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Interpolation for a subclass of H^∞

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Abstract. We introduce and characterize two types of interpolating sequences in the unit disc $\mathbb{D}$ of the complex plane for the class of all functions being the product of two analytic functions in $\mathbb{D}$, one bounded and another regular up to the boundary of $\mathbb{D}$, concretely in the Lipschitz class, and at least one of them vanishing at some point of $\bar{\mathbb{D}}$.

Keywords. Interpolating sequence; bounded analytic function; Lipschitz class.

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1. Introduction

We consider the space H^∞ of bounded analytic functions in $\mathbb{D}$ and the Lipschitz class Λ of analytic functions g in $\mathbb{D}$, continuous in $\bar{\mathbb{D}}$, such that

$$\sup_{z,w\in\mathbb{D}} \frac{|g(z)-g(w)|}{|z-w|} < \infty.$$

We write (z_n) for any sequence in $\mathbb{D}$ having no accumulation points in $\mathbb{D}$. Recall that the pseudo-hyperbolic distance is

$$\psi(z,w) = \frac{|z-w|}{|1-\bar{z}w|}, \quad z,w\in\mathbb{D}.$$

For a given point $z_m \in (z_n)$, we will choose two points in (z_n), that we will denote by $z_{m'}$ and z_m^*, verifying

$$z_{m'} \in \{z\in(z_n)\,/\, \inf_{i\neq m}|z_m - z_i| = |z_m - z|\}$$

and

$$z_m^* \in \{z\in(z_n)\,/\, \inf_{i\neq m}\psi(z_m,z_i) = \psi(z_m,z)\}.$$

For a sequence (z_n) satisfying the Blaschke condition

$$\sum_{n\in\mathbb{N}}(1-|z_n|) < \infty,$$

the Blaschke product with zeros at (z_n) is defined as the function in H^∞,

$$B(z) = \prod_{n\in\mathbb{N}} \frac{|z_n|}{z_n} \frac{z - z_n}{1 - \overline{z_n} z}.$$

The Blaschke products with zeros at $(z_n) \setminus \{z_m\}$ and $(z_n) \setminus \{z_m, z_k\}$ will be denoted by B_m and $B_{m,k}$, respectively. From now on, we will put c for all positive real constants.

A fundamental property of a sequence (z_n) in relation to the separation of its points is to be uniformly separated (US), that is,

$$|B_m(z_m)| \geq c, \quad \forall m \in \mathbb{N} \tag{1}$$

and it is proved in [1] that if a sequence (z_n) verifies

$$|B_{m,m'}(z_m)| \geq c, \quad \forall m \in \mathbb{N}, \tag{2}$$

then it is a union of not more than two US sequences.

Next we collect definitions of interpolating sequences for H^∞ and Λ, as well as results about them that we will need:

DEFINITION 1

(z_n) is called an interpolating sequence for H^∞ if given any bounded sequence of complex numbers (μ_n), there exists $f \in H^\infty$ such that $f(z_n) = \mu_n$, $\forall n \in \mathbb{N}$.

The well-known Carleson's theorem [2] asserts the following:

Theorem 1. *(z_n) is an interpolating sequence for H^∞ if and only if it is US.*

DEFINITION 2

We say that (z_n) is an interpolating sequence in the weak sense for H^∞ if given any sequence of complex numbers (μ_n) verifying

$$|\mu_n - \mu_m| \leq c\, \psi(z_n, z_m), \quad \forall n, m \in \mathbb{N},$$

there exists $f \in H^\infty$ such that $f(z_n) = \mu_n$, $\forall n \in \mathbb{N}$.

The characterization of these interpolating sequences is proved in [5].

Theorem 2. *(z_n) is an interpolating sequence in the weak sense for H^∞ if and only if it is a union of not more than two US sequences.*

Let $\Lambda(z_n)$ be the set of all functions h defined on (z_n) verifying

$$|h(z_n) - h(z_m)| \leq c\, |z_n - z_m|, \quad \forall n, m \in \mathbb{N}.$$

DEFINITION 3

(z_n) is called an interpolating sequence for Λ if given any function $h \in \Lambda(z_n)$, there exists $g \in \Lambda$ such that $g(z_n) = h(z_n)$, $\forall n \in \mathbb{N}$.

It is known that the interpolating sequences for Λ must be a union of not more than two US sequences [4].

We will use two results about the interpolating sequences for Λ when they remain in a certain region of $\mathbb{D}$: the nontangential wedge, defined for a fixed $t \in (0, 1)$ as

$$W_t = \left\{ z \in \mathbb{D} \Bigg/ \frac{1-|z|^2}{|1-z^2|} > t \right\},$$

or the Stolz angle with vertex at a point η in the boundary of $\mathbb{D}$ and aperture $\beta > 0$, defined as

$$S_\beta(\eta) = \left\{ z \in \mathbb{D} \Bigg/ \frac{|z-\eta|}{1-|z|^2} < 1+\beta \right\}.$$

Note that we can confine ourselves (via a rotation) to take $\eta = 1$ or $\eta = -1$. In both cases, the corresponding Stolz angle is included in the nontangential wedge for $t = \frac{1}{2(1+\beta)}$.

Theorem 3.

(i) *If (z_n) in a Stolz angle is a union of not more than two US sequences, then it is an interpolating sequence for Λ.*
(ii) *If (z_n) in a nontangential wedge is US, then it is an interpolating sequence for Λ.*

The assertions (i) and (ii) are proved in [3] and [4], respectively. As a consequence of (i), the geometric characterization of the interpolating sequences for Λ when they remain in Stolz angles is to be a union of not more than two US sequences.

We denote by $H^\infty\Lambda$ the class of all functions in $\mathbb{D}$ which are the product of one function f in H^∞ and another g in Λ, at least one of them vanishing at some point of $\bar{\mathbb{D}}$ (so H^∞ is not included in $H^\infty\Lambda$). It is clear that $H^\infty\Lambda$ is a subclass of H^∞ and therefore, in order to introduce interpolating sequences for $H^\infty\Lambda$, we must impose some restriction to the bounded given sequence.

It is known that if $f \in H^\infty$, then

$$|f(z) - f(w)| \le c\, \psi(z, w), \quad \forall z, w \in \mathbb{D},$$

and, consequently, if $f \in H^\infty$ vanishes on z_m^*,

$$|f(z_m)| \le c\, \psi(z_m, z_m^*), \quad \forall m \in \mathbb{N}. \tag{3}$$

On the other hand, if $g \in \Lambda$ vanishes on $z_{m'}$

$$|g(z_m)| \le c\, |z_m - z_{m'}|, \quad \forall m \in \mathbb{N}.$$

Thus it is natural to pose the following interpolation problem for $H^\infty\Lambda$:

DEFINITION 4

We say that (z_n) is an interpolating sequence in the weak sense for $H^\infty\Lambda$ if given any sequence of complex numbers (λ_n) verifying

$$|\lambda_n| \le c\, \psi(z_n, z_n^*)\, |z_n - z_{n'}|, \quad \forall n \in \mathbb{N}, \tag{4}$$

there exists a product $fg \in H^\infty\Lambda$, such that

(i) f vanishes at a certain point of (z_n) if and only if g vanishes at this given point;
(ii) $(fg)(z_n) = \lambda_n$, $\forall n \in \mathbb{N}$.

Taking into account that $\psi(z_n, z_n^*) < 1$ and $|z_n - z_{n'}| < |1 - \overline{z_n} z_{n'}|$, $\forall n \in \mathbb{N}$, it is possible to pose another interpolation problem for $H^\infty \Lambda$:

DEFINITION 5

We say that (z_n) is an interpolating sequence in the strong sense for $H^\infty \Lambda$ if given any sequence of complex numbers (λ_n) verifying

$$|\lambda_n| \leq c\,|1 - \overline{z_n} z_{n'}|, \quad \forall n \in \mathbb{N}, \tag{5}$$

there exists a product $fg \in H^\infty \Lambda$, such that

(i) if one of the two functions vanishes at a certain point of (z_n), then the other function does not vanish on the sequence (z_n);
(ii) $(fg)(z_n) = \lambda_n$, $\forall n \in \mathbb{N}$.

We characterize these two types of interpolating sequences for $H^\infty \Lambda$ in regions of $\mathbb{D}$. Our results are as follows:

Theorem 4. *(z_n) in a Stolz angle is an interpolating sequence in the weak sense for $H^\infty \Lambda$ if and only if it is a union of not more than two US sequences.*

Theorem 5. *(z_n) in a nontangential wedge is an interpolating sequence in the strong sense for $H^\infty \Lambda$ if and only if it is US.*

We prove these theorems in the next two sections.

It would be interesting to pose other interpolation problems, changing the role of Λ for another space with some regularity up to the boundary of $\mathbb{D}$ or(and) that of H^∞ for a 'nearby' space as, for example, BMOA or the Bloch space. Also there might be looked some result of general type for this model of interpolating sequences, which consists in considering the product of two analytic functions in $\mathbb{D}$ with a different behaviour. Our paper is only a first contribution.

2. Proof of Theorem 4

Proof.

Necessity. Suppose that (z_n) is an interpolating sequence in the weak sense for $H^\infty \Lambda$. For a fixed $m \in \mathbb{N}$, let (λ_n) be such that $\lambda_m = \dfrac{z_m - z_m^*}{1 - \overline{z_m} z_m^*}(z_m - z_{m'})$ and $\lambda_k = 0$, if $k \neq m$. Since (λ_n) verifies (4), there exists a product $fg \in H^\infty \Lambda$, such that both functions f and g vanish on all points of $(z_n) \setminus \{z_m\}$ and $(fg)(z_m) = \lambda_m$.

For a non-zero function $G \in \Lambda$ which vanishes on all points of $(z_n) \setminus \{z_m\}$, it is proved in [4] that

$$|G(z)| \leq c\,|z - z_k|\,|B_{m,k}(z)|, \quad \forall k \in \mathbb{N}, \quad k \neq m.$$

Writing this inequality for $G = g$, $z = z_m$ and $z_k = z_{m'}$, we obtain

$$\frac{\psi(z_m, z_m^*)\,|z_m - z_{m'}|}{|f(z_m)|} = |g(z_m)| \leq c\,|z_m - z_{m'}|\,|B_{m,m'}(z_m)|, \quad \forall m \in \mathbb{N}. \tag{6}$$

Since f vanishes on z_m^*, it satisfies (3), and (2) follows from (6).

Sufficiency. Suppose that (z_n) in a Stolz angle is a union of not more than two US sequences and let (λ_n) be a sequence verifying (4). Let h be the function defined on (z_n) by $h(z_n) = z_n - z_{n'}$, $\forall n \in \mathbb{N}$. By the triangular inequality

$$|h(z_n) - h(z_m)| \leq |z_n - z_{n'}| + |z_m - z_{m'}| \leq 2\,|z_n - z_m|, \quad \forall n, m \in \mathbb{N},$$

it follows that $h \in \Lambda(z_n)$. Then, by Theorem 3(i), there exists $g \in \Lambda$ such that $g(z_n) = h(z_n)$, $\forall n \in \mathbb{N}$.

On the other hand, let $(\mu_n) = \left(\dfrac{\lambda_n}{z_n - z_{n'}}\right)$. By the triangular inequality and (4),

$$|\mu_n - \mu_m| \leq |\mu_n| + |\mu_m| \leq c\,\psi(z_n, z_n^*) + c\,\psi(z_m, z_m^*) \leq c\,\psi(z_n, z_m), \quad \forall n, m \in \mathbb{N}.$$

Then, by Theorem 2, there exists $f \in H^\infty$ such that $f(z_n) = \mu_n$, $\forall n \in \mathbb{N}$. The product $fg \in H^\infty\Lambda$ performs the desired interpolation. □

3. Proof of Theorem 5

Proof.

Necessity. Suppose that (z_n) in a nontangential wedge W_t is an interpolating sequence in the strong sense for $H^\infty\Lambda$. For a fixed $m \in \mathbb{N}$, let (λ_n) be such that $\lambda_m = 1 - \overline{z_m} z_{m'}$ and $\lambda_k = 0$, if $k \neq m$. We denote by Γ the sequence $(z_n) \setminus \{z_m\}$. Since (λ_n) verifies (5), there exists a product $fg \in H^\infty\Lambda$ such that one of the two functions vanishes on all points of Γ, the other does not vanish on Γ and $(fg)(z_m) = \lambda_m$.

In the case where g vanishes on all points of Γ, we continue like in the proof of Theorem 4 where the inequality in (6) becomes

$$\frac{|1 - \overline{z_m} z_{m'}|}{|f(z_m)|} = |g(z_m)| \leq c\,|z_m - z_{m'}|\,|B_{m,m'}(z_m)|, \quad \forall m \in \mathbb{N}. \tag{7}$$

Since $|f(z_m)| \leq c$, (1) follows from (7).

In the case where f vanishes on all points of Γ, we take into account that $f = B_m u$, where $u \in H^\infty$ does not vanish on Γ, and we have

$$\frac{|1 - \overline{z_m} z_{m'}|}{|g(z_m)|} = |f(z_m)| = |B_m(z_m)|\,|u(z_m)| \leq c\,|B_m(z_m)|, \quad \forall m \in \mathbb{N}. \tag{8}$$

We can suppose, without loss of generality, that g vanishes on the points 1 and -1, and then $|g(z_m)| \leq c\,|1 - z_m^2|$. On the other hand, we have $|1 - \overline{z_m} z_{m'}| > \frac{1}{2}(1 - |z_m|^2)$, since

$$|1 - \overline{z_m} z_{m'}| \geq 1 - |z_m|\,|z_{m'}| > 1 - |z_m| = \frac{1 - |z_m|^2}{1 + |z_m|} > \frac{1}{2}(1 - |z_m|^2).$$

Hence, from (8) we obtain

$$|B_m(z_m)| \geq c\,\frac{1 - |z_m|^2}{|1 - z_m^2|}, \quad \forall m \in \mathbb{N}.$$

Since $z_m \in W_t$, the quotient in the right-hand of this inequality is greater than t and it follows (1).

Sufficiency. Suppose that (z_n) in a nontangential wedge is a US sequence and let (λ_n) be a sequence verifying (5). Let h be the function defined on (z_n) by $h(z_n) = 1 - \overline{z_n} z_{n'}$, $\forall n \in \mathbb{N}$. Since

$$\begin{aligned} |h(z_n) - h(z_m)| &= |\overline{z_m} z_{m'} - \overline{z_n} z_{n'}| \leq |\overline{z_m}|\,|z_{m'} - z_{n'}| + |z_{n'}|\,|\overline{z_m} - \overline{z_n}| \\ &< |z_{m'} - z_m| + |z_m - z_n| + |z_n - z_{n'}| + |\overline{z_m} - \overline{z_n}| \\ &\leq 4\,|z_n - z_m|, \quad \forall n, m \in \mathbb{N}, \end{aligned}$$

it follows that $h \in \Lambda(z_n)$. Then, by Theorem 3(ii), there exists $g \in \Lambda$ such that $g(z_n) = h(z_n)$, $\forall n \in \mathbb{N}$.

On the other hand, let $(\mu_n) = \left(\dfrac{\lambda_n}{1 - \overline{z_n} z_{n'}}\right)$. By (5), we have that (μ_n) is a bounded sequence. Then, by Theorem 1, there exists $f \in H^\infty$ such that $f(z_n) = \mu_n$, $\forall n \in \mathbb{N}$. The product $fg \in H^\infty \Lambda$ performs the desired interpolation. $\square$

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