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Schematic Harder–Narasimhan stratification for families of principal bundles and Λ-modules

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Abstract. Let G be a reductive algebraic group over a field k of characteristic zero, let $X \to S$ be a smooth projective family of curves over k, and let E be a principal G bundle on X. The main result of this note is that for each Harder–Narasimhan type τ there exists a locally closed subscheme $S^{\tau}(E)$ of S which satisfies the following universal property. If $f : T \to S$ is any base-change, then f factors via $S^{\tau}(E)$ if and only if the pullback family f^*E admits a relative canonical reduction of Harder–Narasimhan type τ. As a consequence, all principal bundles of a fixed Harder–Narasimhan type form an Artin stack. We also show the existence of a schematic Harder–Narasimhan stratification for flat families of pure sheaves of Λ-modules (in the sense of Simpson) in arbitrary dimensions and in mixed characteristic, generalizing the result for sheaves of $\mathcal{O}$-modules proved earlier by Nitsure. This again has the implication that Λ-modules of a fixed Harder–Narasimhan type form an Artin stack.

Keywords. Principal G-bundle; Harder–Narasimhan type; canonical reduction.

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1. Introduction

Let G be a reductive algebraic group over a field k of characteristic zero, and let $T \subset B \subset G$ be a chosen maximal torus and a Borel subgroup of G. We assume that T is split over k. If X is a curve over k (always assumed to be smooth and projective over k and geometrically irreducible) and if E is a principal G-bundle on X (locally trivial in étale topology), then it is known that E admits a unique *canonical reduction* (P, σ) (see [1–3]). Here, $B \subset P \subset G$ is a standard parabolic subgroup of G and $\sigma : X \to E/P$ is a section which gives a reduction of structure group of E from G to P. Any reduction (P', σ') of E to an arbitrary parabolic subgroup $B \subset P' \subset G$ gives rise to a point of the closed positive Weyl chamber $\bar{C} \subset \mathbb{Q} \otimes X_*(T)$. The particular point defined by the canonical reduction of E is called the *Harder Narasimhan type* (or simply the HN type) of E. We denote it by $\mathrm{HN}(E) \in \bar{C}$. Given any family of curves X/S, where S is a noetherian k-scheme, and a family of principal bundles E over it, the HN type $\mathrm{HN}(E_s)$ of its restriction $E_s = E|_{X_s}$ to the fiber X_s of X over $s \in S$ is known to be an upper semicontinuous function on the underlying topological space $|S|$ of the parameter scheme S (we give a proof later, as it is not easy to find a complete proof in the literature) with respect to a natural partial order on the closed positive Weyl chamber. Thus, for each point $\tau \in \bar{C}$, the level subset $|S|^{\tau}(E)$ of $|S|$ where $\mathrm{HN}(E_s) = \tau$ is locally closed.

Our main result is that for any family E over X/S, each such level set $|S|^{\tau}(E)$ admits a unique structure of a locally closed subscheme $S^{\tau}(E) \subset S$, which has the following universal property. If $T \to S$ is a morphism of k-schemes, such that the base-change E_T which is a principal bundle over X_T admits a *relative canonical reduction* over T with constant HN type τ, then $T \to S$ factors uniquely via $S^{\tau}(E) \hookrightarrow S$.

This note is organized as follows. In § 2, we quickly recall all basic definitions and results about principal bundles that we need. In § 3, we prove our main result and some corollaries.

Section 4 is devoted to the HN stratification of the parameter scheme S for a family E of $\mathcal{O}$-coherent, flat, pure dimensional Λ-modules (in the sense of Simpson [12]) over a family X/S of projective schemes over an arbitrary locally noetherian scheme S (which may have mixed characteristic). Generalizing the main result of Nitsure [9], which is the case when $\Lambda = \mathcal{O}$, we show that there exists a schematic Harder–Narasimhan stratification $\cup_{\tau} S^{\tau}(E)$ of S, with the universal property that an arbitrary base change $T \to S$ factors via a stratum $S^{\tau}(E)$ if and only if the base-change E_T admits a relative HN filtration (as Λ-modules) of constant HN type τ on T.

2. Preliminaries on principal bundles

Let k be a field of characteristic zero, and let $G \supset B \supset T$ be a reductive group, together with a chosen Borel subgroup and a maximal torus (we assume that G is split over k). As usual, $X^*(T)$ and $X_*(T)$ will respectively denote the groups of all characters and all 1-parameter subgroups of T. Let $\Delta \subset X^*(T)$ be the corresponding simple roots. Let $\omega_{\alpha} \in \mathbb{Q} \otimes X^*(T)$ denote the fundamental dominant weight corresponding to $\alpha \in \Delta$, so that $\langle \omega_{\alpha}, \beta^{\vee} \rangle = \delta_{\alpha,\beta}$ where $\beta^{\vee} \in \mathbb{Q} \otimes X_*(T)$ is the simple coroot corresponding to $\beta \in \Delta$. Note that each ω_{α} is a non-negative linear combination (with coefficients in $\mathbb{Q}^{\geq 0}$) of the simple roots α. Recall that *the closed positive Weyl chamber* $\bar{C}$ is the subset of $\mathbb{Q} \otimes X_*(T)$ defined by the condition that $\mu \in \bar{C}$ if and only if $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \Delta$. The standard *partial order* on $\bar{C}$ is defined by putting $\mu \leq \pi$ if $\omega_{\alpha}(\mu) \leq \omega_{\alpha}(\pi)$ for all $\alpha \in \Delta$, and $\chi(\mu) = \chi(\pi)$ for all $\chi : G \to \mathbb{G}_m$.

By definition, all roots and weights in $X^*(T)$ are trivial on the connected component $Z_0(G) \subset T$ of the center of G.

We choose, once for all, a Weyl group invariant positive definite bilinear form on $\mathbb{Q} \otimes X^*(T)$ taking values in $\mathbb{Q}$. This, in particular, will allow us to identify $\mathbb{Q} \otimes X^*(T)$ with $\mathbb{Q} \otimes X_*(T)$.

Let $I \subset \Delta$ be any subset. The following elementary property (which occurs in [3], and which holds in any abstract root system simply because $\oplus_{\beta \in I} \mathbb{Q}\beta$ is the orthogonal complement of $\oplus_{\gamma \in \Delta - I} \mathbb{Q}\omega_{\gamma}$ and the angle between any two simple roots is $\geq \pi/2$ radians) will be useful in what follows.

(**R**) Given any $\alpha \in \Delta - I$, there exist (unique) rational numbers $n_{\beta} \geq 0$ where $\beta \in I$, such that the element $\chi_{\alpha} = \alpha + \sum_{\beta \in I} n_{\beta}\beta$ lies in $\oplus_{\gamma \in \Delta - I} \mathbb{Q}\omega_{\gamma}$.

Canonical reductions of principal bundles

Let $T \subset B \subset G$ be as above. Recall that a principal G-bundle E on a smooth irreducible projective curve X/k is said to be *semistable* if for any reduction $\sigma : X \to E/P$

of the structure group to a parabolic $P \subset G$ and any dominant character $\chi : P \to \mathbb{G}_m$, we have

$$\deg(\chi_*\sigma^*E) \leq 0,$$

where σ^*E is the principal P-bundle on X defined by the reduction σ, and $\chi_*\sigma^*E$ is the $\mathbb{G}_m$-bundle obtained by extending its structure group via $\chi : P \to \mathbb{G}_m$, which is equivalent to a line bundle on X.

A *canonical reduction* of a principal G-bundle E is a pair (P, σ) where P is a standard parabolic subgroup of G (that is, $B \subset P$ for the chosen Borel B) and $\sigma : X \to E/P$ is a reduction of the structure group to P such that the following two conditions (C-1) and (C-2) hold:

(C-1) If $\rho : P \to L = P/U$ is the Levi quotient of P (where U is the unipotent radical of P) then the principal L-bundle $\rho_*\sigma^*E$ is semistable.
(C-2) For any non-trivial character $\chi : P \to \mathbb{G}_m$ whose restriction to the chosen maximal torus $T \subset B \subset P$ is a non-negative linear combination $\sum n_i\alpha_i$ of simple roots $\alpha_i \in \Delta$ (where $n_i \geq 0$, and at least one $n_i \neq 0$), we have $\deg(\chi_*\sigma^*E) > 0$.

Remark 2.1. It was shown by Behrend [2] (see [3] for another proof) that each E has a unique canonical reduction. Moreover, under any extension of base fields, it is easy to see that the canonical reduction base changes.

The standard parabolics $G \supset P \supset B$ of G are in one-to-one correspondence with the subsets of Δ, under which to any P there is associated the subset $I_P \subset \Delta$ of the corresponding inverted simple roots (for example, $I_G = \Delta$ while $I_B = \emptyset$). Let $\hat{P}$ denote the group of all characters $\chi : P \to \mathbb{G}_m$, and let $\hat{P}|_T \subset X^*(T)$ be the group of their restrictions to T. We have an internal direct sum decomposition

$$\mathbb{Q} \otimes X^*(T) = (\mathbb{Q} \otimes \hat{P}|_T) \oplus (\oplus_{\alpha \in I_P}\mathbb{Q}\alpha),$$

where, in turn,

$$\mathbb{Q} \otimes \widehat{P}|_T = (\oplus_{\alpha \in \Delta - I_P}\mathbb{Q}\omega_\alpha) \oplus (\mathbb{Q} \otimes \hat{G}|_T).$$

Given a reduction (P, σ) of E, we get an element $\mu_{(P,\sigma)} \in \mathbb{Q} \otimes X_*(T)$ defined by

$$\langle \chi, \mu_{(P,\sigma)} \rangle = \begin{cases} \deg(\chi_*\sigma^*E), & \text{if } \chi \in \hat{P}, \\ 0, & \text{if } \chi \in I_P. \end{cases}$$

If (P, σ) is the canonical reduction of E, then the element $\mu_{(P,\sigma)}$ is called the *HN type of* E, and is denoted by $\mathrm{HN}(E)$. If $\alpha \in \Delta - I_P$ and $\chi_\alpha = \alpha + \sum_{\beta \in I_P} n_\beta\beta \in \oplus_{\gamma \in \Delta - I_P}\mathbb{Q}\omega_\gamma$, where $n_\beta \in \mathbb{Q}^{\geq 0}$, is the rational character given by the property (R) of root systems mentioned earlier, then

$$\langle \chi_\alpha, \mathrm{HN}(E) \rangle = \deg(\chi_{\alpha *}\sigma^*E) \geq 0$$

for all $\alpha \in \Delta - I_P$ by the condition (C-2) above. As $\langle \beta, \mathrm{HN}(E) \rangle = 0$ for all $\beta \in I_P$, we get $\langle \alpha, \mathrm{HN}(E) \rangle = \langle \chi_\alpha, \mathrm{HN}(E) \rangle \geq 0$ for all $\alpha \in \Delta - I_P$. Hence $\mathrm{HN}(E)$ is in the closed positive Weyl chamber $\bar{C}$, in fact, in the facet of $\bar{C}$ defined by the vanishing of all $\beta \in I_P$.

Note that a principal bundle E of type $\mathrm{HN}(E) = \mu$ is semistable if and only if μ is central, that is, $\mu = a\nu$ for some 1-parameter subgroup $\nu : \mathbb{G}_m \to Z_0(G)$ and $a \in \mathbb{Q}$.

Given the HN-type $\mu = \mathrm{HN}(E)$ of E, we can recover the corresponding standard parabolic P as follows. Let $I_\mu \subset \Delta$ be the set of all simple roots β such that $\langle \beta, \mu \rangle = 0$. Then I_μ is exactly the set of inverted simple roots which defines P. Alternatively, let $n \geq 1$ be any integer such that $\nu = n\mu \in X_*(T)$. The k-valued points of P are all those g for which $\lim_{t\to 0} \nu(t)g\nu(t)^{-1}$ exists in G.

For any standard parabolic P, let $\theta_P = -\sum_{\Delta - I_P} \alpha : P \to \mathbb{G}_m$ be the character which defines the ample line bundle $\det(T_{G/P})$ on G/P. As remarked in [3], for any reduction (P, σ) of a given E, the vector bundle $\sigma^*(T_{E/P})$ is a quotient of $Ad(E)$ and has determinant equal to $\theta_{P*}\sigma^*E$, hence the set of integers $\deg \theta_{P*}\sigma^*E$ as (P, σ) varies is bounded below. Let its minimum element be denoted by d_E.

We will find the following fact useful (see Remark 3.2 of [3]):

(C-3) A parabolic reduction (P, σ) to a standard parabolic P is the canonical reduction of E if and only if $\deg \theta_{P*}\sigma^*E = d_E$ and (C-2) is satisfied.

Therefore by the definition of $\mathrm{HN}(E)$, when (P, σ) is the canonical reduction of E, we get $\langle \theta_P, \mathrm{HN}(E) \rangle = d_E$.

We will also use the following elementary, well-known fact.

(C-4) Let E be a principal G-bundle on a curve X. Let (P, σ) be its canonical reduction, and let (Q, τ) be any reduction to a standard parabolic. Let $\mathrm{HN}(E) = \mu_{(P,\sigma)}$ and $\mu_{(Q,\tau)}$ be the corresponding elements of $\mathbb{Q} \otimes X_*(T)$ (the element $\mathrm{HN}(E)$ lies in the closed positive Weyl chamber $\bar{C}$, but $\mu_{(Q,\tau)}$ need not do so). Then for each $\alpha \in \Delta$ we have the inequality

$$\langle \omega_\alpha, \mathrm{HN}(E) \rangle = \langle \omega_\alpha, \mu_{(P,\sigma)} \rangle \geq \langle \omega_\alpha, \mu_{(Q,\tau)} \rangle,$$

where $\omega_\alpha \in \mathbb{Q} \otimes X^*(T)$ is the fundamental dominant weight corresponding to α. Moreover, if each of the above inequalities is an equality, then $(P, \sigma) = (Q, \tau)$.

Families of principal bundles

We now fix a smooth projective morphism $X \to S$ of relative dimension one, such that each geometric fiber is irreducible, and S is a noetherian scheme over k. By a *family* of principal G-bundles on X/S parametrized by an S-scheme T one means a principal G-bundle E on $X_T = X \times_S T$. The following facts are well-known for any such family.

(1) *Openness of semistability*. All $t \in T$ such that $E_t = E|_{X_t}$ is semistable form an open subset of $|T|$ (see Corollary 3.18 of [10]). The corresponding open subscheme of T will be called the open subscheme of semistable bundles.

(2) *Semicontinuity of HN type*. Given any $\mu \in \bar{C}$, the set $|T|^\mu$ of all $t \in T$ such that $E_t = E|_{X_t}$ is of type μ is a locally closed subset of $|T|$. The closure of any $|T|^\mu$ is contained in the union of all $|T|^\nu$ for $\nu \geq \mu$. This is well-known (a sketch of a proof that (1) implies (2) is given in the next section).

Let E be a family of principal G-bundles on X/S, such that the HN-type $\mathrm{HN}(E_s)$ is constant (say $= \tau \in \bar{C}$) for each $s \in S$. A *relative canonical reduction* for $E/X/S$

is a reduction (P, σ) of the structure group of E to a standard parabolic P over all of X, which restricts to give the canonical reduction of E_s over X_s for each $s \in S$. Note that even if $\mathrm{HN}(E_s)$ is constant, in general such a relative canonical reduction need not exist. However, as we will prove (see Theorem 3.6 and Corollary 3.7 below), it is unique whenever it exists, and it exists in particular when S is reduced.

3. HN stratification for families of principal bundles

We begin by recalling two results of Grothendieck's FGA on Hilbert schemes (see for example [7] for an exposition).

(**A**) Given a projective morphism $W \to S$ there exists a relative Hilbert scheme $\mathrm{Hilb}_{W/S} \to S$ which represents the contravariant functor from S-schemes to *Sets* which to any S-scheme $T \to S$ associates the set of all closed subschemes $Y \subset W_T = W \times_S T$ such that $Y \to T$ is flat. The scheme $\mathrm{Hilb}_{W/S}$ is a disjoint union of open (and closed) subschemes which are projective over S, in particular, the morphism $\mathrm{Hilb}_{W/S} \to S$ satisfies the valuative criterion for properness.

(**B**) Given any projective morphisms $W \to X \to S$ where $X \to S$ is flat, there is an open subscheme $R_{W/X/S} \subset \mathrm{Hilb}_{W/S}$ which parametrizes all sections of $W \to X$ on fibers of $X \to S$. It represents the contravariant functor from S-schemes to *Sets* which associates to any S-scheme $T \to S$ the set of all sections of $W_T \to X_T$.

For any parabolic subgroup $P \subset G$, and a family E of principal G-bundles on X/S, the projection $\pi : E/P \to X$ is a smooth projective morphism. Any character $\chi : P \to \mathbb{G}_m$ defines a line bundle L_χ on E/P which is relatively ample over X whenever χ is anti-dominant. By the result (A) above, we get a relative Hilbert scheme

$$\mathrm{Hilb}_{(E/P)/S} \to S$$

which parametrizes all closed subschemes of the fibers of $E/P \to S$. By the result (B) above, this has an open subscheme

$$R_{(E/P)/X/S} \subset \mathrm{Hilb}_{(E/P)/S}$$

which parametrizes all sections of $E/P \to X$ on fibers of $X \to S$. It represents the contravariant functor from S-schemes to *Sets* which associates to any S-scheme $T \to S$ the set of all reductions $(P, \sigma : X_T \to E_T/P)$ of structure group of E_T from G to P.

Lemma 3.1. *Let X/S be a smooth projective family of geometrically irreducible curves over a noetherian integral scheme S over k. Let K be the function field of S. Let E be a principal G-bundle on X, let $P \subset G$ be a standard parabolic, and let $(P, \tau : X_K \to E_K/P)$ be a canonical reduction of E_K. Then there exists a non-empty open subscheme $U \subset S$ and a section $\sigma : X_U \to E_U/P$ such that (P, σ) is a relative canonical reduction of $E_U/X_U/U$.*

Proof. The reduction (P, τ) defines a K-valued point $\tau \in R_{(E/P)/X/S}$. Under the inclusion $R_{(E/P)/X/S} \subset \mathrm{Hilb}_{(E/P)/S}$, let this point lie in an open and closed subscheme $H^o \subset \mathrm{Hilb}_{(E/P)/S}$ which is projective over S. Let $R^o = R_{(E/P)/X/S} \cap H^o$, so $\tau \in R^o$. As $H^o \to S$ is projective, there exists a nonempty open subscheme $U_1 \subset S$ and a section $\tau_1 : U_1 \to H^o$ which extends τ. Let $U_2 = \tau_1^{-1}(R^o) \subset S$, which is open and is nonempty as it contains the generic point Spec$K$. This gives a reduction $\tau_2 : X_{U_2} \to E_{U_2}/P$ which extends $\tau : X_K \to E_K/P$. Its associated Levi bundle is semistable over the generic point of U_2, so by openness of semistability, the associated Levi bundle is semistable over an open neighbourhood U of the generic point in U_2. Taking $\sigma = \tau_2|_U$, we get the desired relative canonical reduction (P, σ) of E_U over X_U. The condition (C-1) is satisfied by the definition of U, and the condition (C-2) is satisfied by the constancy of deg $\chi_* \sigma_s^* E_s$ as s varies over U. □

Lemma 3.2. *Let X be a noetherian integral scheme of dimension ≥ 1 which is normal. Let $\pi : W \to X$ be a projective morphism. Let $F \subset X$ be a closed subset of dimension zero and let $\sigma : X - F \to W$ be a section of π outside F. Let $V \subset W$ be the closure of $\sigma(X - F)$. If for each $x \in F$, the set $\pi^{-1}(x) \cap V$ is finite then σ prolongs to a global section of $\pi : W \to X$.*

Proof. It is enough to assume that F is a singleton set $\{x\}$ for some closed point $x \in X$. We can take an affine open U in W such that the finite set $\pi^{-1}(x) \cap V$ is inside U. Then by properness of π, the subset $X' = X - \pi(V - U)$ is open in X, and $F \subset X'$. The restricted section $\sigma|_{X'-F} : X' - F \to U$ is given by regular functions as U is affine, so prolongs to F by the normality of X'. □

We now prove a lemma concerning the upper semicontinuity of the HN-types whose analogue for vector bundles was proved by Shatz (see [11]).

Lemma 3.3. *Let R be a discrete valuation ring over k, with quotient field K and residue field k_1. Let X be a smooth projective family of curves over $S =$ Spec R with geometrically irreducible fibers $X_0 = X \otimes_R K$ and $X_1 = X \otimes_R k_1$ over the generic point $\eta_0 =$ Spec K and the special point $\eta_1 =$ Spec k_1 of S. Let E be a principal G-bundle on X. Then the following holds*:

(1) *Between the HN-types of the restrictions $E_0 = E|_{X_0}$ and $E_1 = E|_{X_1}$, we have the inequality* $\mathrm{HN}(E_0) \leq \mathrm{HN}(E_1)$.
(2) *If* $\mathrm{HN}(E_0) = \mathrm{HN}(E_1)$, *then there exists a relative canonical reduction for $E/X/S$.*

Proof.

(1) Let $\alpha : X_0 \to E_0/P_0$ be the canonical reduction of E_0, where P_0 is a standard parabolic in G. Let $\mathrm{Hilb}_{(E/P_0)/R}$ denote the relative Hilbert scheme which parametrizes all closed subschemes of the fibers of $E/P_0 \to$ Spec R. The image $\alpha(X_0)$ of α defines a K-valued point

$$[\alpha(X_0)] \in \mathrm{Hilb}_{(E/P_0)/R}(K).$$

By the valuative criterion of properness, which is satisfied by $\mathrm{Hilb}_{(E/P_0)/R} \to$ Spec R, the K-valued point $[\alpha(X_0)] \in \mathrm{Hilb}_{(E/P_0)/R}(K)$ prolongs uniquely to a R-valued point

$$[Y] \in \mathrm{Hilb}_{(E/P_0)/R}(R)$$

which is represented by a closed subscheme $Y \subset E/P_0$ which is flat over R, and whose fiber Y_0 over $\eta_0 =$ Spec K is $\alpha(X_0)$.

The fiber Y_1 of Y over the special point $\eta_1 =$ Spec(k_1) of S is of dimension 1. The projection $\pi_1 : Y_1 \to X_1$ is surjective as the projection $\pi : Y \to X$ is generically surjective over X, for $\pi_0 : \alpha(X_0) \to X_0$ is surjective, and $Y_0 = \alpha(X_0)$. Note that X is 2-dimensional and regular. Hence by the valuative criterion of properness applied to the projection $\pi : E/P_0 \to X$, the section $\alpha : X_0 \to E_0/P_0$ prolongs to a section

$$\sigma : X - F \to E/P_0 - \pi^{-1}(F),$$

where $F \subset X_1$ is a finite set of closed points of X_1. By closedness of Y, we must have $\sigma(X - F) \subset Y$. By the valuative criterion of properness applied to the projection $\pi_1 : E_1/P_0 \to X_1$, the section $\sigma_1 = \sigma|_{X_1 - F} : X_1 - F \to E_1/P_0$ prolongs to a section $\gamma : X_1 \to E_1/P_0$. Then $\gamma(X_1) \subset Y_1$. But Y_1 is of dimension 1, so $\gamma(X_1)$ is an irreducible component of Y_1. Let Z be the union of the remaining irreducible components of Y_1, if any. Hence we get

$$Y_1 = \gamma(X_1) \cup Z.$$

If $\delta_i : X_i \to E_i/P_i$ is any reduction, then the Hilbert polynomial of $[\delta_i(X_i)]$ with respect to any line bundle L on E/P is the Hilbert polynomial of X_i with respect to $\delta_i^* L$, that is,

$$p_L(\delta_i(X_i))(n) = \dim H^0(X_i, \delta_i^* L^{\otimes n}) \text{ for } n >> 0.$$

If g denotes the genus of X_i, then by Riemann–Roch this gives

$$p_L(\delta_i(X_i))(n) = (\deg \delta_i^* L)n + 1 - g.$$

By flatness of Y over R, the Hilbert polynomial $p_L([Y_1])$ of $[Y_1]$ is the Hilbert polynomial $p_L([\alpha(X_0)])$ of $[\alpha(X_0)]$. As a non-empty Z will have a positive Hilbert polynomial whenever L is relatively ample on $E_1/P_0 \to X_1$, from $Y_1 = \gamma(X_1) \cup Z$ we now get the following crucial fact:

(*) If a line bundle L_χ on E/P_0 is defined by an anti-dominant character $\chi : P_0 \to \mathbb{G}_m$ (so it is relatively ample over X) then we have the inequality of Hilbert polynomials $p_{L_\chi}([\gamma(X_1)]) \leq p_{L_\chi}([\alpha(X_0)])$ which is an equality if and only if Z is empty. Equivalently by Riemann–Roch, we get

$$\deg \chi_* \gamma^* E_1 \leq \deg \chi_* \alpha^* E_0,$$

where equality holds if and only if $Z = \emptyset$.

By the definition of the element $\mu_{(P_0,\sigma)} \in \mathbb{Q} \otimes X_*(T)$ introduced in the course of defining the HN-type of a principal bundle, this means

$$\langle \chi, \mu_{(P_0,\gamma)} \rangle \leq \langle \chi, \mathrm{HN}(E_0) \rangle \text{ for all anti-dominant } \chi : P_0 \to \mathbb{G}_m.$$

Replacing anti-dominant characters by fundamental positive dominant characters ω_α (which changes the direction of the inequality), we get the following:

(**) If I_0 is the set of inverted simple roots defining P_0, we have

$$\langle \omega_\alpha, \mu_{(P_0,\gamma)} \rangle \geq \langle \omega_\alpha, \mathrm{HN}(E_0) \rangle \quad \text{for all } \alpha \in \Delta - I_0.$$

By property (C-4) of canonical reductions we have

$$\langle \omega_\alpha, \mathrm{HN}(E_1) \rangle \geq \langle \omega_\alpha, \mu_{(P_0,\gamma)} \rangle \quad \text{for all } \alpha \in \Delta.$$

Hence we get

$$\langle \omega_\alpha, \mathrm{HN}(E_1)\rangle \geq \langle \omega_\alpha, \mathrm{HN}(E_0)\rangle \quad \text{for all } \alpha \in \Delta - I_0.$$

Note that $\mathrm{HN}(E_0)$ lies in the facet $C(I_0) \subset \bar{C}$ defined by the vanishing of all $\beta \in I_0$. Hence the above inequalities imply that $\mathrm{HN}(E_1) \geq \mathrm{HN}(E_0)$, which proves (1).

(2) As by assumption $\mathrm{HN}(E_0) = \mathrm{HN}(E_1)$, there is a common standard parabolic $P = P_0 \subset G$, and canonical reductions (P, α) for E_0 and (P, β) for E_1. We must show that there exists a section $\sigma : X \to E/P$ such that $\sigma_0 = \alpha$ and $\sigma_1 = \beta$ where σ_i denotes $\sigma|_{X_i}$.

Consider the relatively ample line bundle L on E/P over X which comes from the anti-dominant character θ_P on P. By the property (C-3) of a canonical reduction we must have $d_{E_1} \leq \deg \theta_{P*}\gamma^* E_1$ and $\deg \theta_{P*}\alpha^* E_0 = d_{E_0}$. Hence from (*) above we get

$$d_{E_1} \leq \deg \theta_{P*}\gamma^* E_1 \leq \deg \theta_{P*}\alpha^* E_0 = d_{E_0}.$$

But as by assumption $\mathrm{HN}(E_0) = \mathrm{HN}(E_1)$, we have $d_{E_0} = d_{E_1}$, so we get the equality

$$\deg \theta_{P*}\gamma^* E_1 = \deg \theta_{P*}\alpha^* E_0.$$

Hence we can conclude by (*) that Z is empty, so $Y_1 = \gamma(X_1)$. As $Y_1 = \gamma(X_1)$, the fibres of Y over F are singletons. By Lemma 3.2, this implies that $\sigma : X - F \to E/P$ prolongs to a global section over X, so we can take F to be empty, and regard $\sigma : X \to E/P$ as a global section. As γ is the restriction $\sigma|_{X_1}$, the reduction (P, γ) of E_1 satisfies (C-2) by continuity from $\alpha = \sigma|_{X_0}$. As $d_{E_1} = \deg \theta_{P*}\gamma^* E_1$, it follows by (C-3) that (P, γ) is the canonical reduction of E_1. Thus, $\sigma : X \to E/P$ is the desired relative canonical reduction for E over R. This completes the proof of (2), and of Lemma 3.3. □

Lemmas 3.1 and 3.3(1) allow us to give a proof of the following known result.

PROPOSITION 3.4

Let X be a smooth family of geometrically irreducible curves over a locally noetherian scheme S over k. Let E be a principal G-bundle on X. Then the function $|S| \to \bar{C}$ under which $s \mapsto \mathrm{HN}(E_s)$ is upper semicontinuous. Consequently, for each $\tau \in \bar{C}$, the level subset $|S|^\tau(E)$ is locally closed in $|S|$, and for any maximal element $\tau \in \{\mathrm{HN}(E_s) \mid s \in S\} \subset \bar{C}$, the level set $|S|^\tau(E)$ is closed in $|S|$.

Proof (*Sketch*). It is enough to assume that S is integral. Then by Lemma 3.1 and noetherian induction, it follows that each level subset $|S|^\tau(E)$ is constructible. Lemma 3.3(1) now implies that the map $s \mapsto \mathrm{HN}(E_s)$ is upper semicontinuous. The result follows. □

For the following result, see Proposition 3.7 of Kumar-Narasimhan [5].

Lemma 3.5. *Let X be a smooth geometrically irreducible projective curve over a field k, let E be a principal G-bundle on it, and let (P, σ) be a reduction of E. Let $N_{\sigma(X),E/P}$ be the normal bundle to the image $\sigma(X)$ in E/P. Then $\sigma^* N_{\sigma(X),E/P}$ is the vector bundle on X associated to the principal P-bundle $\sigma^* E$ by the representation of P on* $\mathrm{Lie}(G)/\mathrm{Lie}(P)$ *induced by the adjoint representation of G on* $\mathrm{Lie}(G)$. *If (P, σ) is the canonical reduction, then $H^0(\sigma(X), N_{\sigma(X),E/P}) = 0$.* □

We now come to our main result.

Theorem 3.6 (Main theorem). *Let X/S be a smooth projective family of geometrically irreducible curves over a locally noetherian scheme S/k, and let E be a principal G-bundle on X. Then for each $\tau \in \bar{C}$, there exists a unique locally closed subscheme $S^\tau(E) \subset S$ with the following universal property. Any morphism $f : T \to S$ of k-schemes factors via $S^\tau(E) \subseteq S$ if and only if the pullback $E_T = f^*E$ on X_T admits a global relative canonical reduction $(P, \sigma : X_T \to E_T/P)$ with constant HN type τ on T. Moreover, a global relative canonical reduction for E_T, whenever it exists, is unique.*

Proof. By Proposition 3.4, the subset $|S|^{\leq\tau}(E)$ is open in S. We give it the structure of an open subscheme of S, which we denote as $S^{\leq\tau}(E)$. The level subset $|S|^\tau(E)$ is closed in $S^{\leq\tau}(E)$. In what follows, we endow $|S|^\tau(E)$ with a structure of a closed subscheme $S^\tau(E)$ of $S^{\leq\tau}(E)$ which satisfies the desired universal property. Thus, we can replace S by $S^{\leq\tau}(E)$ for proving the result, and thereby assume that τ is maximal and $|S|^\tau(E)$ is closed in S.

Let $P = P_\tau \subset G$ be the standard parabolic determined by the maximal type τ. Let $R_{(E/P)/X/S} \subset \mathrm{Hilb}_{(E/P)/S}$ be the open subscheme given in the statement (B) at the beginning of § 2. For each anti-dominant character $\chi : P \to \mathbb{G}_m$ of the form $\chi = -\omega_\alpha$, consider the relatively ample line bundle L_χ on $(E/P) \to X$. Let $R^\tau_{(E/P)/X/S} \subset R_{(E/P)/X/S}$ be the open (and closed) subscheme consisting of all points (P, σ) (where $\sigma : X_s \to E_s/P$ is a reduction) such that $\deg = \sigma^* L_\chi \langle \chi, \tau \rangle$. By Riemann–Roch, this amounts to fixing the Hilbert polynomial with respect to each of the finitely many L_χ, so indeed defines an open (and closed) subscheme by the requirement of flatness in the definition of $\mathrm{Hilb}_{(E/P)/S}$.

If R is a discrete valuation ring over k and if $\mathrm{Spec} R \to S$ is a k-morphism under which the generic point maps to a point in $|S|^\tau(E)$, then by closedness of $|S|^\tau(E)$ the special point also maps into $|S|^\tau(E)$. The morphism $R^\tau_{(E/P)/X/S} \to S$ satisfies the valuative criterion for properness, as by Lemma 3.3 there exists a lift $\mathrm{Spec} R \to R^\tau_{(E/P)/X/S}$. By uniqueness of the canonical reduction, the morphism $R^\tau_{(E/P)/X/S} \to S$ is injective. As $R^\tau_{(E/P)/X/S}$ is open in $\mathrm{Hilb}_{(E/P)/S}$, the vertical tangent space to $R^\tau_{(E/P)/X/S} \to S$ at any point (P, σ) over $s \in S$ equals the vertical tangent space to $\mathrm{Hilb}_{(E/P)/S} \to S$ at $[\sigma(X_s)]$. By a standard result on relative Hilbert schemes, this equals

$$H^0(\sigma(X_s), N_{\sigma(X_s), E_s/P}),$$

where $N_{\sigma(X_s), E_s/P}$ is the normal bundle to $\sigma(X_s) \subset E_s/P$. By Lemma 3.5, this is zero. Also note that the canonical reduction (P, σ) for any E_s of HN-type τ exists over X_s itself, that is, (P, σ) is $k(s)$-valued (see Remark 2.1). The above two facts show that the projection $R^\tau_{(E/P)/X/S} \to S$ is unramified and induces an isomorphism on residue fields.

Hence we have proved that $\pi : R^\tau_{(E/P)/X/S} \to S$ is proper, injective, unramified, and induces an isomorphism on residue fields $k(\pi(z)) \to k(z)$ for each $z \in R^\tau_{(E/P)/X/S}$. Hence by Lemma 4 of [9], π is a closed embedding. We put $S^\tau(E) \subset S$ to be its image. By its construction, this has the required universal property. □

The main theorem has the following immediate corollary.

COROLLARY 3.7

Under the hypothesis of Theorem 3.6, *if moreover S is reduced and the HN type is globally constant over S, then there exists a relative canonical reduction over S.*

The algebraic stacks of HN types

The main theorem allows us to show that all principal bundles of a given HN type form an Artin stack. The proof of this is analogous to the corresponding result for $\mathcal{O}$-coherent pure sheaves in [9], so we rapidly sketch it.

For any family of curves X/S as above, we have an Artin stack $\mathrm{Bun}(G/X/S)$ over S, which to any $T \to S$ attaches the groupoid $\mathrm{Bun}(G/X/S)(T)$ of all principal G-bundles on X_T, with pullback (pseudo)-functor defined the usual way. For any $\tau \in \bar{C}$, let $\mathrm{Bun}(G/X/S)^{\tau}(T) \subset \mathrm{Bun}(G/X/S)(T)$ be the full subgroupoid of bundles E_T which admit a global relative canonical reduction with HN type τ. This condition being preserved under pullbacks (where you just pull back the reduction), it defines a sub S-groupoid

$$\mathrm{Bun}(G/X/S)^{\tau} \subset \mathrm{Bun}(G/X/S).$$

This subgroupoid is actually a stack, as the following effective descent condition is satisfied.

Lemma 3.8. *Let T be an S-scheme and let E be an object of* $\mathrm{Bun}(G/X/S)^{\tau}(T)$. *Let $f : T' \to T$ be a faithfully flat quasi-compact morphism. If the pullback f^*E is in* $\mathrm{Bun}(G/X/S)^{\tau}(T')$, *then E is in* $\mathrm{Bun}(G/X/S)^{\tau}(T)$.

Proof (*Sketch*). This amounts to saying that the relative canonical reduction (P, σ') over T' can be descended to T. Under the two projections $\pi_1, \pi_2 : T'' \rightrightarrows T'$, where $T'' = T' \times_T T'$, the pullbacks of (P, σ') are identical by the uniqueness of a relative canonical reduction. As a reduction is just a section of E/P, by the Grothendieck result on effective descent for a closed subscheme of a scheme (see Remark 3.9 below), the reduction (P, σ') descends to give a reduction (P, σ) of E_T, which is canonical as it is pointwise so. □

Remark 3.9 (*Fundamental Algebraic Geometry*). Let $X \to T$ be a projective morphism and let $f : T' \to T$ be a faithfully flat quasi-compact morphism. Let $Y' \subset X_{T'} = X \times_T T'$ be a closed subscheme such that under the two projections $\pi_1, \pi_2 : T'' \rightrightarrows T'$, where $T'' = T' \times_T T'$, the two pullbacks of Y' are identical as closed subschemes of $X_{T''}$. Then there exists a unique closed subscheme $Y \subset X$ such that $Y' = Y_{T'}$.

Theorem 3.10. *The stack* $\mathrm{Bun}(G/X/S)^{\tau}$ *is an Artin stack, which is a locally closed substack of* $\mathrm{Bun}(G/X/S)$.

Proof. Theorem 3.6 implies (just as in the proof of Theorem 8 in [9]) that the inclusion 1-morphism $\mathrm{Bun}(G/X/S)^{\tau} \hookrightarrow \mathrm{Bun}(G/X/S)$ is representable and it is a locally closed embedding. As $\mathrm{Bun}(G/X/S)$ is known to be an Artin stack, the result follows. □

4. HN stratification for families of Λ-modules

Simpson introduced in [12] the notion of Λ-modules, which is a common generalization of important examples such as $\mathcal{O}$-modules, Higgs sheaves, integrable connections, integrable logarithmic connections, etc. Further, by working over the parameter scheme $S = \mathbf{A}^1_k$, the notion of Λ-modules incorporates Deligne's idea of continuously interpolating between Higgs bundles and integrable connections.

The result we prove here showing the existence of a schematic HN stratification for families of Λ-modules therefore applies to all the interesting special cases of Λ-modules that are stated above, including the interpolation between Higgs bundles and integrable connections. They are a generalization of the results in [9] for $\mathcal{O}$-modules. The main theorem is given together with more expository details in a chapter of the Ph.D. thesis (unpublished) of Gurjar [4], based on this joint work.

In subsection 4.1 we will recall basic definitions and facts which – explicitly or essentially–occur in the literature (mainly in [12]) or are 'standard knowledge'. The main theorem is proved in subsection 4.2.

4.1 *Preliminaries on Λ-modules*

Let S be a locally noetherian base scheme (need not even be equicharacteristic). Let $f : X \to S$ be a projective, faithfully flat morphism with a chosen relatively ample line bundle $\mathcal{O}_{X/S}(1)$. Simpson introduced in [12] the concept of a split almost-polynomial sheaf of rings of differential operators Λ on X over S. By definition, such a Λ is a sheaf of rings on X (not necessarily commutative), together with a ring homomorphism $i : \mathcal{O}_X \to \Lambda$, a filtration of Λ by subsheaves of abelian groups $\Lambda_0 \subset \Lambda_1 \subset \cdots$, and a sheaf homomorphism $\zeta : \mathrm{Gr}_1(\Lambda) \to \Lambda_1$ which satisfies the following properties (1) to (8).

(1) $\Lambda = \bigcup_{i=0}^{\infty} \Lambda_i$ and $\Lambda_i \cdot \Lambda_j \subset \Lambda_{i+j}$.
(2) The image of the given homomorphism $i : \mathcal{O}_X \to \Lambda$ is equal to Λ_0 (we will denote the resulting homomorphism $\mathcal{O}_X \to \Lambda_0$ again by i). (Note that the above two conditions make each Λ_i a left as well as a right $\mathcal{O}_X$-submodule of Λ).
(3) The image of $f^{-1}(\mathcal{O}_S)$ by the composite $f^{-1}\mathcal{O}_S \xrightarrow{f^\sharp} \mathcal{O}_X \xrightarrow{i} \Lambda$ is contained in the centre of Λ.
(4) The left and right $\mathcal{O}_X$-module structures on $\mathrm{Gr}_i(\Lambda) = \Lambda_i/\Lambda_{i-1}$ are equal.
(5) The sheaves of $\mathcal{O}_X$-modules $\mathrm{Gr}_i(\Lambda)$ are coherent.
(6) The sheaf of graded $\mathcal{O}_X$-algebras $\mathrm{Gr}(\Lambda) = \oplus_{i=0}\mathrm{Gr}_i(\Lambda)$ is generated by $\mathrm{Gr}_1(\Lambda)$ in the sense that the homomorphism $\mathrm{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathrm{Gr}_1(\Lambda) \to \mathrm{Gr}_i(\Lambda)$ is surjective.
(7) The condition that Λ is *almost polynomial*: The homomorphism $i : \mathcal{O}_X \to \Lambda_0$ should be an isomorphism, $\mathrm{Gr}_1(\Lambda)$ should be locally free over $\mathcal{O}_X$, and the graded ring $\mathrm{Gr}(\Lambda)$ should be naturally isomorphic to the symmetric $\mathcal{O}_X$-algebra on $\mathrm{Gr}_1(\Lambda)$ under the homomorphism induced by the multiplication on Λ.
(8) The condition that Λ is *split*: The given homomorphism $\zeta : \mathrm{Gr}_1(\Lambda) \to \Lambda_1$ is a homomorphism of left $\mathcal{O}_X$-modules which splits the projection $\Lambda_1 \to \mathrm{Gr}_1(\Lambda)$.

By abuse of notation, the data $(\Lambda, \Lambda_j, i, \zeta)$ will be denoted simply by Λ.

We now fix an algebra Λ as above. As Simpson explains in [12], given any $T \to S$, we functorially get a split almost-polynomial algebra Λ_T on X_T/T, which we call as the *pullback of* Λ *under* $T \to S$. Its underlying sheaf of $\mathcal{O}_{X_T}$-modules is the pullback of the left $\mathcal{O}_X$-module Λ. The data of the multiplication on Λ_T, the filtration $(\Lambda_T)_j$, the map $i_T : \mathcal{O}_{X_T} \to \Lambda_T$ and the splitting ζ_T are defined functorially.

Λ-modules, semistability, HN types

Let $(Y, \mathcal{O}_Y(1))$ be a projective scheme over a base field k, together with an ample line bundle $\mathcal{O}_Y(1)$ on it. Let Λ be a split almost-polynomial sheaf of rings of differential operators on Y over k. Unless otherwise indicated, by a Λ-module we will mean a sheaf E of left Λ-modules on Y, which is assumed to be coherent as an $\mathcal{O}_Y$-module, where the $\mathcal{O}_Y$-module structure on E is induced via $i : \mathcal{O}_Y \to \Lambda$. Simpson defines a Λ-modules E on Y to be *semistable* with respect to $\mathcal{O}_Y(1)$ if E is coherent and pure as an $\mathcal{O}_Y$-module, and moreover for any Λ-submodule $F \subset E$ the inequality $r(E)P(F) \leq r(F)P(E)$ holds, where for any coherent $\mathcal{O}_Y$-module M whose support is d-dimensional, we denote by $P(M)(m) = \sum_i (-1)^i \dim_k H^i(Y, M(m))$ the Hilbert polynomial of M, and by $r(M)$ the non-negative integer such that the leading coefficient of $P(M)$ is $r(M)/d!$. We put $r(M) = 0$ if $M = 0$.

The set HNT of all abstract HN types for $\mathcal{O}_Y$-coherent, pure d-dimensional Λ-modules on Y is the same as the set of all abstract HN types for coherent pure d-dimensional $\mathcal{O}_Y$-modules, which we now recall from [9]. A polynomial $f \in \mathbb{Q}[\lambda]$ is called a *numerical polynomial* if $f(\mathbb{Z}) \subset \mathbb{Z}$. If a nonzero numerical polynomial f has degree d, it can be uniquely expanded as $f = (r(f)/d!)\lambda^d +$ lower degree terms, where $r(f) \in \mathbb{Z}$. If $f = 0$, we put $r(f) = 0$. There is a *total order* $\leq$ on $\mathbb{Q}[\lambda]$ under which $f \leq g$ if $f(m) \leq g(m)$ for all sufficiently large integers m. The *set of all HN types*, denoted by HNT, is the set of all finite sequences $(f_1, \ldots, f_p)$ of numerical polynomials in $\mathbb{Q}[\lambda]$, where p is allowed to vary over all integers ≥ 1, such that the following three conditions are satisfied:

(1) We have $0 < f_1 < \cdots < f_p$ in $\mathbb{Q}[\lambda]$,
(2) the polynomials f_i are all of the same degree, say d, and
(3) the following inequalities are satisfied

$$\frac{f_1}{r(f_1)} > \frac{f_2 - f_1}{r(f_2) - r(f_1)} > \cdots > \frac{f_p - f_{p-1}}{r(f_p) - r(f_{p-1})}.$$

As described in [9], to each HN type $(f_1, \ldots, f_p)$ there corresponds a certain subset (HN polygon)

$$\mathrm{HNP}(f_1, \ldots, f_p) \subset \mathbb{Z} \times \mathbb{Q}[\lambda]$$

which is the union of the segments $\overline{x_0x_1} \cup \overline{x_1x_2} \cup \cdots \cup \overline{x_{p-1}x_p}$ where $x_0 = (0, 0)$ and $x_i = (r(f_i), f_i)$ for $1 \leq i \leq p$. A point $(a, f) \in \mathbb{Z} \times \mathbb{Q}[\lambda]$ is said to *lie under* another point $(b, g) \in \mathbb{Z} \times \mathbb{Q}[\lambda]$ if $a = b$ in $\mathbb{Z}$ and $f \leq g$ in $\mathbb{Q}[\lambda]$. A point $(a, f) \in \mathbb{Z} \times \mathbb{Q}[\lambda]$ is said to *lie under the polygon* $\mathrm{HNP}(g_1, \ldots, g_q)$ if there exists some $(b, g) \in \mathrm{HNP}(g_1, \ldots, g_q)$ such that the point (a, f) lies under the point (b, g). There is a *partial order* $\leq$ on HNT, under which $(f_1, \ldots, f_p) \leq (g_1, \ldots, g_q)$ if for each $1 \leq i \leq p$, the point $(r(f_i), f_i)$ lies under $\mathrm{HNP}(g_1, \ldots, g_q)$.

When a Λ-module E is $\mathcal{O}_Y$-coherent of pure dimension $d \geq 0$ but not necessarily semistable, it admits a unique strictly increasing filtration $0 = \mathrm{HN}_0(E) \subset \mathrm{HN}_1(E) \subset \cdots \subset \mathrm{HN}_\ell(E) = E$ by Λ-submodules $\mathrm{HN}_i(E)$ such that for each $1 \leq i \leq \ell$, the graded piece $\mathrm{Gr}_i(E) = \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$ is a semistable Λ-module of pure dimension d, and the inequalities

$$\frac{P(\mathrm{Gr}_1(E))}{r(\mathrm{Gr}_1(E))} > \cdots > \frac{P(\mathrm{Gr}_\ell(E))}{r(\mathrm{Gr}_\ell(E))}$$

hold. This filtration is called the *Harder–Narasimhan filtration* of E (in the sense of Gieseker semistability). The first step $\mathrm{HN}_1(E)$ is called the *maximal destabilizing subsheaf* of E. The integer ℓ (also written as $\ell(E)$) is called as the *length* of the

Harder–Narasimhan filtration of E. In these terms, a nonzero $\mathcal{O}_Y$-coherent pure-dimensional Λ-module E is semistable if and only if its Harder–Narasimhan filtration is of length $\ell(E) = 1$. The ordered $\ell(E)$-tuple

$$\mathrm{HN}(E) = (P(\mathrm{HN}_1(E)), \ldots, P(\mathrm{HN}_\ell(E))) \in \mathrm{HNT}$$

is called the *Harder–Narasimhan type* of E. The reader should note that the phrases 'HN filtration' and 'HN type' in what follows are to be understood in the sense of Λ-modules.

Remark 4.1. If $(f_1, \ldots, f_p) \in \mathrm{HNT}$, then $(f_2 - f_1, \ldots, f_p - f_1)$ is again in HNT. Let E be an $\mathcal{O}_Y$-coherent pure-dimensional Λ-module on Y with $\mathrm{HN}(E) \leq (f_1, \ldots, f_p) \in \mathrm{HNT}$. If $E' \subset E$ is a Λ-submodule with $P(E') = f_1$, then we must have $\mathrm{HN}_1(E) = E'$, that is, such an E' is automatically the maximal destabilizing subsheaf of E. The quotient $E'' = E/E'$ is pure, with $\mathrm{HN}(E'') \leq (f_2 - f_1, \ldots, f_p - f_1)$. Moreover $\mathrm{Hom}_\Lambda(E', E'') = 0$, where Hom_Λ denotes the global Λ-homomorphisms.

Remark 4.2. If $(Y, \mathcal{O}_Y(1))$ is a projective scheme over a field k and if K is any extension field of k, then a Λ-module E on Y is semistable with respect to $\mathcal{O}_Y(1)$ if and only if its base-change $E_K = E \otimes_k K$ to Y_K is a semistable Λ_K-module with respect to $\mathcal{O}_{Y_K}(1) = \mathcal{O}_Y(1) \otimes_k K$. Consequently, if E is any $\mathcal{O}_Y$-coherent pure-dimensional Λ-module on Y then the Harder–Narasimhan filtration $\mathrm{HN}_i(E_K)$ is just the pullback $\mathrm{HN}_i(E) \otimes_k K$ of the Harder–Narasimhan filtration of E.

Families and Quot schemes for Λ-modules

Let $f : X \to S$ be a projective, faithfully flat morphism with a chosen relatively ample line bundle $\mathcal{O}_{X/S}(1)$, where S is a locally noetherian scheme. Let there be given a split almost-polynomial sheaf of rings of differential operators Λ on X over S. By a *family* of Λ-modules on X/S we will mean an $\mathcal{O}_X$-coherent sheaf E of (left) Λ-modules on X, such that as an $\mathcal{O}_X$-module, E is flat over the base S. We say that the family E of Λ-modules is of *pure dimension* d if moreover each restriction $E|_{X_s}$ as an $\mathcal{O}_{X_s}$-module (where X_s is the schematic fiber over $s \in S$) is a coherent $\mathcal{O}_{X_s}$-module of pure dimension d. The scheme S is called the *parameter scheme* of the family.

If E is a family of Λ-modules on X/S, and $q : E \to F$ is a surjection of $\mathcal{O}_X$-modules where F is a coherent $\mathcal{O}_X$-module, we say that q is a Λ-*quotient* if the kernel of q (which is *a priori* only a coherent $\mathcal{O}_X$-submodule of E) is a Λ-submodule of E. In that case, note that F acquires the Λ-module structure of the quotient $E/\ker(q)$.

Lemma 4.3 [12]. *Given any coherent $\mathcal{O}_X$-module quotient $q : E \to F$ of the family E of Λ-modules on X/S, there exists a unique closed subscheme $S_\Lambda \subset S$ which has the following universal property. Any morphism $T \to S$ of schemes factors via S_Λ if and only if the $\mathcal{O}_{X_T}$-module pullback $q_T : E_T \to F_T$ on X_T is a Λ_T quotient in the above sense.*

Proof. The proof occurs within the proof of Theorem 3.8 of [12]. □

With X/S, $\mathcal{O}_{X/S}(1)$ and Λ as above, let E be a family of Λ-modules on X/S and let $f \in \mathbb{Q}[\lambda]$ be any polynomial. Let $Q = \mathrm{Quot}_{E/X/S}^{f, \mathcal{O}_X(1)}$ be the relative Quot scheme of coherent $\mathcal{O}_X$-module quotients of E_s with Hilbert polynomial f with respect to $\mathcal{O}_{X/S}(1)$. By definition, for any $T \to S$, a T-valued point of Q is an equivalence class of quotients as $\mathcal{O}_{X_T}$-modules $q_T : E_T \to F$ where F is a coherent $\mathcal{O}_{X_T}$-module flat over T, and two such quotients $q_T : E_T \to F$ and $q'_T : E_T \to F'$ are deemed to be equivalent if there exists an $\mathcal{O}_{X_T}$-module isomorphism $\phi : F \to F'$ such that $q'_T = \phi \circ q_T$. Moreover, it is required that for each $t \in T$, the Hilbert polynomial of $F_t = F|_{X_t}$ with respect to $\mathcal{O}_{X_t}(1)$ is the given polynomial $f \in \mathbb{Q}[\lambda]$, where $\mathcal{O}_{X_t}(1)$ denotes the pullback of $\mathcal{O}_{X/S}(1)$ to the fibre X_t of X_T/T. The existence of such a scheme Q over S, and its projectivity over S, are fundamental theorems of Grothendieck (see for example [7] for an expository account). As Q represents the above functor, applying the Yoneda lemma we get a universal family of quotients $q : E_Q \to F$ on X_Q.

Remark 4.4 (*Quot scheme for Λ-modules*). With X/S, $\mathcal{O}_{X/S}(1)$ and Λ as above, let E be a family of Λ-modules on X/S and let $f \in \mathbb{Q}[\lambda]$ be any polynomial. Let $Q = \mathrm{Quot}^{f,\mathcal{O}_X(1)}_{E/X/S}$ be the relative Quot scheme of coherent $\mathcal{O}_X$-module quotients of E with Hilbert polynomial f with respect to $\mathcal{O}_{X/S}(1)$. Let $Q_\Lambda \subset Q$ be its closed subscheme defined by Lemma 4.3. By its construction, Q_Λ represents the contravariant functor from S-schemes to $Sets$, which associates to any S-scheme T the set of all $\mathcal{O}_{X_T}$-coherent Λ_T-module quotient of E_T which are flat over T and whose restriction to each schematic fibre X_t for $t \in T$ has Hilbert polynomial f with respect to $\mathcal{O}_{X_t}(1)$. The universal quotient $q : E_Q \to F$ on X_{Q_Λ} will be the restriction (as $\mathcal{O}$-modules) of the universal quotient on X_Q.

PROPOSITION 4.5 (Vertical tangent spaces to the Quot scheme Q_Λ)

Let k be a field, X a projective scheme over k, Λ a split almost polynomial sheaf of rings of differential operators on X, E an $\mathcal{O}_X$-coherent Λ-module, and $q_0 : E \to F_0$ a surjective $\mathcal{O}_X$-homomorphism of coherent $\mathcal{O}_X$-modules such that q_0 is a Λ-quotient as defined earlier. Let $q_0 \in Q_\Lambda$ again denote the corresponding k-valued point. Then the tangent space at q_0 to Q_Λ is given by

$$T_{q_0}(Q_\Lambda) = \mathrm{Hom}_\Lambda(\ker(q_0), F_0).$$

Proof. Let Q be the quot scheme of coherent $\mathcal{O}$-quotients of E, and let $Q_\Lambda \subset Q$ be its closed subscheme of Λ-quotients, as above. As proved by Grothendieck, the tangent space $T_{q_0}Q$ is given by

$$T_{q_0}Q = \mathrm{Hom}_{\mathcal{O}_X}(\ker(q_0), F_0),$$

where the right-hand side is the k-vector space of all global $\mathcal{O}_X$-linear homomorphisms $\ker(q_0) \to F_0$ (see for example [8] for an exposition).

Note that any $v \in T_{q_0}Q$ is just a $k[\epsilon]/(\epsilon^2)$-valued point of Q which specializes to q_0 under $\epsilon \mapsto 0$. Let $X[\epsilon] = X \otimes_k k[\epsilon]/(\epsilon^2)$, and let $E[\epsilon]$ be the $\mathcal{O}$-module pullback of E to $X[\epsilon]$. Then in terms of valued points, v can be regarded as a pair $(q : E[\epsilon] \to F,\ i : F_0 \to F|_X)$ where F is a coherent $\mathcal{O}_{X[\epsilon]}$-module that is flat over $k[\epsilon]/(\epsilon^2)$, q is an $\mathcal{O}_{X[\epsilon]}$-linear surjection which restricts to q_0 modulo ϵ, and i is an $\mathcal{O}_X$-linear isomorphism. Two such pairs $(q : E[\epsilon] \to F, i)$ and $(q' : E[\epsilon] \to F',\ i')$ define the same $v \in T_{q_0}Q$ if and only if there exists a $\mathcal{O}_{X[\epsilon]}$-linear isomorphism $F \to F'$ which takes (q, i) to (q', i').

As a sheaf of abelian groups $E[\epsilon] = E \oplus \epsilon E$ and its germs of local sections are of the form $(\alpha, \epsilon\beta)$ where α and β are germs of local sections of E. Given any

$v \in \mathrm{Hom}_{\mathcal{O}_X}(\ker(q_0), F_0)$, let $G \subset E[\epsilon]$ be the sub sheaf of abelian groups whose germs of local sections are of the form $(\alpha, \epsilon\beta)$ such that $q_0(\alpha) = 0$ and

$$v(\alpha) = q_0(\beta).$$

Then G is a coherent $\mathcal{O}_{X[\epsilon]}$-submodule of $E[\epsilon]$ which canonically restricts to $\ker(q_0)$ modulo $\epsilon = 0$, and the quotient $F = E[\epsilon]/G$ is flat over $k[\epsilon]/(\epsilon^2)$, with $F_0 = F/\epsilon F$ (flatness of F over $k[\epsilon]/(\epsilon^2)$ is equivalent to the map $\epsilon : F/\epsilon F \to \epsilon F$ being an isomorphism). Then the resulting deformation $(q : E[\epsilon] \to F, i) \in Q(k[\epsilon]/(\epsilon^2))$ represents v.

Note that $E[\epsilon]$ is naturally a $\Lambda[\epsilon]$-module. The point $v \in T_{q_0}Q$ lies in $T_{q_0}(Q_\Lambda)$ if and only if the deformation $(q : E[\epsilon] \to F, i)$ as concretely described above lies in $Q_\Lambda(k[\epsilon]/(\epsilon^2)) \subset Q(k[\epsilon]/(\epsilon^2))$. This is clearly the case if and only if G is a $\Lambda[\epsilon]$-submodule of $E[\epsilon]$, which is equivalent to the condition that $v : \ker(q_0) \to F_0$ is Λ-linear. □

Semicontinuity of HN type

The well-known Narasimhan–Ramanathan argument that semistability is a Zariski open condition for a flat family of pure dimensional coherent sheaves of $\mathcal{O}$-modules, which uses the existence and projectivity of quot schemes, works equally well for a pure dimensional family of Λ-modules when the scheme Q_Λ given by Remark 4.4 is used in place of the quot scheme for $\mathcal{O}$-quotients.

Given a family E of Λ-modules of pure dimension d, we can define a function

$$|S| \to \mathrm{HNT} : s \mapsto \mathrm{HN}(E_s)$$

on the underlying topological space $|S|$ of the parameter scheme S, taking values in the partially ordered set HNT of all HN-types. For a family E of $\mathcal{O}$-modules (which is the special case $\Lambda = \mathcal{O}_X$), Shatz proved that the above function is upper semicontinuous. Actually, Shatz considered HN types in the sense of μ-semistability rather than in the sense of Gieseker semistability, but his proof readily works for HN types in the sense of Gieseker semistability. The key ingredient is again the existence and projectivity of an appropriate quot scheme for $\mathcal{O}$-modules. When we use the schemes Q_Λ in place of quot schemes, the proof of Shatz equally works for our case of families of Λ-modules, with HN types in the sense of Gieseker Λ-semistability (this is given in explicit detail in Gurjar [4]). In particular, we get the following.

Remark 4.6. Let E be a pure dimensional family of Λ-modules on X/S where S is locally noetherian. For any $\tau \in \mathrm{HNT}$, the corresponding level set

$$|S|^\tau(E) = \{s \in |S| \text{ such that } \mathrm{HN}(E_s) = \tau\}$$

is locally closed in $|S|$, the subset $|S|^{\leq\tau}(E) = \bigcup_{\alpha\leq\tau} |S|^\alpha(E) \subset |S|$ is open in $|S|$, and $|S|^\tau(E)$ is closed in $|S|^{\leq\tau}(E)$.

4.2 *Schematic HN stratification*

Let E be a family of pure dimensional Λ-modules on X/S, with notation as before. A *relative HN filtration* for E will mean a strictly increasing filtration $0 = \mathrm{HN}_0(E) \subset$

$\mathrm{HN}_1(E) \subset \cdots \subset \mathrm{HN}_\ell(E) = E$ by Λ-submodu les $\mathrm{HN}_i(E)$ such that for each $1 \leq i \leq \ell$, the graded piece $\mathrm{Gr}_i(E) = \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$ (regarded as a coherent $\mathcal{O}_X$-module) is flat over S, and for each $s \in S$, the restriction of the filtration to the fibre X_s of X/S (which is a filtration of $E_s = E|_{X_s}$ given the flatness condition) is the HN filtration for the Λ_s-module E_s.

Theorem 4.7. *Let $X \to S$ be projective, faithfully flat, with a chosen relatively ample line bundle $\mathcal{O}_{X/S}(1)$, where S is a locally noetherian scheme. Let Λ be a split almost-polynomial sheaf of rings of differential operators on X/S. Let E be a left Λ-module on X, which as an $\mathcal{O}_X$-module is coherent and flat over S, and such that the restriction $E_s = E|_{X_s}$ is a pure $\mathcal{O}_{X_s}$-module for each $s \in S$. Then we have the following*:

(1) *Each HN stratum $|S|^\tau(E)$ of the Λ-module E has a unique structure of a locally closed subscheme $S^\tau(E)$ of S, with the following universal property*: *a morphism $T \to S$ factors via $S^\tau(E)$ if and only if the pullback E_T on $X \times_S T$ admits a relative HN filtration of type τ.*
(2) *A relative HN filtration on E, if it exists, is unique.*
(3) *For any morphism $f : T \to S$ of locally noetherian schemes, the schematic stratum $T^\tau(E_T) \subset T$ for E_T equals the schematic inverse image of $S^\tau(E)$ under f.*

Proof. With all the preparation that we have made, the proof is now exactly the Λ-module analogue of the proof of Theorem 5 in [9]. We give a sketch for completeness. Let $\tau = (f_1, \ldots, f_\ell) \in \mathrm{HNT}$ have length ℓ. If $\ell = 1$, we take $S^\tau(E)$ to be the semi-stable stratum $S^{\mathrm{ss}}(E)$, which is an open subscheme of S. This satisfies all requirements, as semistability is indeed an open condition on the parameter scheme as a special case of the semicontinuity of $\mathrm{HN}(E_s)$, and semistability is preserved by arbitrary base changes by Remark 4.2. For $\ell \geq 2$, we proceed by induction. Let $S^{\leq\tau}(E) \subset S$ be the open subscheme where $\mathrm{HN}(E_s) \leq \tau$. Then the subset $|S|^\tau(E) \subset S^{\leq\tau}(E)$ is closed. We will give it the structure of a closed subscheme $S^\tau(E)$ of the scheme $S^{\leq\tau}(E)$, such that the conclusion of the theorem is satisfied. The type τ is a maximal type for the restriction of the family E to the inverse image of $S^{\leq\tau}(E)$ in X. Thus, it is enough to prove the above theorem for the special case where τ is the global maximum type for the given family. Hence we now assume that τ is the global maximum type for our given family E on X/S.

Consider the relative quot scheme

$$Q_\Lambda \subset \mathrm{Quot}_{E/X/S}^{f_\ell - f_1, \mathcal{O}_{X/S}(1)}$$

of $\mathcal{O}$-coherent Λ-quotients of fibers of E, with Hilbert polynomial $f_\ell - f_1$ where $\tau = (f_1, \ldots, f_\ell)$. Let $q \in Q_\Lambda$ be any point, and let $q \mapsto s \in S$. Let $k(s) \hookrightarrow k(q)$ be the resulting residue field extension. Let $E_q = E|_{X_q}$, and let $q : E_q \to F$ also denote the corresponding Λ-quotient represented by q. Then $\ker(q) = \mathrm{HN}_1(E_q)$ by Remark 4.2. By Remark 4.2, the quotient q is the pullback of the quotient $E_s \to E_s/\mathrm{HN}_1(E_s)$ which is defined over X_s. Hence the residue field extension $k(s) \to k(q)$ is trivial. By the uniqueness of $\mathrm{HN}_1(E_s)$, there exists at most one such q over s. The fibre of $\pi : Q_\Lambda \to S$ over s is the Quot scheme

$$\pi^{-1}(s) = (\mathrm{Quot}_{E_s/X_s/k(s)}^{f_\ell - f_1,\, \mathcal{O}_{X_s}(1)})_\Lambda$$

of all Λ-quotients of E_s with Hilbert polynomial $f_\ell - f_1$. By Proposition 4.5, the tangent space $T_q(\pi^{-1}(s))$ to the fibre $\pi^{-1}(s)$ at q is given by

$$T_q(\pi^{-1}(s)) = \mathrm{Hom}_{\Lambda_s}(\mathrm{HN}_1(E_q), E_q/\mathrm{HN}_1(E_q)).$$

This is zero by Remark 4.1. Hence $\pi : Q_\Lambda \to S$ is unramified. By Lemma 4 of [9], any morphism $f : T \to S$ between locally noetherian schemes is a closed embedding if (and only if) f is proper, injective, unramified and induces an isomorphism $k(f(t)) \to k(t)$ of residue fields for all $t \in T$. Hence $\pi : Q_\Lambda \to S$ is a closed imbedding.

Now consider the universal Λ-quotient $E_{Q_\Lambda} \to E''$ on $X_{Q_\Lambda} = X \times_S Q_\Lambda$. By Remark 4.1, for all $q \in Q_\Lambda$ the sheaf E''_q on X_q is pure-dimensional, with

$$\mathrm{HN}(E''_q) \leq \tau'' = (f_2 - f_1, \ldots, f_\ell - f_1).$$

In particular, we have $Q_\Lambda^{\leq \tau''}(E'') = Q_\Lambda$. The Harder–Narasimhan type τ'' has length $\ell - 1$, hence by induction on the length, the closed subset $|Q_\Lambda|^{\tau''}(E'')$ of Q_Λ has the structure of a closed subscheme $Q_\Lambda^{\tau''}(E'') \subset Q_\Lambda$ which has the desired universal property for E''. We regard Q_Λ as a closed subscheme of S via π, and we finally define the closed subscheme $S^\tau(E) \subset S$ by putting

$$S^\tau(E) = Q_\Lambda^{\tau''}(E'') \subset Q_\Lambda \subset S.$$

By their construction it is clear (as in the proof of Theorem 5 of [9]) that the subschemes $S^\tau(E)$, and the resulting schematic stratification of S, have the desired properties. □

COROLLARY 4.8 (Case of constant HN type over a reduced base)

Let X be a faithfully flat projective scheme over a locally noetherian base scheme S, with a relatively ample line bundle $\mathcal{O}_{X/S}(1)$ and a split almost polynomial sheaf of rings of differential operators Λ on X/S. Let E be a left Λ-module on X, which as an $\mathcal{O}_X$-module is coherent and flat over S, and such that the restriction $E_s = E|_{X_s}$ is a pure $\mathcal{O}_{X_s}$-module for each $s \in S$ of a fixed Harder–Narasimhan type $\tau \in$ HNT as a Λ_s-module. Suppose moreover that S is reduced. Then $S = S^\tau(E)$, that is, E admits a unique relative Harder–Narasimhan filtration.

Λ-modules of HN type τ form an algebraic stack

Let X be a faithfully flat projective scheme over a locally noetherian base scheme S, with a split almost polynomial sheaf of rings of differential operators Λ on X/S. For any S-scheme T, let $\Lambda\text{-Coh}_{X/S}(T)$ denote the groupoid whose objects are all families E of left Λ_T-module on X_T, such that as an $\mathcal{O}_{X_T}$-module E is coherent and flat over T, and such that the restriction $E_t = E|_{X_t}$ is a pure $\mathcal{O}_{X_t}$-module for each $t \in T$. The morphisms in this groupoid are Λ_T-linear isomorphisms. For any S-morphism $T' \to T$, we have a natural pullback functor $\Lambda\text{-Coh}_{X/S}(T) \to \Lambda\text{-Coh}_{X/S}(T')$, which makes $\Lambda\text{-Coh}_{X/S}$ into an S-groupoid. As in the case of $\mathcal{O}$-modules (see, for example, 2.4.4 of [6]), it can be seen that $\Lambda\text{-Coh}_{X/S}$ is an Artin stack over S.

Theorem 4.7 has the following Corollary, with proofs again as in the case of $\mathcal{O}$-modules in [9]. We omit the details.

COROLLARY 4.9

Let X be a faithfully flat projective scheme over a locally noetherian base scheme S, with a relatively ample line bundle $\mathcal{O}_{X/S}(1)$ and a split almost polynomial sheaf of rings of differential operators Λ on X/S. Let τ be any Harder–Narasimhan type. Then all families of Λ-modules on X/S which are flat families of coherent pure $\mathcal{O}$-modules on X/S with a fixed Harder–Narasimhan type τ as Λ-modules form an algebraic stack $\Lambda\text{-Coh}^{\tau}_{X/S}$ over S, which is a locally closed substack of the algebraic stack $\Lambda\text{-Coh}_{X/S}$ of all families of Λ-modules on X/S which are flat families of coherent pure $\mathcal{O}$-modules on X/S.

It should be noted that whenever a boundedness theorem holds for semistable Λ-modules, each stack $\Lambda\text{-Coh}^{\tau}_{X/S}$ admits an atlas of finite type over S, as in Proposition 9 of [9].

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