrm{d}t \leq \left[\frac{M_2^2}{2\gamma} + \frac{2}{\gamma}c_1(M_2)\right]\omega. \tag{2.11}$$

It then follows from (2.11) that there exists a time $t_2 \in (0, \omega)$ such that

$$\|\nabla\Delta u(\cdot, t_2)\|^2 \le \frac{M_2^2}{2\gamma} + \frac{2}{\gamma}c_1(M_2). \tag{2.12}$$

Applying (2.9), it follows from (2.12) that

$$\|\nabla u_N(\cdot, t_2)\|^2 \le \frac{\gamma}{2}\left[\frac{M_2^2}{2\gamma} + \frac{2}{\gamma}c_1(M_2)\right] + c_1(M_2). \tag{2.13}$$

Integrating (2.10) again over $[t_2, t+\omega](\forall t \in [0, \omega])$ and using (2.13), we deduce that

$$\sup_{0\le t\le\omega} \|\nabla u_N(\cdot, t)\|^2 \le \|\nabla u_N(\cdot, t_2)\|^2 + (M_2^2 + 4c_1(M_2)]\omega \le c_2(M_2). \tag{2.14}$$

Adding (2.8) and (2.14), we complete the proof of Lemma 2.1. □

Employing the Leray–Schauder fixed-point argument and using the standard method as in [21], we can prove that there exists at least one solution $(u_{Nj}(t))_{j=1}^N \in C_\omega[0, \omega]$ for problem (2.2). In the case of $\theta = 1$, the solution to problem (2.2) is the solution of problem (2.1). By the linear ODE theory, we get $(u_{Nj}(t))_{j=1}^N \in C_\omega^1[0, \omega]$.

Lemma 2.2. *Suppose that the assumptions of Lemma* 2.1 *hold and function* $f(x,t)$ *satisfies* $f \in C_\omega(\mathbf{R}; H^3_{per}(\Omega))$, $f_t \in C_\omega(\mathbf{R}; H^1_{per}(\Omega))$, *then*

$$\sup_{0\le t\le\omega} (\|u_{Nt}(\cdot, t)\|^2_{H^3} + \|u_N(\cdot, t)\|^2_{H^5}) \le c_2(M_2),$$

where $M_2 = \sup_{0\le t\le\omega}(\|f\|_{H^3} + \|f_t\|_{H^1})$ *is sufficiently small. Here and in the sequel,* $c_i(M_2)(i = 1, 2, \ldots)$ *is nondecreasing with respect to* M_2 *and* $\lim_{M_2\to 0} c_i(M_2) = c_i(0) = 0$, $c_i(M_2)$ *is independent of* N.

Proof. Multiplying both sides of eq. (2.1) by $\lambda_j^2 u_{Nj}(t)$, and summing up the products over $j = 1, 2, \ldots, N$, we derive that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_N\|^2 + \gamma\|\Delta^2 u_N\|^2 - \int_\Omega \nabla\cdot(|\nabla u_N|^2\nabla u_N)\Delta^2 u_N \mathrm{d}x = \int_\Omega f\Delta^2 u_N \mathrm{d}x.$$

Using Nirenberg's inequality, we get

$$\|\nabla u_N\|_8^8 \le (c_1'\|\Delta^2 u_N\|^{\frac14}\|\nabla u_N\|^{\frac34} + c_2'\|\nabla u_N\|)^8 \le \varepsilon\|\Delta^2 u_N\|^2 + c_3(M_2),$$
$$\|\Delta u_N\|_4^4 \le (c_1'\|\Delta^2 u_N\|^{\frac12}\|\nabla u_N\|^{\frac12} + c_2'\|\nabla u_N\|)^4 \le \varepsilon\|\Delta^2 u_N\|^2 + c_5(M_2).$$

Hence

$$\begin{aligned}
&-\int_\Omega \nabla\cdot(|\nabla u_N|^2\nabla u_N)\Delta^2 u_N \mathrm{d}x \\
&\qquad = -3\int_\Omega |\nabla u_N|^2\Delta u_N\Delta^2 u_N \mathrm{d}x \le \frac{\gamma}{4}\|\Delta^2 u_N\|^2 \\
&\qquad\quad + \frac{9}{2\gamma}\|\nabla u_N\|_8^8 + \frac{9}{2\gamma}\|\Delta u_N\|_4^4 \\
&\qquad \le \frac{\gamma}{4}\|\Delta^2 u_N\|^2 + \frac{9}{\gamma}\varepsilon\|\Delta^2 u_N\|^2 + \frac{9}{2\gamma}(c_3(M_2) + c_5(M_2)).
\end{aligned}$$

By Hölder's inequality, we obtain

$$\int_\Omega f\Delta u_N \mathrm{d}x \le \frac{\gamma}{4}\|\Delta^2 u_N\|^2 + \frac{1}{\gamma}\|f\|^2 \le \frac{\gamma}{4}\|\Delta^2 u_N\|^2 + \frac{M_2^2}{\gamma}.$$

Summing up, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_N\|^2 + c_6\|\Delta^2 u_N\|^2 \le c_7(M_2), \tag{2.15}$$

where ε is sufficiently small, it satisfies $c_6 = \gamma - \frac{18}{\gamma}\varepsilon > 0$.

Integrating (2.15) over $[0, \omega]$, we have

$$\int_0^\omega \|\Delta^2 u_N\|^2 \mathrm{d}t \le \frac{c_7(M_2)}{c_6}\omega. \tag{2.16}$$

It then follows from (2.16) that there exists a time $t_3 \in (0, \omega)$ such that

$$\|\Delta^2 u(\cdot, t_3)\|^2 \le \frac{c_7(M_2)}{c_6}. \tag{2.17}$$

On the other hand, we have

$$\|\Delta u_N(\cdot, t_3)\|^2 \le \frac{1}{2}\|\Delta^2 u_N(\cdot, t_3)\|^2 + \frac{1}{2}\|u_N\|^2 \le c_8(M_2). \tag{2.18}$$

Integrating (2.15) again over $[t_3, t+\omega](\forall t \in [0, \omega])$ and using (2.18), we deduce that

$$\sup_{0\le t\le\omega} \|\Delta u_N(\cdot, t)\|^2 \le \|\Delta u_N(\cdot, t_3)\|^2 + 2c_7(M_2)\omega \le c_9(M_2). \tag{2.19}$$

Based on Sobolev's embedding theorem, we have

$$\|u_N(\cdot, t)\|_{C(\bar{\Omega})} \le c\|u_N(\cdot, t)\|_{H^2} \le c_{10}(M_2). \tag{2.20}$$

Multiplying both sides of eq. (2.1) by $\lambda^3 u_{Nj}(t)$, and summing up the products over $j = 1, 2, \ldots, N$, we derive that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u_N\|^2 + \gamma\|\nabla\Delta^2 u_N\|^2 \;&=\; \int_\Omega \nabla\Delta^2 u_N \Delta(|\nabla u_N|^2\nabla u_N)\mathrm{d}x \\ &\quad + \int_\Omega \nabla f \nabla \Delta u_N^2 \mathrm{d}x. \end{aligned}$$

Using Nirenberg's inequality, we deduce that

$$\begin{aligned} \|\nabla u_N\|_4^4 &\le (c_1'\|\nabla\Delta^2 u_N\|^{\frac{1}{8}}\|\nabla u_N\|^{\frac{7}{8}} + c_2'\|\nabla u_N\|)^4 \\ &\le \varepsilon\|\nabla\Delta^2 u_N\|^2 + c_{11}(M_2), \\ \|\Delta u_N\|_8^8 &\le (c_1'\|\nabla\Delta^2 u_N\|^{\frac{1}{4}}\|\Delta u_N\|^{\frac{3}{4}} + c_2'\|\Delta u_N\|)^8 \\ &\le \varepsilon\|\nabla\Delta^2 u_N\|^2 + c_{12}(M_2), \\ \|\nabla u_N\|_8^8 &\le (c_1'\|\nabla\Delta^2 u_N\|^{\frac{3}{16}}\|\Delta u_N\|^{\frac{13}{16}} + c_2'\|\Delta u_N\|)^8 \\ &\le \varepsilon\|\nabla\Delta^2 u_N\|^2 + c_{13}(M_2), \\ \|\nabla\Delta u_N\|_4^4 &\le (c_1'\|\nabla\Delta^2 u_N\|^{\frac{1}{2}}\|\Delta u_N\|^{\frac{1}{2}} + c_2'\|\Delta u_N\|)^8 \\ &\le \varepsilon\|\nabla\Delta^2 u_N\|^2 + c_{15}(M_2), \end{aligned}$$

where ε is an arbitrary positive constant. Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u_N\|^2+\gamma\|\nabla\Delta^2 u_N\|^2\\
&\quad\le \frac{\gamma}{2}\|\nabla\Delta^2 u_N\|^2+\frac{1}{\gamma}\int_\Omega(\Delta(|\nabla u_N|^2\nabla u_N))^2\mathrm{d}x+\frac{1}{\gamma}\|\nabla f\|^2\\
&\quad\le \frac{\gamma}{2}\|\nabla\Delta^2 u_N\|^2+\frac{18}{\gamma}\int_\Omega|\nabla u_N|^4|\nabla\Delta u_N|^2\mathrm{d}x\\
&\qquad+\frac{72}{\gamma}\int_\Omega|\nabla u_N|^2|\Delta u_N|^4\mathrm{d}x+\frac{1}{\gamma}\|\nabla f\|^2\\
&\quad\le \frac{\gamma}{2}\|\nabla\Delta u_N\|^2+\frac{9}{\gamma}\|\nabla u_N\|_8^8+\frac{9}{\gamma}\|\nabla\Delta u_N\|_4^4\\
&\qquad+\frac{36}{\gamma}\|\nabla u_N\|_4^4+\frac{36}{\gamma}\|\Delta u_N\|_8^8+\frac{M_2^2}{\gamma}\\
&\quad\le c_{16}\|\nabla\Delta^2 u_N\|^2+c_{17}(M_2),
\end{aligned}$$

where $c_{16}=\frac{\gamma}{2}+\frac{90}{\gamma}\varepsilon$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u_N\|^2+2(\gamma-c_{16})\|\nabla\Delta^2 u_N\|^2\le 2c_{17}(M_2), \tag{2.21}$$

where ε is sufficiently small, it satisfies $\gamma-c_{16}=\frac{\gamma}{2}-\frac{90}{\gamma}\varepsilon>0$. Integrating (2.21) over $[0,\omega]$, we have

$$\int_0^\omega\|\Delta^2 u_N\|^2\mathrm{d}t\le\frac{c_{17}(M_2)}{\gamma-c_{16}}\omega. \tag{2.22}$$

It then follows from (2.22) that there exists a time $t_4\in(0,\omega)$ such that

$$\|\nabla\Delta^2 u(\cdot,t_4)\|^2\le\frac{c_{17}(M_2)}{\gamma-c_{16}}. \tag{2.23}$$

On the other hand, we have

$$\|\nabla\Delta u_N(\cdot,t_4)\|^2\le\frac{1}{2}\|\nabla\Delta^2 u_N(\cdot,t_4)\|^2+\frac{1}{2}\|\nabla u_N\|^2\le c_{18}(M_2). \tag{2.24}$$

Integrating (2.21) again over $[t_4,t+\omega](\forall t\in[0,\omega])$ and using (2.24), we deduce that

$$\sup_{0\le t\le\omega}\|\nabla\Delta u_N(\cdot,t)\|^2\le\|\nabla\Delta u_N(\cdot,t_4)\|^2+4c_{17}(M_2)\omega\le c_{19}(M_2). \tag{2.25}$$

Based on Sobolev' embedding theorem, we have

$$\|u_N(\cdot,t)\|_{C^1(\bar{\Omega})}\le c\|u_N(\cdot,t)\|_{H^3}\le c_{20}(M_2). \tag{2.26}$$

Multiplying both sides of eq. (2.1) by $\lambda_j^4u_{Nj}(t)$, and summing up the products over $j=1,2,\ldots,N$, we derive that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^2 u_N\|^2+\gamma\|\Delta^3 u_N\|^2=&\int_\Omega\nabla\cdot\Delta(|\nabla u_N|^2\nabla u_N)\Delta^3 u_N\mathrm{d}x\\
&+\int_\Omega\Delta f\Delta^3 u_N\mathrm{d}x.
\end{aligned}$$

By Nirenberg's inequality, we conclude that

$$\|\Delta^2 u_N\|^2 \leq (c_1'\|\Delta^3 u_N\|^{\frac{1}{3}}\|\nabla\Delta u_N\|^{\frac{2}{3}} + c_2'\|\nabla\Delta u_N\|)^2 \leq \varepsilon\|\Delta^3 u_N\|^2 + c_{21}(M_2),$$
$$\|\Delta u_N\|_6^6 \leq (c_1'\|\Delta^3 u_N\|^{\frac{1}{6}}\|\Delta u_N\|^{\frac{5}{6}} + c_2'\|\Delta u_N\|)^6 \leq \varepsilon\|\Delta^3 u_N\|^2 + c_{22}(M_2),$$
$$\|\nabla u_N\|_8^8 \leq (c_1'\|\Delta^3 u_N\|^{\frac{3}{20}}\|\nabla u_N\|^{\frac{17}{20}} + c_2'\|\nabla u_N\|)^8 \leq \varepsilon\|\Delta^3 u_N\|^2 + c_{23}(M_2),$$
$$\|\Delta u_N\|_8^8 \leq (c_1'\|\Delta^3 u_N\|^{\frac{3}{16}}\|\Delta u_N\|^{\frac{13}{16}} + c_2'\|\Delta u_N\|)^8 \leq \varepsilon\|\Delta^3 u_N\|^2 + c_{24}(M_2),$$
$$\|\nabla\Delta u_N\|_4^4 \leq (c_1'\|\Delta^3 u_N\|^{\frac{3}{8}}\|\Delta u_N\|^{\frac{5}{8}} + c_2'\|\Delta u_N\|)^4 \leq \varepsilon\|\Delta^3 u_N\|^2 + c_{25}(M_2),$$

where ε is an arbitrary positive constant. Therefore

$$\begin{aligned}
&\int_\Omega \nabla\cdot\Delta(|\nabla u_N|^2\nabla u_N)\Delta^3 u_N \mathrm{d}x \\
&\quad= \int_\Omega (3|\nabla u_N|^2\Delta^2 u_N + 6|\Delta u_N|^3 + 18\nabla u_N\Delta u_N\nabla\Delta u_N)\Delta^3 u_N \mathrm{d}x \\
&\quad\leq \frac{\gamma}{4}\|\Delta^3 u_N\|^2 + \frac{27}{\gamma}\int_\Omega |\nabla u_N|^4|\Delta^2 u_N|^2\mathrm{d}x + \frac{72}{\gamma}\int_\Omega |\Delta u_N|^6\mathrm{d}x \\
&\qquad+ \frac{972}{\gamma}\int_\Omega |\nabla u_N\Delta u_N\nabla\Delta u_N|^2\mathrm{d}x \\
&\quad\leq \frac{\gamma}{4}\|\Delta^3 u_N\|^2 + \frac{27}{\gamma}\|\nabla u_N\|_\infty^4\|\Delta^2 u_N\|^2 + \frac{72}{\gamma}\|\Delta u_N\|_6^6 \\
&\qquad+ \frac{486}{\gamma}\|\nabla\Delta u_N\|_4^4 + \frac{243}{\gamma}\|\nabla u_N\|_8^8 + \frac{243}{\gamma}\|\Delta u_N\|_8^8 \\
&\quad\leq \frac{\gamma}{4}\|\Delta^3 u_N\|^2 + \frac{27}{\gamma}[c_{20}(M_2)]^4(\varepsilon\|\Delta^3 u_N\|^2 \\
&\qquad+ c_{21}(M_2)) + \frac{1044\varepsilon}{\gamma}\|\Delta^3 u_N\|^2 \\
&\qquad+ \frac{72}{\gamma}c_{22}(M_2) + \frac{486}{\gamma}c_{25}(M_2) + \frac{243}{\gamma}(c_{23}(M_2) + c_{24}(M_2)).
\end{aligned}$$

On the other hand, we have

$$\int_\Omega \Delta f\Delta^3 u_N \mathrm{d}x \leq \frac{1}{\gamma}\|\Delta f\|^2 + \frac{\gamma}{4}\|\Delta^3 u_N\|^2 \leq \frac{M_2^2}{\gamma} + \frac{\gamma}{4}\|\Delta^3 u_N\|^2.$$

Summing up, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^2 u_N\|^2 + c_{26}\|\Delta^3 u_N\|^2 \leq c_{27}(M_2), \tag{2.27}$$

where ε is sufficiently small, it satisfies $c_{26} = \gamma - \left(\frac{54}{\gamma}[c_{20}(M_2)]^4 + \frac{2088}{\gamma}\right)\varepsilon > 0$. Integrating (2.27) over $[0, \omega]$, we have

$$\int_0^\omega \|\Delta^3 u_N\|^2\mathrm{d}t \leq \frac{c_{27}(M_2)}{c_{26}}\omega. \tag{2.28}$$

It then follows from (2.28) that there exists a time $t_5 \in (0, \omega)$ such that

$$\|\Delta^3 u(\cdot, t_5)\|^2 \leq \frac{c_{27}(M_2)}{c_{26}}. \tag{2.29}$$

On the other hand, we have

$$\|\Delta^2 u_N(\cdot, t_5)\|^2 \leq \frac{1}{2}\|\Delta^3 u_N(\cdot, t_5)\|^2 + \frac{1}{2}\|\Delta u_N\|^2 \leq c_{28}(M_2). \tag{2.30}$$

Integrating (2.27) again over $[t_5, t+\omega](\forall t \in [0,\omega])$ and using (2.30), we deduce that

$$\sup_{0\le t\le\omega} \|\Delta^2 u_N(\cdot,t)\|^2 \le \|\Delta^2 u_N(\cdot,t_5)\|^2 + 2c_{27}(M_2)\omega \le c_{29}(M_2). \tag{2.31}$$

Based on Sobolev' embedding theorem, we have

$$\|u_N(\cdot,t)\|_{C^2(\bar\Omega)} \le c\|u_N(\cdot,t)\|_{H^4} \le c_{30}(M_2). \tag{2.32}$$

Multiplying both sides of eq. (2.1) by $\lambda^5 u_{Nj}(t)$, and summing up the products over $j = 1, 2, \dots, N$, we derive that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta^2 u_N\|^2 + \gamma\|\nabla\Delta^3 u_N\|^2 = \int_\Omega \Delta^2(|\nabla u_N|^2\nabla u_N)\nabla\Delta^3 u_N \mathrm{d}x + \int_\Omega \nabla\Delta f_t \nabla\Delta^3 u_N \mathrm{d}x.$$

By Nirenberg's inequality, we deduce that

$$\|\nabla^{3+j}u_N\| \le c_1'\|\nabla\Delta^3 u_N\|^{\frac{j}{4}}\|\nabla\Delta u_N\|^{1-\frac{j}{4}} + c_2'\|\nabla\Delta u_N\|, \quad j = 1, 2, 3$$

and

$$\|\nabla\Delta u_N\|_4 \le c_1'\|\nabla\Delta^3 u_N\|^{\frac{1}{8}}\|\nabla\Delta u_N\|^{\frac{7}{8}} + c_2'\|\nabla\Delta u_N\|.$$

Hence, a simple calculation shows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta^2 u_N\|^2 + \gamma\|\nabla\Delta^3 u_N\|^2 \\
&= \int_\Omega (3|\nabla u_N|^2\nabla\Delta^2 u_N + 24\nabla u_N\Delta u_N\Delta^2 u_N + 18\nabla u_N|\nabla\Delta u_N|^2 \\
&\quad + 36|\Delta u_N|^2\nabla\Delta u_N + \nabla\Delta f)\nabla\Delta^3 u_N \mathrm{d}x \\
&\le (3\|\nabla u_N\|_\infty^2\|\nabla\Delta^2 u_N\| + 24\|\nabla u_N\Delta u_N\|_\infty\|\Delta^2 u_N\| \\
&\quad + 18\|\nabla u_N\|_\infty\|\nabla\Delta u_N\|_4^2 + 36\|\Delta u_N\|_\infty^2\|\nabla\Delta u_N\| \\
&\quad + \|\nabla\Delta f\|)\|\nabla\Delta^3 u_N\|\mathrm{d}x \\
&\le \frac{\gamma}{2}\|\nabla\Delta^3 u_N\|^2 + c_{31}(M_2).
\end{aligned}$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta^2 u_N\|^2 + \gamma\|\nabla\Delta^3 u_N\|^2 \le 2c_{31}(M_2). \tag{2.33}$$

Integrating (2.33) over $[0,\omega]$, we have

$$\int_0^\omega \|\nabla\Delta^3 u_N\|^2 \mathrm{d}t \le \frac{2c_{31}(M_2)}{\gamma}\omega. \tag{2.34}$$

It then follows from (2.34) that there exists a time $t_6 \in (0,\omega)$ such that

$$\|\nabla\Delta^3 u_N(\cdot,t_6)\|^2 \le \frac{2c_{31}(M_2)}{\gamma}. \tag{2.35}$$

On the other hand, we have

$$\|\nabla\Delta^2 u_N(\cdot,t_6)\|^2 \le \frac{1}{2}\|\nabla\Delta^3 u_N(\cdot,t_6)\|^2 + \frac{1}{2}\|\nabla\Delta u_N\|^2 \le c_{32}(M_2). \tag{2.36}$$

Integrating (2.33) again over $[t_6, t+\omega](\forall t \in [0,\omega])$ and using (2.36), we deduce that

$$\sup_{0\le t\le\omega} \|\nabla\Delta^2 u_N(\cdot,t)\|^2 \le \|\nabla\Delta^2 u_N(\cdot,t_6)\|^2 + 4c_{31}(M_2)\omega \le c_{33}(M_2). \quad (2.37)$$

Based on Sobolev's embedding theorem, we have

$$\|u_N(\cdot,t)\|_{C^3(\bar{\Omega})} \le c\|u_N(\cdot,t)\|_{H^5} \le c_{34}(M_2). \quad (2.38)$$

Multiplying both sides of eq. (2.1) by $u'_{Nj}(t)$, and summing up the products over $j = 1, 2, \ldots, N$, we derive that

$$\begin{aligned}\|u_{Nt}\|^2 &= (-\gamma\Delta^2 u_N + \nabla\cdot(|\nabla u_N|^2\nabla u_N) + f, u_{Nt})\\ &= (-\gamma\Delta^2 u_N + 3|\nabla u_N|^2\Delta u_N + f, u_{Nt})\\ &\le \frac{1}{2}\|u_{Nt}\|^2 + \frac{3\gamma^2}{2}\|\Delta^2 u_N\|^2 + \frac{27}{2}\|\nabla u_N\|_\infty^4\|\Delta u_N\|^2 + \frac{3}{2}\|f\|^2\\ &\le \frac{1}{2}\|u_{Nt}\|^2 + c_{35}(M_2),\end{aligned}$$

that is,

$$\sup_{0\le t\le\omega} \|u_{Nt}(\cdot,t)\|^2 \le 2c_{35}(M_2). \quad (2.39)$$

Differentiating (2.1) with respect to t, we get

$$(u_{Ntt} + \gamma\Delta^2 u_{Nt} - 3|\nabla u_N|^2\Delta u_{Nt} - 6\nabla u_N\Delta u_N\nabla u_{Nt}, y_j) = (f_t, y_j), \quad (2.40)$$

where $j = 1, \ldots, N$.

Multiplying both sides of eq. (2.40) by $\lambda_j^3 u'_{Nj}(t)$, and summing up the products over $j = 1, 2, \ldots, N$, we derive that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u_{Nt}\|^2 + \gamma\|\nabla\Delta^2 u_{Nt}\|^2 + 3\int_\Omega |\nabla u_N|^2\Delta u_{Nt}\Delta^3 u_{Nt}\mathrm{d}x\\ &\quad +6\int_\Omega \nabla u_N\Delta u_N\nabla u_{Nt}\Delta^3 u_{Nt}\mathrm{d}x + \int_\Omega f_t\Delta^3 u_{Nt}\mathrm{d}x = 0.\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u_{Nt}\|^2 + \gamma\|\nabla\Delta^2 u_{Nt}\|^2\\ &\quad = 3(|\nabla u_N|^2\nabla\Delta u_{Nt} + 2\nabla u_N\Delta u_N\Delta u_{Nt}, \nabla\Delta^2 u_{Nt})\\ &\qquad + 6(|\Delta u_N|^2\nabla u_{Nt} + \nabla u_N\nabla\Delta u_N\nabla u_{Nt} + \nabla u_N\Delta u_N\Delta u_{Nt}, \nabla\Delta^2 u_{Nt})\\ &\qquad + (\nabla f_t, \nabla\Delta u_{Nt})\\ &\quad \le (3\|\nabla u_N\|_\infty^2\|\nabla\Delta u_{Nt}\| + 6\|\nabla u_N\Delta u_N\|_\infty\|\Delta u_{Nt}\|\\ &\qquad + 6\|\Delta u_N\|_\infty^2\|\nabla u_{Nt}\| + 6\|\nabla u_N\|_\infty\|\nabla\Delta u_N\|\|\nabla u_{Nt}\|_\infty\\ &\qquad + 6\|\nabla u_N\Delta u_N\|_\infty\|\Delta u_{Nt}\| + \|\nabla f_t\|)\|\nabla\Delta^2 u_{Nt}\|.\end{aligned}$$

Using Nirenberg's inequality, we derive that

$$\|\nabla u_{Nt}\|_\infty \le c'_1\|\nabla\Delta^2 u_{Nt}\|^{\frac{2}{5}}\|u_{Nt}\|^{\frac{3}{5}} + c'_2\|u_{Nt}\| \quad (2.41)$$

and

$$\|\nabla^j u_{Nt}\| \le c_1'\|\nabla\Delta^2 u_{Nt}\|^{\frac{j}{5}}\|u_{Nt}\|^{\frac{5-j}{5}} + c_2'\|u_{Nt}\|, \quad j = 1, 2, 3. \tag{2.42}$$

Applying (2.41) and (2.42) and using Hölder's inequality, a simple calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u_{Nt}\|^2 + \gamma\|\nabla\Delta^2 u_{Nt}\|^2 \le c_{36}(M_2). \tag{2.43}$$

Integrating (2.43) over $[0, \omega]$, by means of the integration of mean value theorem, it follows that there is a $t_6 \in (0, \omega)$ such that

$$\|\nabla\Delta^2 u_{Nt}(\cdot, t_6)\|^2 \le \frac{c_{36}(M_2)}{\gamma}.$$

It then follows from (2.42) and the above inequality that

$$\|\nabla\Delta u_{Nt}(\cdot, t_6)\|^2 \le c_{37}(M_2).$$

Integrating (2.43) over $[t_6, t+\omega](\forall t \in [0, \omega]$, we see that

$$\sup_{0\le t\le\omega} \|\nabla\Delta u_{Nt}\|^2 \le \|\nabla\Delta u_{Nt}(\cdot, t_6)\|^2 + 2c_{36}(M_2)\omega \le c_{38}(M_2). \tag{2.44}$$

Combining (2.19), (2.25), (2.31), (2.37), (2.39), (2.44) and the result of Lemma 2.1, we complete the proof. □

Lemma 2.3. *Suppose that the assumptions of Lemma* 2.2 *is satisfied, and the function* f *satisfies* $f \in C_\omega(\mathbf{R}; H^6(\Omega))$, $f_t \in C_\omega(\mathbf{R}; H^4(\Omega))$ *then the approximate solution of problems* (1.1)–(1.3) *satisfy*

$$\sup_{0\le t\le\omega} (\|u_{Ntt}\|^2_{H^2} + \|u_{Nt}\|^2_{H^6} + \|u_N\|^2_{H^8}) \le c_{39}(M_3),$$

where $M_3 = \sup_{0\le t\le\omega}(\|f\|_{H^6} + \|f_t\|_{H^4})$. *Here and in the sequel,* $c_i(M_3)$ *denotes constants independent of* N.

Proof. Multiplying both sides of (2.1) by $\lambda_j^9 u_{Nj}(t)$ and integrating it over Ω, multiplying both sides of (2.40) by $\lambda_j^6 u_{Nj}'$, u_{Nj}'', $\lambda_j^2 u_{Nj}''$ and integrating it over Ω, using Nirenberg's inequality, Hölder's inequality, by means of the integration mean value theorem, we can easily complete the proof of Lemma 2.3. Since the proof is so classical, we omit it. □

3. Existence and uniqueness of solutions for problems (1.1)–(1.3)

Theorem 3.1. *Suppose the assumptions in Lemma* 2.2 *is satisfied, then there exists a generalized time-periodic solution*

$$u(x,t) \in L^2_\omega(\mathbf{R}; H^5(\Omega)), \quad u_t(x,t) \in L^2_\omega(\mathbf{R}; H^3(\Omega)), \tag{3.1}$$

for problems (1.1)–(1.3), *which satisfy*

$$\int_0^\omega \int_\Omega (u_t + \gamma\Delta^2 u - \nabla\cdot(|\nabla u|^2\nabla u) - f)\varphi \mathrm{d}x\mathrm{d}t = 0,$$
$$\forall\varphi \in L^2_\omega(\mathbf{R}; L^2(\Omega)). \tag{3.2}$$

Especially, if M_2 *is sufficiently small, the solution is unique.*

Proof. Based on Lemma 2.2 and Sobolev's embedding theorem, we obtain the following estimate:

$$\sup_{0\le t\le\omega}\left(\|u_{Nt}(\cdot,t)\|_{C^{1,\lambda}(\bar{\Omega})}+\|u_N\|_{C^{3,\lambda}(\bar{\Omega})}\right)\le c_{40}(M_2), \tag{3.3}$$

where $\lambda\in(0,\frac{1}{2}]$. It then follows from (3.3) and Ascoli–Arzelá's theorem that there exists a function $u(x,t)$ and a subsequence of $\{u_N(x,t)\}$, still denoted by $\{u_N(x,t)\}$, such that when $N\to+\infty$, $\{u_N(x,t)\}$, $u_{Nx}(x,t)\}$ uniformly converge to $u(x,t)$ and $u_x(x,t)$ on $[0,\omega]\times\Omega$. Based on the result of Lemma 2.2, when $N\to+\infty$, the subsequences $\{u_N(x,t)\}$, $\{u_{Nx_1}(x,t)\}$, $\{u_{Nx_2}(x,t)\}$ uniformly converge to $u(x,t)$, $u_{x_1}(x,t)$ and $u_{x_2}(x,t)$ on $[0,\omega]\times\bar{\Omega}$, respectively, $\{u_{Nt}\}$, $\{\Delta u_N\}$ and $\{\Delta^2 u_N\}$ weakly converge to u_t, Δu and $\Delta^2 u$ in $L^2_\omega(\mathbf{R};H^3(\Omega)$, $L^2_\omega(\mathbf{R};H^3(\Omega))$ and $L^2_\omega(\mathbf{R};H^1(\Omega))$. By virtue of the result of Lemma 2.2, we get

$$\|u_N\|_{L^2_\omega(\mathbf{R};H^5(\Omega))}+\|u_{Nt}\|_{L^2_\omega(\mathbf{R};H^3(\Omega))}\le c_{41}(M_2).$$

Set

$$W=\{u|u\in L^2_\omega(\mathbf{R};H^5(\Omega)),u_t\in L^2_\omega(\mathbf{R};L^2(\Omega))\}.$$

Aubin's compact lemma implies that the embedding $W\hookrightarrow L^2_\omega(\mathbf{R};H^2(\Omega))$ is compact. Owing to the assumptions, we know that there exists a subsequence of $\{u_N(x,t)\}$ still denoted by $\{u_N(x,t)\}$ such that when $N\to+\infty$, $\{u_N(x,t)\}$ is convergent in $L^2_\omega(\mathbf{R};H^2(\Omega))$.

Setting $F(s)=|s|^2s$, according to the previous subsequence $\{u_N(x,t)\}$, we conclude that $\{\nabla\cdot[F(\nabla u_N)]\}=\{\nabla\cdot(|\nabla u_N|^2\nabla u_N)\}$ weakly converges to $\nabla\cdot[F(\nabla u)]=\nabla\cdot(|\nabla u|^2\nabla u)$ in $L^2_\omega(\mathbf{R};L^2(\Omega))$. In fact, for any $\omega\in L^2_\omega(\mathbf{R};L^2(\Omega))$, by (3.3), we have

$$\begin{aligned}
&\left|\int_0^\omega(\nabla\cdot[F(\nabla u_N)]-\nabla\cdot[F(\nabla u)],w)\mathrm{d}t\right|\\
&\quad\le\int_0^\omega\int_\Omega(|F'(\nabla u_N)-F'(\nabla u)||\Delta u_N|\\
&\qquad+|F'(\nabla u)||\Delta u_N-\Delta u|)|\omega|\mathrm{d}x\mathrm{d}t\\
&\quad\le\int_0^\omega\int_\Omega(|F''(\theta\nabla u_N+(1-\theta)\nabla u)||\nabla u_N-\nabla u||\Delta u_N||\omega|)\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_0^\omega\int_\Omega\int_\Omega(|F'(\nabla u)||\Delta u_N-\Delta u||\omega|)\mathrm{d}x\mathrm{d}t\\
&\quad\le c_{42}(M_2)\int_0^\omega\int_\Omega(|\nabla u_N-\nabla u|+|\Delta u_N-\Delta u|)|\omega|\mathrm{d}x\mathrm{d}t\\
&\quad\le c_{42}(M_2)[\|\nabla u_N-\nabla u\|_{L^2((0,\omega)\times\Omega)}\\
&\qquad+\|\Delta u_N-\Delta u\|_{L^2((0,\omega)\times\Omega}]\|\omega\|_{L^2((0,\omega)\times\Omega)},
\end{aligned}$$

where $\theta\in(0,1)$. By the above inequality, we know that there exists a subsequence $\{u_N(x,t)\}$ such that $\{\nabla\cdot[F(\nabla u_N)]\}$ weakly converges to $\nabla\cdot[F(\nabla u)]$ in $L^2(\mathbf{R};L^2\Omega)$. Then, problems (1.1)–(1.3) admit a generalized time-periodic solution $u(x,t)$, which satisfies (3.1) and (3.2).

Now, we are going to prove the uniqueness of the solution. Suppose that $u(x,t)$ and $v(x,t)$ are two solutions of (1.1)–(1.3). Let $\xi(x,t) = u(x,t) - v(x,t)$, then $\xi(x,t)$ satisfies the following problem:

$$\begin{cases} \xi_t + \gamma\Delta^2\xi + \nabla\cdot[F(\nabla u)] - \nabla\cdot[F(\nabla v)] = 0, \quad x\in\Omega, \quad t\in\mathbf{R}, \\ \xi(\cdot,t) \text{ is } L \text{ periodic}, \quad t\in\mathbf{R}, \\ \xi(t+\omega) = \xi(x,t), \quad t\in\mathbf{R}. \end{cases} \tag{3.4}$$

Multiplying both side of eq. (3.4) by ξ, integrating the products over (Ω) and using the mean value theorem, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 + \gamma\|\Delta\xi\|^2 &= \int_\Omega F(\nabla u) - F(\nabla v)\nabla\xi\mathrm{d}x \\ &= \int_\Omega F'(\theta\nabla u + (1-\theta)\nabla v)\Delta\xi\nabla\xi\mathrm{d}x \\ &\le c_{43}(M_2)\|\Delta\xi\|\|\nabla\xi\| \le \frac{\gamma}{4}\|\Delta\xi\|^2 + \frac{1}{\gamma}[c_{43}(M_2)]^2\|\nabla\xi\|^2 \\ &= \frac{\gamma}{4}\|\Delta\xi\|^2 - \frac{1}{\gamma}[c_{43}(M_2)]^2(\xi,\Delta\xi) \\ &\le \frac{\gamma}{2}\|\Delta\xi\|^2 + \frac{1}{\gamma^3}[c_{43}(M_2)]^4\|\xi\|^2. \end{aligned} \tag{3.5}$$

Using Poincaré's inequality, we get

$$\|\xi\|^2 \le C^*\|\nabla\xi\|^2 \le \frac{(C^*)^2}{2}\|\Delta\xi\|^2 + \frac{1}{2}\|\xi\|^2,$$

which means

$$\|\xi\|^2 \le (C^*)^2\|\Delta\xi\|^2.$$

It then follows from (3.5) and the above inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 + \left(\frac{\gamma}{(C^*)^2} - \frac{2}{\gamma^3}[c_{43}(M_2)]^4\right)\|\xi\|^2 \le 0.$$

Taking M_2 sufficiently small such that $\frac{\gamma}{(C^*)^2} - \frac{2}{\gamma^3}[c_{43}(M_2)]^4 > 0$, using Gronwall's inequality, we have

$$\|\xi(\cdot,t)\|^2 \le \|\xi(\cdot,0)\|^2 \mathrm{e}^{-\left(\frac{\gamma}{(C^*)^2} - \frac{2}{\gamma^3}[c_{43}(M_2)]^4\right)t}, \quad \forall t > 0.$$

Since $\xi(x,t)$ is time-periodic, for any $t\in\mathbf{R}$, there exists a natural number N_0 such that $t + N_0\xi > 0$ and

$$\|\xi(\cdot,t)\|^2 = \|\xi(\cdot,t+N_0\omega)\|^2 \le \|\xi(\cdot,0)\|^2 \mathrm{e}^{-\left(\frac{\gamma}{(C^*)^2} - \frac{2}{\gamma^3}[c_{43}(M_2)]^4\right)N\omega}, \quad \forall N > N_0,$$

that is,

$$\|\xi(\cdot,t)\|^2 = 0, \quad \forall t\in\mathbf{R}.$$

Then, Theorem 3.1 is proved. □

Theorem 3.2. *Suppose that the assumptions in Lemma* 2.3 *is satisfied*, *problems* (1.1)–(1.3) *admit a uniqueness classical time-periodic solution* $u(x,t)$.

Proof. Using the result of Lemma 2.3, by Sobolev's embedding theorem, we obtain

$$\sup_{0\leq t\leq\omega}(\|u_{Ntt}\|^2_{C^{0,\lambda}(\bar{\Omega})}+\|u_{Nt}\|^2_{C^{4,\lambda}(\bar{\Omega})}+\|u_N\|^2_{C^{7,\lambda}(\bar{\Omega})})\leq c_{44}(M_3),$$

where $\lambda\in(0,\frac{1}{2}]$. Applying Ascoli–Arzelá theorem, we know there exists a function $u(x,t)$ and a subsequence of $\{u_N(x,t)\}$, which is still denoted by $\{u_N(x,t)\}$, such that, when $N\to+\infty$, $\{\nabla^j u_N(x,t)\}$ and $\{u_{Nt}(x,t)\}$ uniformly converge to $\nabla^j u(x,t)$ and $u_t(x,t)$, where $j=0,1,2,3,4$. Therefore, it is easy to check that $u(x,t)$ is a classical time-periodic solution of problems (1.1)–(1.3). On the other hand, the uniqueness of solution is proved in Theorem 3.1. Hence, we complete the proof. □

4. Conclusion

By using Galerkin method and Leray–Schauder fixed point toeorem, we studied the existence and uniqueness of time-periodic generalized solutions and time-periodic classical solutions to a fourth-order nonlinear thin film equation in 2D case. On the other hand, there are some other fourth-order or sixth-order thin film equations which are recently being discussed in literature, e.g. equations with stronger nonlinearities such as (see [1])

$$u_t+\Delta^2 u=\Delta(|\nabla u|^2). \tag{4.1}$$

We will use the same method as used in this article to consider the time-periodic solutions for the 3D case of problems (1.1)–(1.3) and other thin film equations such as (4.1).

Acknowledgements

This work is supported by the Fundamental Research Funds for the Central Universities (JUSRP 11407). The authors would like to express their deep thanks to the referee's valuable suggestions which helped in improving the manuscript.

References

[1] Blömker D, Gugg C and Raible M, Thin-film-growth models: roughness and correlation functions, *European J. Appl. Math.* **4** (2002) 385–402
[2] Elliott C M and Zheng S, On the Cahn-Hilliard equation, *Arch. Rational Mech. Anal.* **96** (1986) 339–357
[3] Fontes M and Verdier O, Time-periodic solutions of the Burgers equation, *J. Math. Fluid Mech.* **11** (2009) 303–323
[4] Fu Y and Guo B, Time periodic solution of the viscous Camassa-Holm equation, *J. Math. Anal. Appl.* **313** (2006) 311–321
[5] King B B, Stein O and Winkler M, A fourth order parabolic equation modeling epitaxial thin film growth, *J. Math. Anal. Appl.* **286** (2003) 459–490
[6] Kohn R V and Yan X, Upper bound on the coarsening rate for an epitaxial growth model, *Comm. Pure Appl. Math.* **56** (2003) 1549–1564
[7] Kucera P, The time-periodic solutions of the Navier-Stokes equations with mixed boundary conditions, *Discrete Contin. Dyn. Syst. Ser. S*, **3** (2010) 325–337

[8] Li Y, Yin J and Jin C, A Cahn-Hilliard type equation with periodic gradient-dependent potentials and sources, *Math. Methods Appl. Sci.* **33** (2010) 263–272

[9] Li Y and Yin J, The viscous Cahn-Hilliard equation with periodic potentials and sources, *J. Fixed Point Theory Appl.* **9** (2011) 63–84

[10] Liu C, Regularity of solutions for a fourth order parabolic equation, *Bull. Belg. Math. Soc. Simon Stevin* **13(3)** (2006) 527–535

[11] Liu C, A fourth order parabolic equation with nonlinear principal part, *Nonlinear Anal.* **68** (2008) 393–401

[12] Luca R, Time periodic solutions for a class of nonlinear boundary value problems, *Fixed Point Theory* **9** (2008) 173–187

[13] Silvestre A L, Existence and uniqueness of time-periodic solutions with finite kinetic energy for the Navier-Stokes equations in R^3, *Nonlinearity* **25** (2012) 37–55

[14] Wang Y and Chen G, Time-periodic solution to a Ginzburg-Landau model equation in population problems, *Acta Math. Sci. Ser. A Chin. Ed.* **26** (2006) 258–266

[15] Wang Y and Zhang Y, Time-periodic solution to a nonlinear parabolic type equation of higher order, *Acta Math. Appl. Sin. Engl. Ser.* **24** (2008) 129–140

[16] Xu C and Tang T, Stability analysis of large time-stepping methods for epitaxial growth models, *SIAM J. Numer. Anal.* **44** (2006) 1759–1779

[17] Yin L, Li Y, Huang R and Yin J, Time periodic solutions for a Cahn-Hilliard type equation, *Math. Comput. Modelling* **48** (2008), 11–18

[18] Yin J, Li Y and Huang R, The Cahn-Hilliard type equations with periodic potentials and sources, *Appl. Math. Comput.* **211** (2009) 211–221

[19] Zangwill A, Some causes and a consequence of epitaxial roughening, *J. Cryst. Growth* **163** (1996) 8–21

[20] Zhao X and Liu C, The existence of global attractor for a fourth-order parabolic equation, *Appl. Anal.* **92** (2013) 44–59

[21] Zhou Y and Fu H, The nonlinear hyperbolic systems of higher order of generalized Sine-Gordon type, *Acta Math. Sinica* **26** (1983) 234–249 (in Chinese)