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Some Hermite–Hadamard type inequalities for geometrically quasi-convex functions

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Abstract. In the paper, we introduce a new concept 'geometrically quasi-convex function' and establish some Hermite–Hadamard type inequalities for functions whose derivatives are of geometric quasi-convexity.

Keywords. Hermite–Hadamard's integral inequality; geometrically quasi-convex function.

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1. Introduction

Throughout this paper, we use the following notations

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty). \tag{1.1}$$

DEFINITION 1.1

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.2}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

DEFINITION 1.2 [5]

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be quasi-convex if

$$f(\lambda x + (1-\lambda)y) \le \sup\{f(x), f(y)\} \tag{1.3}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on $[a, b]$ and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

The inequality (1.4) is the well known Hermite–Hadamard inequality and it has been refined or generalized for convex, s-convex, and quasi-convex functions and other kinds of functions by a number of mathematicians. Some of them can be reformulated as follows.

Theorem 1.1 (Theorem 2.2 of [6]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.5}$$

Theorem 1.2 (Theorems 1 and 2 of [13]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ is convex on $[a, b]$ for $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q} \tag{1.6}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{1.7}$$

Theorem 1.3 (Theorem 2.3 of [11]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$. If the mapping $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$ for $p > 1$, then*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| &\le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\{[|f'(a)|^{p/(p-1)}\\ &+3|f'(b)|^{p/(p-1)}]^{1-1/p} + [3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}]^{1-1/p}\}.\end{aligned} \tag{1.8}$$

Theorem 1.4 (Theorems 1 and 2 of [9]). *Assume that $a, b \in \mathbb{R}$ with $a < b$ and that $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b).*

(1) *If $|f'|$ is quasi-convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)\sup\{|f'(a)|, |f'(b)|\}}{4}. \tag{1.9}$$

(2) *If $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$ for $p > 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\qquad \le \frac{b-a}{2(p+1)^{1/p}}\sup\{|f'(a)|, |f'(b)|\}.\end{aligned} \tag{1.10}$$

Theorem 1.5 (Theorems 2.3 and 2.4 of [1]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$.*

(1) *If $|f'|^p$ is quasi-convex on $[a, b]$ for $p > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4(p+1)^{1/p}}\left(\sup\left\{|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(b)|\right\}\right). \tag{1.11}$$

(2) *If $|f'|^q$ is quasi-convex on $[a, b]$ for $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{8}\sup\left\{|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(b)|\right\}. \tag{1.12}$$

In recent years, some other kinds of Hermite–Hadamard type inequalities were generated, for example, [2–4, 10, 14–26]. For more systematic information, please refer to monographs [7, 8, 12] and related references therein.

In this paper, we will introduce a new concept 'geometrically quasi-convex function' and establish some integral inequalities of Hermite–Hadamard type for functions whose derivatives are of geometric quasi-convexity.

2. Definition and lemmas

In this section, we introduce the notion 'geometrically quasi-convex function' and establish an integral identity.

DEFINITION 2.1

A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}_0$ is said to be geometrically quasi-convex on I if

$$f(x^\lambda y^{1-\lambda}) \le \sup\{f(x), f(y)\} \tag{2.1}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Remark 1. If $f(x)$ is decreasing and geometrically quasi-convex on $I \subseteq \mathbb{R}_+$, then it is quasi-convex on I. If $f(x)$ is increasing and quasi-convex on $I \subseteq \mathbb{R}_+$, then it is geometrically quasi-convex on I.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L([a, b])$, then*

$$\frac{(\ln b)f(b) - (\ln a)f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\mathrm{d}x = \int_0^1 a^{1-t}b^t \ln(a^{1-t}b^t) f'(a^{1-t}b^t)\mathrm{d}t. \tag{2.2}$$

Proof. Letting $x = a^{1-t}b^t$ for $t \in [0, 1]$ and integrating by parts give

$$\begin{aligned}
&(\ln b - \ln a)\int_0^1 a^{1-t}b^t \ln(a^{1-t}b^t) f'(a^{1-t}b^t)\mathrm{d}t \\
&= \int_0^1 \ln(a^{1-t}b^t) f'(a^{1-t}b^t)\mathrm{d}(a^{1-t}b^t) \\
&= \int_a^b (\ln x) f'(x)\mathrm{d}x \\
&= (\ln x) f(x)|_{x=a}^{x=b} - \int_a^b \frac{f(x)}{x}\mathrm{d}x \\
&= (\ln b) f(b) - (\ln a) f(a) - \int_a^b \frac{f(x)}{x}\mathrm{d}x.
\end{aligned}$$

Lemma 2.1 is proved. □

Lemma 2.2. *For $b > a > 0$, we have*

$$M(a,b) = \int_0^1 |\ln(a^{1-t}b^t)|\mathrm{d}t = \begin{cases} \dfrac{\ln a + \ln b}{2}, & a \ge 1, \\ \dfrac{(\ln a)^2 + (\ln b)^2}{\ln b - \ln a}, & a < 1 < b, \\ -\dfrac{\ln a + \ln b}{2}, & b \le 1 \end{cases} \tag{2.3}$$

and

$$N(a,b) = \int_0^1 a^{1-t}b^t |\ln(a^{1-t}b^t)| = \begin{cases} \dfrac{b\ln b - a\ln a - (b-a)}{\ln b - \ln a}, & a \ge 1, \\ \dfrac{b\ln b + a\ln a + 2 - b - a}{\ln b - \ln a}, & a < 1 < b, \\ \dfrac{b - a - (b\ln b - a\ln a)}{\ln b - \ln a}, & b \le 1. \end{cases} \tag{2.4}$$

Proof. This follows from a straightforward computation of definite integrals. □

3. Some Hermite–Hadamard type inequalities

In this section, we will establish some integral inequalities of Hermite–Hadamard type for functions whose derivatives are of geometric quasi-convexity.

Theorem 3.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I° and $f' \in L([a,b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is geometrically quasi-convex on $[a,b]$, then*

$$\left|\frac{(\ln b) f(b) - (\ln a) f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\mathrm{d}x\right| \\ \leq N(a,b)\sup\{|f'(a)|, |f'(b)|\}, \tag{3.1}$$

where $N(a,b)$ is defined by (2.4).

Proof. From Lemma 2.1, it follows that

$$\left|\frac{(\ln b) f(b) - (\ln a) f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\mathrm{d}x\right| \\ \leq \int_0^1 a^{1-t}b^t |\ln(a^{1-t}b^t)||f'(a^{1-t}b^t)|\mathrm{d}t. \tag{3.2}$$

Using the geometric quasi-convexity of $|f'(x)|$ on $[a,b]$ yields

$$|f'(a^{1-t}b^t)| \leq \sup\{|f'(a)|, |f'(b)|\}, \quad 0 \leq t \leq 1.$$

From this inequality and Lemma 2.2, it follows that

$$\int_0^1 a^{1-t}b^t |\ln(a^{1-t}b^t)||f'(a^{1-t}b^t)|\mathrm{d}t \\ \leq \sup\{|f'(a)|, |f'(b)|\}\int_0^1 a^{1-t}b^t |\ln(a^{1-t}b^t)|\mathrm{d}t \\ = N(a,b)\sup\{|f'(a)|, |f'(b)|\}. \tag{3.3}$$

Substituting (3.3) into inequality (3.2) and simplifying establishes the inequality (3.1). Theorem 3.1 is thus proved. □

COROLLARY 3.2

Let $b > a > 0$ and $r \in \mathbb{R}$ and let

$$I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{3.4}$$

$$L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b, \end{cases} \tag{3.5}$$

and

$$L_r(a,b) = \begin{cases} \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, & r \neq -1, 0, \\ L(a,b), & r = -1, \\ I(a,b), & r = 0 \end{cases} \tag{3.6}$$

denote respectively the exponential, logarithmic, and generalized logarithmic means of two positive numbers a and b.

(1) *If* $a \geq 1$ *and* $r > 0$, *then*

$$\ln I(a^{r+1}, b^{r+1}) \leq \frac{(r+1)b^r}{[L_r(a,b)]^r} \ln I(a,b). \tag{3.7}$$

(2) *If* $b \leq 1$ *and* $r < -1$, *then*

$$\ln I(a^{r+1}, b^{r+1}) \leq -\frac{|r+1|a^r}{[L_r(a,b)]^r} \ln I(a,b). \tag{3.8}$$

Proof. Set $f(x) = x^{r+1}$ for $x \in \mathbb{R}_+$ and $r \in \mathbb{R}$ with $r \neq -1$. If $y > x > 0$,

$$|f'(x^t y^{1-t})| = |r+1|(x^t y^{1-t})^r \leq \begin{cases} |r+1|y^r, & r \geq 0, \\ |r+1|x^r, & r < 0. \end{cases}$$

This shows that the function $|f'(x)| = |r+1|x^r$ is geometrically quasi-convex on $\mathbb{R}_+$ for $r \in \mathbb{R}$ with $r \neq -1$. On the other hand,

$$\begin{aligned} b^{r+1}\ln b - a^{r+1}\ln a &= \frac{1}{r+1}\ln\left[\frac{(b^{r+1})^{b^{r+1}}}{(a^{r+1})^{a^{r+1}}}\right] \\ &= \frac{b^{r+1}-a^{r+1}}{r+1}[\ln I(a^{r+1}, b^{r+1}) + 1] \end{aligned}$$

and

$$\int_a^b \frac{f(x)}{x}\,\mathrm{d}x = \int_a^b x^r\,\mathrm{d}x = \frac{b^{r+1}-a^{r+1}}{r+1}.$$

Substituting these scalars into Theorem 3.1 yields the required results. □

Theorem 3.3. *Let* $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ *be differentiable on* I° *and* $f' \in L([a,b])$ *for* $a, b \in I^\circ$ *with* $a < b$. *If* $|f'(x)|^q$ *is geometrically quasi-convex on* $[a,b]$ *for* $q > 1$, *then*

$$\begin{aligned} &\left|\frac{(\ln b)f(b) - (\ln a)f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| \leq [M(a,b)]^{1/q} \\ &\quad \times \left[\frac{q-1}{q} N(a^{q/(q-1)}, b^{q/(q-1)})\right]^{1-1/q} \sup\{|f'(a)|, |f'(b)|\}, \end{aligned} \tag{3.9}$$

where $M(u,v)$ *and* $N(u,v)$ *are defined by* (2.3) *and* (2.4).

Proof. By Lemma 2.1, Hölder's inequality, and the geometric quasi-convexity of $|f'(x)|^q$ on $[a,b]$, we have

$$\begin{aligned} &\left|\frac{(\ln b)f(b) - (\ln a)f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\right| \\ &\quad \leq \int_0^1 a^{1-t}b^t|\ln(a^{1-t}b^t)||f'(a^{1-t}b^t)|\,\mathrm{d}t \end{aligned}$$

$$\leq \left[\int_0^1 a^{q(1-t)/(q-1)} b^{qt/(q-1)} |\ln(a^{1-t}b^t)| \mathrm{d}t\right]^{1-1/q} \times \left[\int_0^1 |\ln(a^{1-t}b^t)||f'(a^{1-t}b^t)|^q \mathrm{d}t\right]^{1/q}$$

$$\leq \left[\int_0^1 a^{q(1-t)/(q-1)} b^{qt/(q-1)} |\ln(a^{1-t}b^t)| \mathrm{d}t\right]^{1-1/q} \times \left[\int_0^1 |\ln(a^{1-t}b^t)| \mathrm{d}t\right]^{1/q} \sup\{|f'(a)|, |f'(b)|\},$$

where using Lemma 2.2 shows

$$\int_0^1 a^{q(1-t)/(q-1)} b^{qt/(q-1)} |\ln(a^{1-t}b^t)| \mathrm{d}t = \frac{(q-1)^2}{q^2} N(a^{q/(q-1)}, b^{q/(q-1)})$$

and

$$\int_0^1 |\ln(a^{1-t}b^t)| \mathrm{d}t = M(a, b).$$

The proof of Theorem 3.3 is complete. □

Theorem 3.4. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I° and $f' \in L([a, b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'(x)|^q$ is geometrically quasi-convex on $[a, b]$ for $q > 1$ and $q > \ell > 0$, then*

$$\left|\frac{(\ln b) f(b) - (\ln a) f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} \mathrm{d}x\right| \leq \left(\frac{q-1}{q-\ell}\right)^{1-1/q} \left(\frac{1}{\ell}\right)^{1/q} [N(a^\ell, b^\ell)]^{1/q} [N(a^{(q-\ell)/(q-1)}, b^{(q-\ell)/(q-1)})]^{1-1/q} \times \sup\{|f'(a)|, |f'(b)|\}. \tag{3.10}$$

where $N(u, v)$ is defined by (2.4).

Proof. From Lemma 2.1, Hölder's inequality, and the geometric quasi-convexity of $|f'(x)|^q$ on $[a, b]$ and by Lemma 2.2 it follows that

$$\left|\frac{(\ln b) f(b) - (\ln a) f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} \mathrm{d}x\right| \leq \int_0^1 a^{1-t} b^t |\ln(a^{1-t}b^t)||f'(a^{1-t}b^t)| \mathrm{d}t$$

$$\leq \left[\int_0^1 a^{(q-\ell)(1-t)/(q-1)} b^{(q-\ell)/(q-1)t} |\ln(a^{1-t}b^t)| \mathrm{d}t\right]^{1-1/q} \times \left[\int_0^1 a^{\ell(1-t)} b^{\ell t} |\ln(a^{1-t}b^t)||f'(a^{1-t}b^t)|^q \mathrm{d}t\right]^{1/q}$$

$$
\begin{aligned}
&\le \left[\int_0^1 a^{(q-\ell)(1-t)/(q-1)} b^{(q-\ell)t/(q-1)} |\ln(a^{1-t}b^t)| \mathrm{d}t\right]^{1-1/q} \\
&\quad \times \left[\int_0^1 a^{\ell(1-t)} b^{\ell t} |\ln(a^{1-t}b^t)| \mathrm{d}t\right]^{1/q} \sup\{|f'(a)|, |f'(b)|\} \\
&= \left(\frac{q-1}{q-\ell}\right)^{1-1/q} \left(\frac{1}{\ell}\right)^{1/q} [N(a^{(q-\ell)/(q-1)}, b^{(q-\ell)/(q-1)})]^{1-1/q} \\
&\quad \times [N(a^\ell, b^\ell)]^{1/q} \sup\{|f'(a)|, |f'(b)|\}.
\end{aligned}
$$

The proof of Theorem 3.4 is complete. □

Theorem 3.5. *Let $f : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}_0$ be a geometrically quasi-convex function on $[a, b]$ and $f \in L([a, b])$. Then*

$$
f((ab)^{1/2}) \le \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \mathrm{d}x \le \sup\{f(a), f(b)\}. \tag{3.11}
$$

Proof. Since

$$
(ab)^{1/2} = a^{(1-t)/2} b^{t/2} a^{t/2} b^{(1-t)/2}
$$

for $0 \le t \le 1$, by the geometric quasi-convexity of $f(x)$ on $[a, b]$, we have

$$
f((ab)^{1/2}) \le \sup\{f(a^{1-t}b^t), f(a^t b^{1-t})\} \le \sup\{f(a), f(b)\}
$$

and

$$
\int_0^1 f(a^{1-t}b^t) \mathrm{d}t = \int_0^1 f(a^t b^{1-t}) \mathrm{d}t = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \mathrm{d}x.
$$

The proof of Theorem 3.5 is complete. □

Theorem 3.6. *Let $f, g : [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}_0$ be geometrically quasi-convex functions on $[a, b]$ and $fg \in L([a, b])$. Then*

$$
\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} g(x) \mathrm{d}x \le \sup\{f(a)g(a), f(a)g(b), f(b)g(a), f(b)g(b)\}.
$$

Proof. Letting $x = a^{1-t}b^t$ for $0 \le t \le 1$ and using the geometric quasi-convexity of $f(x)$ and $g(x)$ on $[a, b]$ yields

$$
\begin{aligned}
&\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} g(x) \mathrm{d}x \\
&\quad = \int_0^1 f(a^{1-t}b^t) g(a^{1-t}b^t) \mathrm{d}t \le \sup\{f(a), f(b)\} \sup\{g(a), g(b)\}.
\end{aligned}
$$

The proof of Theorem 3.6 is complete. □

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