Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 3, August 2014, pp. 301–313.

Positive integer solutions of the diophantine equation $x^2 - L_n xy + (-1)^n y^2 = \pm 5^r$

REFİK KESKİN[1] and ZAFER ŞİAR[2]

[1]Sakarya University, Merkezi, 54180 Sakarya, Turkey
[2]Bingöl University, Rektörlüğü, 12000 Bingöl, Turkey
E-mail: rkeskin@sakarya.edu.tr; zsiar@bingol.edu.tr

MS received 5 April 2013; revised 11 June 2013

Abstract. In this paper, we consider the equation $x^2 - L_n xy + (-1)^n y^2 = \pm 5^r$ and determine the values of n for which the equation has positive integer solutions x and y. Moreover, we give all positive integer solutions of the equation $x^2 - L_n xy + (-1)^n y^2 = \pm 5^r$ when the equation has positive integer solutions.

Keywords. Fibonacci numbers; Lucas numbers; diophantine equations

Mathematics Subject Classification. 11B37, 11B39.

1. Introduction

The Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Fibonacci numbers for negative subscripts are defined as $F_{-n} = (-1)^{n+1} F_n$ for $n \geq 1$. Similarly the Lucas sequence $\{L_n\}$ is defined as $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Lucas numbers for negative subscripts are defined as $L_{-n} = (-1)^n L_n$ for $n \geq 1$. For more information about Fibonacci and Lucas sequences, one can refer [7,8].

In [3], the authors determined when the equations $x^2 - L_n xy + (-1)^n y^2 = 5$ and $x^2 - L_n xy + (-1)^n y^2 = -5$ have positive integer solutions. They showed that the equation $x^2 - L_n xy + (-1)^n y^2 = 5$ have positive integer solutions only when $n = 1, 2, 3, 4$ and the equation $x^2 - L_n xy + (-1)^n y^2 = -5$ have integer solutions only when $n = 1, 2, 3$. Moreover, they found all positive integer solutions of the equations $x^2 - xy - y^2 = 5^r$ and $x^2 - xy - y^2 = -5^r$ when r is a positive integer.

In this study, we will consider the diophantine equations

$$x^2 - L_n xy + (-1)^n y^2 = 5^r \tag{1.1}$$

and

$$x^2 - L_n xy + (-1)^n y^2 = -5^r, \tag{1.2}$$

where $n > 0$ and $r > 1$ are natural numbers. We will determine when eqs (1.1) and (1.2) have positive integer solutions. Then we will find all positive integer solutions of eqs (1.1) and (1.2).

2. Preliminaries

In this section we present some known theorems concerning Fibonacci and Lucas numbers. Then, in the next section we give our main theorems.

The following two theorems are given in [6].

Theorem 2.1. *Let $m > 3$ be an integer and $F_n = F_m x^2$ for some integer x. Then $n = m$.*

Theorem 2.2. *Let $m \geq 2$ be an integer and $L_n = L_m x^2$ for some integer x. Then $n = m$.*

Now we give a theorem from [2].

Theorem 2.3. *If $F_m = x^2$, then $m = 0, 1, 2, 12$. If $F_m = 2x^2$, then $m = 0, 3, 6$. If $L_m = x^2$, then $m = 1, 3$ and if $L_m = 2x^2$, then $m = 0, 6$.*

The following two theorems are given in [6].

Theorem 2.4. *There is no integer x such that $L_n = 2L_m x^2$ for $m > 1$.*

Theorem 2.5. *If $F_n = 2F_m x^2$ and $m \geq 3$, then $m = 3$, $x^2 = 36$, and $n = 12$ or $m = 6$, $x^2 = 9$, and $n = 12$.*

The proofs of the following three theorems can be found in [1,5] and [8].

Theorem 2.6. *Let $m, n \in \mathbb{N}$ and $m \geq 3$. Then $F_m \mid F_n$ if and only if $m \mid n$.*

Theorem 2.7. *Let $m, n \in \mathbb{N}$ and $m \geq 2$. Then $L_m \mid L_n$ if and only if $m \mid n$ and n/m is an odd integer.*

Theorem 2.8. *Let $m, n \in \mathbb{N}$ and $m \geq 2$. Then $L_m \mid F_n$ if and only if $m \mid n$ and n/m is an even integer.*

The following theorem can be found in [4].

Theorem 2.9. *Let $m, n \in \mathbb{N}$. Then $F_n \mid L_m$ if and only if $n = 1, m = k$; $n = 2$, $m = k$; $n = 3$, $m = 3k$; $n = 4$, $m = 4k + 2$ for some integer k.*

The proof of the following lemma can be done by induction.

Lemma 1. *$L_{2^k} \equiv 3 \pmod 4$ for the all positive integers k.*

The proof of the following lemma is easy and therefore we omit it.

Lemma 2. *Let m be an odd integer. Suppose that $x^2 \equiv -a^2 \pmod m$ for some integers x and a. Then $m \equiv 1 \pmod 4$.*

The following theorem is given in [5].

Theorem 2.10. *Let $n \in \mathbb{N} \cup \{0\}$ and $k, m \in \mathbb{Z}$. Then*

$$F_{2mn+k} \equiv (-1)^{(m+1)n} F_k \pmod{L_m}. \tag{2.1}$$

The following theorem is given in [3].

Theorem 2.11. *Let $k \geq 1$. Then all nonnegative integer solutions of the equation $u^2 - 5v^2 = 4 \cdot 5^k$ are given by*

$$(u, v) = \begin{cases} (5^{(k+1)/2} F_{2m+1}, 5^{(k-1)/2} L_{2m+1}), & k \text{ is an odd integer;} \\ (5^{k/2} L_{2m}, 5^{k/2} F_{2m}), & k \text{ is an even integer,} \end{cases}$$

and all nonnegative integer solutions of the equation $u^2 - 5v^2 = -4 \cdot 5^k$ are given by

$$(u, v) = \begin{cases} (5^{(k+1)/2} F_{2m}, 5^{(k-1)/2} L_{2m}), & k \text{ is an odd integer;} \\ (5^{k/2} L_{2m+1}, 5^{k/2} F_{2m+1}), & k \text{ is an even integer,} \end{cases}$$

where $m \geq 0$.

Now we give some identities concerning Fibonacci and Lucas numbers, which will be needed in the sequel:

$$F_{2n} = F_n L_n, \tag{2.2}$$

$$F_{5n} = 5F_n(5F_n^4 + (-1)^n 5F_n^2 + 1), \tag{2.3}$$

$$L_n^2 - 5F_n^2 = 4(-1)^n, \tag{2.4}$$

$$F_{5n} = F_n(L_n^4 - 3(-1)^n L_n^2 + 1), \tag{2.5}$$

$$5 \nmid L_n, \tag{2.6}$$

$$F_m L_n + F_n L_m = 2F_{m+n}, \tag{2.7}$$

$$F_m L_n - F_n L_m = 2(-1)^n F_{m-n}, \tag{2.8}$$

$$L_m L_n + 5F_n F_m = 2L_{m+n}, \tag{2.9}$$

$$L_m L_n - 5F_n F_m = 2(-1)^n L_{m-n}, \tag{2.10}$$

$$L_{n+m}^2 - L_n L_{n+m} L_m + (-1)^n L_m^2 = 5(-1)^{m+1} F_n^2, \tag{2.11}$$

$$F_{n+m}^2 - L_n F_{n+m} F_m + (-1)^n F_m^2 = (-1)^m F_n^2, \tag{2.12}$$

$$(F_m, L_n) = \begin{cases} L_{(m,\, n)}, & \text{if } m/(m,\, n) \text{ is even and } n/(m,\, n) \text{is odd;} \\ 1 \text{ or } 2, & \text{otherwise,} \end{cases} \tag{2.13}$$

$$(F_{m,}\ F_n) = F_{(m,\, n)}. \tag{2.14}$$

3. Main theorems

For the sake of completeness, we will give the following theorem [3]:

Theorem 3.1. *The equation $x^2 - L_n xy + (-1)^n y^2 = 5$ has positive integer solutions only when $n = 1, 2, 3, 4$ and all positive integer solutions of the equation $x^2 - L_n xy + (-1)^n y^2 = 5$ are given by $(x, y) = (L_{n+m}/F_n, L_m/F_n)$ with odd positive integer m.*

The equation $x^2 - L_n xy + (-1)^n y^2 = -5$ has positive integer solutions only when $n = 1, 2, 3$ and all positive integer solutions of the equation $x^2 - L_n xy + (-1)^n y^2 = -5$ are given by $(x, y) = (L_{n+m}/F_n, L_m/F_n)$ with even nonnegative integer m.

3.1 *Solutions of the diophantine equation $x^2 - L_n xy + (-1)^n y^2 = \pm 5^r$*

From now on, we will assume that $n > 0$ is an even integer and k is a positive integer.

Theorem 3.2. *The equation $x^2 - L_n xy + y^2 = -5^{2k}$ has positive integer solutions if and only if $n = 2$ and all positive integer solutions of the equation $x^2 - L_2 xy + y^2 = -5^{2k}$ are given by $(x, y) = (5^k F_{2m-1}, 5^k F_{2m+1})$ with $m \geq 0$.*

Proof. Assume that $x^2 - L_n xy + y^2 = -5^{2k}$ for some positive integers x and y. Without loss of generality, we may suppose that $y \geq x$. Multiplying both sides of the equation by 4, we get

$$4x^2 - 4L_n xy + 4y^2 = -4 \cdot 5^{2k}.$$

Completing the square gives

$$(2x - L_n y)^2 - (L_n^2 - 4)y^2 = -4 \cdot 5^{2k}. \tag{3.1}$$

Since n is an even integer, it is seen that $L_n^2 - 4 = 5F_n^2$ from identity (2.4). Thus, from eq. (3.1), we get

$$(2x - L_n y)^2 - 5(F_n y)^2 = -4 \cdot 5^{2k}.$$

Since $y \geq x$ and $L_n > 2$, it follows that $L_n y > 2x$. Therefore by Theorem 2.11, we obtain $L_n y - 2x = 5^k L_{2m+1}$ and $F_n y = 5^k F_{2m+1}$ with $m \geq 0$.

If $5 | F_n$, then by Theorem 2.6, it follows that $5 | n$. Since n is an even integer, we get $10 | n$, i.e., $n = 10s$ for some positive integer s. Thus, we get $F_{10s} y = 5^k F_{2m+1}$. From identity (2.2), it follows that $F_{5s} L_{5s} y = 5^k F_{2m+1}$ and also since $(5, L_{5s}) = 1$ by (2.6), it is seen that $L_{5s} |\ F_{2m+1}$. Thus by Theorem 2.8, it follows that $(2m + 1)/5s$ is an even integer, which is impossible.

If $5 \nmid F_n$, then $F_n | F_{2m+1}$ and therefore by Theorem 2.6 , it follows that $n | 2m+1$ for $n \geq 3$. But since n is an even integer, this is impossible. Therefore $n = 2$. Then, we get $y = 5^k F_{2m+1}$ with $m \geq 0$. Substituting this value of y into $L_n y - 2x = 5^k L_{2m+1}$, we obtain $(x, y) = (5^k F_{2m-1}, 5^k F_{2m+1})$ by (2.8). Conversely, if $(x, y) = (5^k F_{2m-1}, 5^k F_{2m+1})$ with $m \geq 0$, then from identity (2.12), it follows that $x^2 - L_2 xy + y^2 = -5^{2k}$. This completes the proof. □

Since the proof of the following theorem is similar to that of Theorem 3.2, we omit its proof.

Theorem 3.3. *The equation $x^2 - L_n xy + y^2 = -5^{2k+1}$ has positive integer solutions if and only if $n = 2$ or $n = 4$. Moreover, all positive integer solutions of the equation $x^2 - L_2 xy + y^2 = -5^{2k+1}$ are given by $(x, y) = (5^k L_{2m-2}, 5^k L_{2m})$ with $m \geq 0$ and all positive integer solutions of the equation $x^2 - L_4 xy + y^2 = -5^{2k+1}$ are given by $(x, y) = (5^k L_{2m-4}/3, 5^k L_{2m}/3)$ with odd integer $m \geq 0$.*

Theorem 3.4. *All positive integer solutions of the equation* $x^2 - L_n xy + y^2 = 5^{2k}$ *are given by* $(x, y) = (5^k F_{n(t-1)}/F_n, 5^k F_{nt}/F_n)$ *with* $t > 1$.

Proof. Assume that $x^2 - L_n xy + y^2 = 5^{2k}$ for some positive integers x and y. Without loss of generality, we may suppose that $y \geq x$. Multiplying both sides of the equation by 4, we get

$$4x^2 - 4L_n xy + 4y^2 = 4 \cdot 5^{2k}.$$

Completing the square gives

$$(2x - L_n y)^2 - (L_n^2 - 4)y^2 = 4 \cdot 5^{2k}. \tag{3.2}$$

Since n is an even integer, it is seen that $L_n^2 - 4 = 5F_n^2$ from identity (2.4). Thus, from eq. (3.2), we get

$$(2x - L_n y)^2 - 5(F_n y)^2 = 4 \cdot 5^{2k}.$$

Since $y \geq x$ and $L_n > 2$, it follows that $L_n y > 2x$. Therefore by Theorem 2.11, we obtain $L_n y - 2x = 5^k L_{2m}$ and $F_n y = 5^k F_{2m}$ with $m > 0$.

Now if $5|F_n$, then by Theorem 2.6, it follows that $5|n$. Since n is an even integer, we get $10|n$. Thus $n = 10s$ for some positive integer s. Then, we get $F_{10s} y = 5^k F_{2m}$. From identity (2.2), it follows that

$$F_{5s} L_{5s} y = 5^k F_{2m}.$$

Moreover, since $(5, L_{5s}) = 1$ by (2.6), it is seen that $L_{5s}|\ F_{2m}$. Thus by Theorem 2.8, it follows that $2m/5s = 2t$, i.e., $2m = 10st$ for some integer t. And thus, we get $n|2m$, i.e., $F_n|F_{2m}$ by Theorem 2.6. If $5 \nmid F_n$, then $F_n|F_{2m}$ and therefore $n|2m$ by Theorem 2.6. Therefore it follows that $y = 5^k F_{2m}/F_n$ with $m > 0$. Substituting this value of y into $L_n y - 2x = 5^k L_{2m}$, we obtain $(x, y) = \left(5^k F_{2m-n}/F_n, 5^k F_{2m}/F_n\right)$ by (2.8), where $2m > n$. Since $n|2m$, it follows that $2m = nt$ for some integer $t > 1$. This shows that $(x, y) = \left(5^k F_{n(t-1)}/F_n, 5^k F_{nt}/F_n\right)$ with $t > 1$. Conversely, if $(x, y) = \left(5^k F_{n(t-1)}/F_n, 5^k F_{nt}/F_n\right)$ with $t > 1$, then from identity (2.12), it follows that $x^2 - L_n xy + y^2 = 5^{2k}$. This completes the proof. □

Lemma 3. *Let* n, m *be positive integers. There is no integer* x *such that* $F_n = 5F_m x^2$ *for* $m \geq 3$.

Proof. Assume that $F_n = 5F_m x^2$ for some integer x. If $m = 3$, then we get $F_n = 10x^2 = 2F_5 x^2$, which is impossible by Theorem 2.5. Now assume that $m > 3$. Since $F_m|F_n$, we get $n = mt$ for some integer t by Theorem 2.6. Moreover, since $5|F_n$, we get $5|n$. Thus $5|mt$. Firstly, assume that $5|t$. Then $t = 5s$ for some integer s. Thus we get $F_n = F_{mt} = F_{5ms} = 5F_{ms}(5F_{ms}^4 + (-1)^{ms} 5F_{ms}^2 + 1) = 5F_m x^2$ by (2.3). Then it follows that $(F_{ms}/F_m)(5F_{ms}^4 \pm 5F_{ms}^2 + 1) = x^2$. Since $(F_{ms}/F_m,\ 5F_{ms}^4 \pm 5F_{ms}^2 + 1) = 1$, we get $F_{ms}/F_m = u^2$ for some integer u. This shows that $s = 1$ by Theorems 2.1 and 2.3. Thus $n = 5m$.

On the other hand, we get $F_{5m} = F_m(L_m^4 - 3(-1)^m L_m^2 + 1)$ by (2.5). Thus it follows that

$$L_m^4 \pm 3L_m^2 + 1 = 5x^2. \tag{3.3}$$

A simple computation shows that $(2x)^2 - 5(\frac{2L_m^2 \pm 3}{5})^2 = -1$. All positive integer solutions of the equation $u^2 - 5v^2 = -1$ are given by $(u, v) = (L_{3z}/2,\ F_{3z}/2)$ with odd positive integer z. Thus $(2L_m^2 \pm 3)/5 = F_{3z}/2$. Then $4{L_m}^2 = 5F_{3z} \pm 6$. Since $m > 3$, it follows that $z > 3$. Thus $z = 4q \pm 1$ for some positive integer q. Then $z = 2 \cdot 2^k a \pm 1$ for some odd positive integer a with $k \geq 1$. So

$$(2L_m)^2 = 5F_{3z} \pm 6 = \pm 6 + 5F_{2 \cdot 2^k 3a \pm 3} \equiv \pm 6 - 5F_{\pm 3} \equiv -4, -16 (\bmod L_{2^k})$$

by (2.1). This is impossible by Lemmas 1 and 2. Secondly, assume that $5 \nmid t$. Then $5|m$. By using (2.3), it can be shown that if a is a positive integer, then $F_{5^s a} = 5^s F_a (5b_1 + 1)(5b_2 + 1) \ldots (5b_s + 1)$ for some integers $b_1, b_2, \ldots, b_s$. Since $5|m$, we have $m = 5^j u$ with $5 \nmid u$. Thus $F_m = 5^j F_u \prod_{i=1}^{j} (5a_i + 1)$ and therefore $F_n = F_{mt} = F_{5^j ut} = 5^j F_{ut} \prod_{i=1}^{j} (5b_i + 1)$. Then we get $F_n = 5F_m x^2 = 5^{j+1} F_u \prod_{i=1}^{j} (5a_i + 1)\, x^2 = 5^j F_{ut} \prod_{i=1}^{j} (5b_i + 1)$. This is impossible since $5 \nmid F_{ut}$. This completes the proof. □

Now we can give the following corollaries easily.

COROLLARY 1

Let r, m be nonnegative integers. If $F_m = 5^r$, then $r = 0$, $m = 1,\ 2$ and $r = 1, m = 5$ and if $F_m = 2 \cdot 5^r$, then $r = 0$ and $m = 3$.

COROLLARY 2

Let n, m be positive integers, k be a nonnegative integer, and $F_n = 5^k F_m$. Then $k = 0$, $m = n$, $(m, n) = (1, 2),\ (2, 1)$ or $k = 1, n = 5, m = 1,\ 2$.

Theorem 3.5. *The equation $x^2 - L_n xy + y^2 = 5^{2k+1}$ has positive integer solutions if and only if $n = 2$ or $n = 10$. Moreover, all positive integer solutions of the equation $x^2 - L_2 xy + y^2 = 5^{2k+1}$ are given by $(x, y) = (5^k L_{2m+1}, 5^k L_{2m+3})$ with $m \geq 0$ and all positive integer solutions of the equation $x^2 - L_{10} xy + y^2 = 5^{2k+1}$ are given by $(x, y) = (5^{k-1}(L_{10m-5}/11), 5^{k-1}(L_{10m+5}/11))$ with $m \geq 1$.*

Proof. Assume that $x^2 - L_n xy + y^2 = 5^{2k+1}$ for some positive integers x and y. Without loss of generality we may suppose that $y \geq x$. Multiplying both sides of the equation by 4, we get

$$4x^2 - 4L_n xy + 4y^2 = 4 \cdot 5^{2k+1}.$$

Completing the square gives

$$(2x - L_n y)^2 - (L_n^2 - 4)y^2 = 4 \cdot 5^{2k+1}. \quad (3.4)$$

Since n is an even integer, it is seen that $L_n^2 - 4 = 5F_n^2$ from identity (2.4). Thus, from eq. (3.4), we get

$$(2x - L_n y)^2 - 5(F_n y)^2 = 4 \cdot 5^{2k+1}.$$

Since $y \geq x$ and $L_n > 2$, it follows that $L_n y > 2x$. Therefore by Theorem 2.11, we obtain $L_n y - 2x = 5^{k+1} F_{2m+1}$ and $F_n y = 5^k L_{2m+1}$ with $m \geq 0$.

If $5|F_n$, then by Theorem 2.6, it follows that $5|n$. Since n is an even integer, we get $10|n$. Thus $n = 10s$ for some positive integer s. Then, we get $F_{10s}y = 5^k L_{2m+1}$. From identity (2.2), it follows that

$$F_{5s}L_{5s}y = 5^k L_{2m+1} \tag{3.5}$$

and also since $(5, L_{5s}) = 1$ by (2.6), it is seen that $L_{5s}|\, L_{2m+1}$. Thus by Theorem 2.7, it follows that $2m + 1 = 5st$ for some odd positive integer t. Then from eq. (3.5), we get

$$F_{5s}L_{5s}y = 5^k L_{5st}. \tag{3.6}$$

Now let $(F_{5s}, L_{5st}) = d$. It follows that $d = 1$ or $d = 2$ by (2.13). Assume that $d = 1$. Then from eq. (3.6), we get $F_{5s}|5^k$. Thus $F_{5s} = 5^r$ with $1 \le r \le k$. By Corollary 1, it is seen that $s = r = 1$. Then it follows that $n = 10$. Therefore $y = 5^{k-1}(L_{2m+1}/11)$ with $m \ge 0$ and $5|2m + 1$. Substituting this value of y into $L_n y - 2x = 5^{k+1}F_{2m+1}$, we obtain $(x, y) = (5^{k-1}(L_{2m-9}/11), 5^{k-1}(L_{2m+1}/11))$, where $m \ge 7$. Since $5|2m + 1$, then we see that $m = 5a + 2$ for some positive integer a. Thus we get $(x, y) = (5^{k-1}(L_{10a-5}/11), 5^{k-1}(L_{10a+5}/11))$ with $a \ge 1$. Assume that $d = 2$. Then it follows that $(F_{5s}/2, L_{5st}/2) = 1$. Thus from eq. (3.6), we get $(F_{5s}/2)|5^k$. Thus $F_{5s} = 2 \cdot 5^r$ with $1 \le r \le k$. This is impossible by Corollary 1.

If $5 \nmid F_n$, then $F_n|L_{2m+1}$. Therefore by Theorem 2.9, it follows that $n = 2$ or $n = 4$. But when $n = 4$, it is seen that $3|L_{2m+1}$, i.e., $L_2|L_{2m+1}$, which is impossible by Theorem 2.7. When $n = 2$, we get $(x, y) = (5^k L_{2m-1}, 5^k L_{2m+1})$ with $m \ge 1$. We can take x and y as $(x, y) = (5^k L_{2m+1}, 5^k L_{2m+3})$ with $m \ge 0$. Conversely, if $(x, y) = (5^k L_{2m+1}, 5^k L_{2m+3})$, then by (2.11) it follows that $x^2 - L_2xy + y^2 = 5^{2k+1}$ and if $(x, y) = (5^{k-1}(L_{10m-5}/11), 5^{k-1}(L_{10m+5}/11))$, then by (2.11) it follows that $x^2 - L_{10}xy + y^2 = 5^{2k+1}$. This completes the proof. □

COROLLARY 3

If k is even, then all positive integer solutions of the equation $x^2 - 3xy^2 + y^4 = -5^{2k}$ are given by $(x, y) = (2 \cdot 5^k, 5^{k/2})$ or $(x, y) = (5^k, 5^{k/2})$ and if k is odd, then all positive integer solutions of the equation $x^2 - 3xy^2 + y^4 = -5^{2k}$ are given by $(x, y) = (13 \cdot 5^k, 5^{(k+1)/2})$ or $(x, y) = (2 \cdot 5^k, 5^{(k+1)/2})$.

Proof. Assume that $x^2 - 3xy^2 + y^4 = -5^{2k}$ for some positive integers x and y. Then by Theorem 3.2, it is seen that $(x, y^2) = (5^k F_{2m-1}, 5^k F_{2m+1})$ or $(x, y^2) = (5^k F_{2m+1}, 5^k F_{2m-1})$ with $m \ge 0$. Assume that $y^2 = 5^k F_{2m+1}$. If k is even, then $F_{2m+1} = z^2$ for some integer z. By Theorem 2.3, it is seen that $m = 0$ and therefore $(x, y) = (5^k, 5^{k/2})$. If k is odd, then $F_{2m+1} = 5z^2 = F_5 z^2$ for some integer z. By Theorem 2.1, it follows that $m = 2$ and therefore $(x, y) = (2 \cdot 5^k, 5^{(k+1)/2})$. Assume that $y^2 = 5^k F_{2m-1}$. If k is even, then $F_{2m-1} = z^2$ for some integer z. By Theorem 2.3, it is seen that $m = 1$ and therefore $(x, y) = (2 \cdot 5^k, 5^{k/2})$. If k is odd, then $F_{2m-1} = 5z^2 = F_5 z^2$ for some integer z. By Theorem 2.1, it follows that $m = 3$ and therefore $(x, y) = (13 \cdot 5^k, 5^{(k+1)/2})$. This completes the proof. □

COROLLARY 4

The equation $x^2 - 3xy^2 + y^4 = -5^{2k+1}$ has no positive integer solution.

Proof. Assume that $x^2 - 3xy^2 + y^4 = -5^{2k+1}$ for some positive integers x and y. Then by Theorem 3.3, it follows that $(x, y^2) = (5^k L_{2m-2}, 5^k L_{2m})$ or $(x, y^2) =$

$(5^k L_{2m}, 5^k L_{2m-2})$ with $m \geq 0$. Thus we get $y^2 = 5^k L_{2m}$ or $y^2 = 5^k L_{2m-2}$. Since $5 \nmid L_{2m}$ and $5 \nmid L_{2m-2}$ by (2.6), it can be seen that k is even in both cases. Then $L_{2m} = u^2$ or $L_{2m-2} = v^2$ for some integers u and v. This is impossible by Theorem 2.3. This completes the proof. □

COROLLARY 5

If k is odd, then there is no positive integer solution of the equation $x^2 - 7xy^2 + y^4 = -5^{2k+1}$. *If k is even, then all positive integer solutions of the equation* $x^2 - 7xy^2 + y^4 = -5^{2k+1}$ *are given by* $(x, y) = (5^k, 5^{k/2})$ *or* $(x, y) = (6 \cdot 5^k, 5^{k/2})$.

Proof. Assume that $x^2 - 7xy^2 + y^4 = -5^{2k+1}$ for some positive integers x and y. Then by Theorem 3.3, it follows that $(x, y^2) = (5^k L_{2m-4}/3, 5^k L_{2m}/3)$ or $(x, y^2) = (5^k L_{2m}/3, 5^k L_{2m-4}/3)$ with odd integer $m \geq 0$. Assume that $y^2 = 5^k(L_{2m}/3)$. Since $5 \nmid L_{2m}$ by (2.6), k is an even integer. Then $L_{2m} = 3z^2 = L_2 z^2$ for some integer z. By Theorem 2.2, we get $m = 1$. Therefore it follows that $(x, y) = (5^k, 5^{k/2})$. Assume that $y^2 = 5^k(L_{2m-4}/3)$. Since $5 \nmid L_{2m-4}$ by (2.6), k is an even integer. Then $L_{2m-4} = 3z^2 = L_2 z^2$ for some integer z. By Theorem 2.2, we get $m = 3$. Therefore it follows that $(x, y) = (6 \cdot 5^k, 5^{k/2})$. This completes the proof. □

COROLLARY 6

If k is odd, then there is no positive integer solution of the equation $x^2 - L_n xy^2 + y^4 = 5^{2k}$. *If k is even, then all positive integer solutions of the equation* $x^2 - L_2 xy^2 + y^4 = 5^{2k}$ *are given by* $(x, y) = (3 \cdot 5^k, 5^{k/2})$, $(x, y) = (377 \cdot 5^k, 12 \cdot 5^{k/2})$ *or* $(x, y) = (55 \cdot 5^k, 12 \cdot 5^{k/2})$ *and the equation* $x^2 - L_n xy^2 + y^4 = 5^{2k}$ *has only one positive integer solution given by* $(x, y) = (5^k L_n, 5^{k/2})$ *for* $n > 2$.

Proof. Assume that $x^2 - L_n xy^2 + y^4 = 5^{2k}$ for some positive integers x and y. Then by Theorem 3.4, it follows that $(x, y^2) = (5^k F_{n(t-1)}/F_n, 5^k F_{nt}/F_n)$ or $(x, y^2) = (5^k F_{nt}/F_n, 5^k F_{n(t-1)}/F_n)$ with $t > 1$. Assume that $y^2 = 5^k(F_{nt}/F_n)$. If k is even, then $F_{nt} = F_n z^2$ for some integer z. By Theorem 2.1, it follows that $t = 1$ for $n > 3$. But this is impossible since $t > 1$. When $n = 2$, from equality $F_{nt} = F_n z^2$, it is seen that $F_{2t} = z^2$. This shows that $t = 6$ by Theorem 2.3. If $t = 6$, then we get $(x, y) = (55 \cdot 5^k, 12 \cdot 5^{k/2})$. If k is odd, then it can be seen that $F_{nt} = 5F_n z^2$ for some integer z. But this is impossible for $n \geq 3$ by Lemma 3. When $n = 2$, we get $F_{2t} = 5z^2 = F_5 z^2$, which is impossible by Theorem 2.1. Assume that $y^2 = 5^k(F_{n(t-1)}/F_n)$. If k is even, then $F_{n(t-1)} = F_n z^2$ for some integer z. By Theorem 2.1, it follows that $t = 2$ for $n > 3$. Thus we get $(x, y) = (5^k L_n, 5^{k/2})$ for $n > 3$. When $n = 2$, from equality $F_{n(t-1)} = F_n z^2$, it is seen that $F_{2t-2} = z^2$. This shows that $t = 2$ or $t = 7$ by Theorem 2.3. If $t = 2$, then we get $(x, y) = (3 \cdot 5^k, 5^{k/2})$ and if $t = 7$, then $(x, y) = (377 \cdot 5^k, 12 \cdot 5^{k/2})$. If k is odd, then it can be seen that $F_{n(t-1)} = 5F_n z^2$ for some integer z. But this is impossible for $n \geq 3$ by Lemma 3. When $n = 2$, we get $F_{2t-2} = 5z^2 = F_5 z^2$, which is impossible by Theorem 2.1. This completes the proof. □

Since the proofs of the following corollaries are similar, we omit their proofs.

COROLLARY 7

If k is odd, then there is no positive integer solution of the equation $x^2 - 3xy^2 + y^4 = 5^{2k+1}$. *If k is even, then all positive integer solutions of the equation* $x^2 - 3xy^2 +$

$y^4 = 5^{2k+1}$ are given by $(x, y) = (4 \cdot 5^k, 5^{k/2})$, $(x, y) = (11 \cdot 5^k, 2 \cdot 5^{k/2})$ or $(x, y) = (5^k, 2 \cdot 5^{k/2})$.

COROLLARY 8

If k is even, then there is no positive integer solution of the equation $x^2 - 123xy^2 + y^4 = 5^{2k+1}$. If k is odd, then the equation $x^2 - 123xy^2 + y^4 = 5^{2k+1}$ has only one positive integer solution given by $(x, y) = (124 \cdot 5^{k-1}, 5^{(k-1)/2})$.

3.2 *Solutions of the diophantine equation $x^2 - L_n xy - y^2 = \pm 5^r$*

The following theorem is given in [3].

Theorem 3.6. *Let $k \geq 1$. Then all nonnegative integer solutions of the equation $x^2 - xy - y^2 = 5^k$ are given by*

$$(x, y) = \begin{cases} (5^{(k-1)/2} L_{2m+2}, 5^{(k-1)/2} L_{2m+1}), & k \text{ is an odd integer;} \\ (5^{k/2} F_{2m+1}, 5^{k/2} F_{2m}), & k \text{ is an even integer} \end{cases}$$

and all nonnegative integer solutions of the equation $x^2 - xy - y^2 = -5^k$ are given by

$$(x, y) = \begin{cases} (5^{(k-1)/2} L_{2m+1}, 5^{(k-1)/2} L_{2m}), & k \text{ is an odd integer;} \\ (5^{k/2} F_{2m+2}, 5^{k/2} F_{2m+1}), & k \text{ is an even integer} \end{cases}$$

where $m \geq 0$.

From now on, we will assume that n is an odd integer greater than 1.

Theorem 3.7. *The equation $x^2 - L_n xy - y^2 = -5^{2k+1}$ has positive integer solutions if and only if $n = 3$ or $n = 5$. Moreover, all positive integer solutions of the equation $x^2 - L_3 xy - y^2 = -5^{2k+1}$ are given by $(x, y) = (5^k L_{6m+3}/2, 5^k L_{6m}/2)$ with $m \geq 0$ and all positive integer solutions of the equation $x^2 - L_5 xy - y^2 = -5^{2k+1}$ are given by $(x, y) = (5^{k-1} L_{2m+1}, 5^{k-1} L_{2m-4})$ with $m \geq 0$.*

Proof. Assume that $x^2 - L_n xy - y^2 = -5^{2k+1}$ for some positive integers x and y. Multiplying both sides of the equation by 4, we get

$$4x^2 - 4L_n xy - 4y^2 = -4 \cdot 5^{2k+1}.$$

Completing the square gives

$$(2x - L_n y)^2 - (L_n^2 + 4)y^2 = -4 \cdot 5^{2k+1}. \tag{3.7}$$

Since n is an odd integer, we get $L_n^2 + 4 = 5F_n^2$ from identity (2.4). Thus, from eq. (3.7), we have

$$(2x - L_n y)^2 - 5(F_n y)^2 = -4 \cdot 5^{2k+1}.$$

Therefore by Theorem 2.11, we obtain $|2x - L_n y| = 5^{k+1} F_{2m}$ and $F_n y = 5^k L_{2m}$ with $m \geq 0$. It follows that $(F_n, L_{2m}) = 1$ or 2 by (2.13). Assume that $(F_n, L_{2m}) = 1$. Then $F_n | 5^k$ and therefore $F_n = 5^r$ with $1 \leq r \leq k$. By Corollary 1, it is seen that $r = 1$ and $n = 5$.

Thus we get $y = 5^{k-1}L_{2m}$ with $m \geq 0$. Substituting this value of y into $|2x - L_n y| = 5^{k+1}F_{2m}$ and using (2.9) and (2.10), we obtain $(x, y) = (5^{k-1}L_{2m+5}, 5^{k-1}L_{2m})$ with $m \geq 0$ or $(x, y) = (5^{k-1}L_3, 5^{k-1}L_2)$ or $(x, y) = (5^{k-1}L_1, 5^{k-1}L_4)$. Therefore we can take x and y as $(x, y) = (5^{k-1}L_{2m+1}, 5^{k-1}L_{2m-4})$ with $m \geq 0$. Assume that $(F_n, L_{2m}) = 2$. Then $3|m$ and thus $m = 3s$ for some positive integer s. Also it follows that $(F_n/2, L_{2m}/2) = 1$. Thus $F_n/2|5^k$ and therefore $F_n = 2 \cdot 5^r$ with $0 \leq r \leq k$. By Corollary 1, it is seen that $r = 0$ and $n = 3$. Then we get $y = 5^k L_{6s}/2$. Substituting this value of y into $|2x - L_n y| = 5^{k+1}F_{6s}$, we obtain $(x, y) = (5^k L_{6s+3}/2, 5^k L_{6s}/2)$ with $s \geq 0$. Conversely, if $(x, y) = (5^{k-1}L_{2m+1}, 5^{k-1}L_{2m-4})$ with $m \geq 0$, then by (2.11), it follows that $x^2 - L_5 xy - y^2 = -5^{2k+1}$ and if $(x, y) = (5^k L_{6s+3}/2, 5^k L_{6s}/2)$ with $s \geq 0$, then by (2.11), it follows that $x^2 - L_3 xy - y^2 = -5^{2k+1}$. This completes the proof. □

Since the proof of the following theorem is similar to that of Theorem 3.7, we omit its proof.

Theorem 3.8. *The equation $x^2 - L_n xy - y^2 = 5^{2k+1}$ has positive integer solutions if and only if $n = 3$ or $n = 5$. Moreover, all positive integer solutions of the equation $x^2 - L_3 xy - y^2 = 5^{2k+1}$ are given by $(x, y) = (5^k L_{3m+3}/2, 5^k L_{3m}/2)$ with odd integer $m \geq 0$ and all positive integer solutions of the equation $x^2 - L_5 xy - y^2 = 5^{2k+1}$ are given by $(x, y) = (5^{k-1}L_{2m+6}, 5^{k-1}L_{2m+1})$ with $m \geq 0$.*

Theorem 3.9. *If $n \neq 5$, then all positive integer solutions of the equation $x^2 - L_n xy - y^2 = -5^{2k}$ are given by $(x, y) = (5^k F_{n(t+1)}/F_n, 5^k F_{nt}/F_n)$ with odd integer $t > 0$ and if $n = 5$, then all positive integer solutions of the equation $x^2 - L_5 xy - y^2 = -5^{2k}$ are given by $(x, y) = (5^{k-1}F_{2m+2}, 5^{k-1}F_{2m-3})$ with $m \geq 0$.*

Proof. Assume that $x^2 - L_n xy - y^2 = -5^{2k}$ for some positive integers x and y. Multiplying both sides of the equation by 4, we get

$$4x^2 - 4L_n xy - 4y^2 = -4 \cdot 5^{2k}.$$

Completing the square gives

$$(2x - L_n y)^2 - (L_n^2 + 4)y^2 = -4 \cdot 5^{2k}. \tag{3.8}$$

Since n is an odd integer, it is seen that $L_n^2 + 4 = 5F_n^2$ from identity (2.4). Thus, from eq. (3.8), we get

$$(2x - L_n y)^2 - 5(F_n y)^2 = -4 \cdot 5^{2k}.$$

Therefore by Theorem 2.11, we obtain $|2x - L_n y| = 5^k L_{2m+1}$ and $F_n y = 5^k F_{2m+1}$ with $m \geq 0$. If $d = (n, 2m + 1)$, then $(F_n, F_{2m+1}) = F_d$ by (2.14). It follows that $F_n = F_d a$ and $F_{2m+1} = F_d b$ for some positive integers a and b, with $(a, b) = 1$. Then, from equality $F_n y = 5^k F_{2m+1}$, we obtain $ay = 5^k b$. Since $(a, b) = 1$, it follows that $a|5^k$ and therefore $a = 5^r$ with $0 \leq r \leq k$. Thus we get $F_n = 5^r F_d$. By Corollary 2, we obtain $r = 0$, $n = d$ or $r = 1$, $n = 5$ and $d = 1$. Assume that $r = 0$, $n = d$. Then $n|2m + 1$ and therefore $F_n|F_{2m+1}$ by Theorem 2.6. Thus we get $y = 5^k(F_{2m+1}/F_n)$ with $m \geq 0$. Substituting this value of y into $|2x - L_n y| = 5^k L_{2m+1}$, we obtain $(x, y) = (5^k F_{n+2m+1}/F_n, 5^k F_{2m+1}/F_n)$ by (2.7). Since $n|2m + 1$, it follows that $2m + 1 = nt$ for some odd integer $t > 0$. Thus we can write $(x, y) = (5^k F_{n(t+1)}/F_n, 5^k F_{nt}/F_n)$. Assume that $r = 1$, $n = 5$ and $d = 1$. Then we get $y = 5^{k-1}F_{2m+1}$. Substituting this value of y into $|2x - L_n y| = 5^k L_{2m+1}$ and using (2.7) and (2.8), we obtain $(x, y) = (5^{k-1}F_{2m+6}, 5^{k-1}F_{2m+1})$ with $m \geq 0$ or $(x, y) = (5^{k-1}F_4, 5^{k-1}F_1)$ or $(x, y) = (5^{k-1}F_2, 5^{k-1}F_3)$. We can take x and y as $(x, y) = (5^{k-1}F_{2m+2}, 5^{k-1}F_{2m-3})$ with $m \geq 0$. Conversely, if $(x, y) = (5^k F_{n(t+1)}/F_n, 5^k F_{nt}/F_n)$ with odd integer $t > 0$, then by (2.12), it follows that $x^2 - L_n xy - y^2 = -5^{2k}$ for $n \neq 5$ and if $(x, y) = (5^{k-1}F_{2m+2}, 5^{k-1}F_{2m-3})$ with $m \geq 0$, then by (2.12), it follows that $x^2 - L_5 xy - y^2 = -5^{2k}$. This completes the proof. □

Since the proof of the following theorem is similar to that of Theorem 3.9, we omit its proof.

Theorem 3.10. *If $n \neq 5$, then all positive integer solutions of the equation $x^2 - L_n xy - y^2 = 5^{2k}$ are given by $(x, y) = (5^k F_{n(t+1)}/F_n, 5^k F_{nt}/F_n)$ with even integer $t > 0$ and if $n = 5$, then all positive integer solutions of the equation $x^2 - L_5 xy - y^2 = 5^{2k}$ are given by $(x, y) = (5^{k-1}F_{2m+5}, 5^{k-1}F_{2m})$ with $m > 0$.*

COROLLARY 9

If k is odd, then there is no positive integer solution of the equation $x^2 - 4xy^2 - y^4 = -5^{2k+1}$. If k is even, then all positive integer solutions of the equation $x^2 - 4xy^2 - y^4 = -5^{2k+1}$ are given by $(x, y) = (2 \cdot 5^k, 5^{k/2})$ or $(x, y) = (38 \cdot 5^k, 3 \cdot 5^{k/2})$.

Proof. Assume that $x^2 - 4xy^2 - y^4 = -5^{2k+1}$ for some positive integers x and y. Then by Theorem 3.7, it follows that $(x, y^2) = (5^k L_{6m+3}/2, 5^k L_{6m}/2)$ with $m \geq 0$. Thus we get $y^2 = 5^k L_{6m}/2$. Since $5 \nmid L_{6m}$ by (2.6), it can be seen that k is even. Then $L_{6m} = 2z^2$ for some integer z. By Theorem 2.3, we get $m = 0$ or $m = 1$ and therefore $(x, y) = (2 \cdot 5^k, 5^{k/2})$ or $(x, y) = (38 \cdot 5^k, 3 \cdot 5^{k/2})$. This completes the proof. □

COROLLARY 10

The equation $x^2 - 11xy^2 - y^4 = -5^{2k+1}$ has no positive integer solution.

Proof. Assume that $x^2 - 11xy^2 - y^4 = -5^{2k+1}$ for some positive integers x and y. Then by Theorem 3.7, it follows that $(x, y^2) = (5^{k-1}L_{2m+1}, 5^{k-1}L_{2m-4})$ with $m \geq 0$. Thus we get $y^2 = 5^{k-1}L_{2m-4}$. Since $5 \nmid L_{2m-4}$ by (2.6), it is seen that k is odd. Then $L_{2m-4} = z^2$ for some integer z. By Theorem 2.3, this is impossible. This completes the proof. □

COROLLARY 11

The equation $x^2 - 4xy^2 - y^4 = 5^{2k+1}$ has no positive integer solution.

Proof. Assume that $x^2 - 4xy^2 - y^4 = 5^{2k+1}$ for some positive integers x and y. Then by Theorem 3.8, it follows that $(x, y^2) = (5^k L_{3m+3}/2, 5^k L_{3m}/2)$ with odd integer $m \geq 0$. Thus we get $y^2 = 5^k L_{3m}/2$. Since $5 \nmid L_{3m}$ by (2.6), it is seen that k is even. Then $L_{3m} = 2z^2$ for some integer z. Since m is odd, this is impossible by Theorem 2.3. This completes the proof. □

COROLLARY 12

If k is even, then there is no positive integer solution of the equation $x^2 - 11xy^2 - y^4 = 5^{2k+1}$. If k is odd, then all positive integer solutions of the equation $x^2 - 11xy^2 - y^4 = 5^{2k+1}$ are given by $(x, y) = (18 \cdot 5^{k-1}, 5^{(k-1)/2})$ or $(x, y) = (47 \cdot 5^{k-1}, 2 \cdot 5^{(k-1)/2})$.

Proof. Assume that $x^2 - 11xy^2 - y^4 = 5^{2k+1}$ for some positive integers x and y. Then by Theorem 3.8, it follows that $(x, y^2) = (5^{k-1}L_{2m+6}, 5^{k-1}L_{2m+1})$. Thus we get $y^2 = 5^{k-1}L_{2m+1}$. Since $5 \nmid L_{2m+1}$ by (2.6), it is seen that k is odd. Then $L_{2m+1} = z^2$ for some integer z. By Theorem 2.3, we get $m = 0$ or $m = 1$ and therefore $(x, y) = (18 \cdot 5^{k-1}, 5^{(k-1)/2})$ or $(x, y) = (47 \cdot 5^{k-1}, 2 \cdot 5^{(k-1)/2})$. This completes the proof. □

Since the proofs of the following corollaries are similar, we omit their proofs.

COROLLARY 13

If k is odd, then there is no positive integer solution of the equation $x^2 - L_nxy^2 - y^4 = -5^{2k}$. If k is even, then the equation $x^2 - L_nxy^2 - y^4 = -5^{2k}$ has only one positive integer solution given by $(x, y) = (L_n 5^k, 5^{k/2})$.

COROLLARY 14

If k is even, then the equation $x^2 - L_nxy^2 - y^4 = 5^{2k}$ has positive integer solution only when $n = 3$ and this solution is given by $(x, y) = (17 \cdot 5^k, 2 \cdot 5^{k/2})$. If k is odd, then the equation $x^2 - L_nxy^2 - y^4 = 5^{2k}$ has positive integer solution only when $n = 5$ and these solutions are given by $(x, y) = (1597 \cdot 5^{k-1}, 12 \cdot 5^{(k-1)/2})$ or $(x, y) = (13 \cdot 5^{k-1}, 5^{(k-1)/2})$.

COROLLARY 15

The equation $x^4 - 4x^2y - y^2 = -5^{2k+1}$ has no positive integer solution.

COROLLARY 16

If k is even, then there is no positive integer solution of the equation $x^4 - 11x^2y - y^2 = -5^{2k+1}$. If k is odd, then all positive integer solutions of the equation $x^4 - 11x^2y - y^2 = -5^{2k+1}$ are given by $(x, y) = (5^{(k-1)/2}, 7 \cdot 5^{k-1})$ or $(x, y) = (2 \cdot 5^{(k-1)/2}, 3 \cdot 5^{k-1})$.

COROLLARY 17

If k is odd, then there is no positive integer solution of the equation $x^4 - 4x^2y - y^2 = 5^{2k+1}$. If k is even, then the equation $x^4 - 4x^2y - y^2 = 5^{2k+1}$ has one positive integer solution given by $(x, y) = (3 \cdot 5^{k/2}, 2 \cdot 5^k)$.

COROLLARY 18

The equation $x^4 - 11x^2y - y^2 = 5^{2k+1}$ has no positive integer solution.

COROLLARY 19

If k is even, then the equation $x^4 - L_nx^2y - y^2 = -5^{2k}$ has positive integer solution only when $n = 3$ and this solution is given by $(x, y) = (2 \cdot 5^{k/2}, 5^k)$. If k is odd, then the equation $x^4 - L_nx^2y - y^2 = -5^{2k}$ has positive integer solution only when $n = 5$ and these solutions are given by $(x, y) = (5^{(k-1)/2}, 2 \cdot 5^{k-1})$ or $(x, y) = (12 \cdot 5^{(k-1)/2}, 13 \cdot 5^{k-1})$.

COROLLARY 20

The equation $x^4 - L_nx^2y - y^2 = 5^{2k}$ has no positive integer solution.

References

[1] Carlitz L, A note on Fibonacci numbers, *Fibonacci Quart.* **1** (1964) 15–28
[2] Cohn J H E, Square Fibonacci numbers, etc., *Fibonacci Quart.* **2.2** (1964) 109–113
[3] Demirtürk B and Keskin R, Integer solutions of some diophantine equations via Fibonacci and Lucas numbers, *J. Integer Seq.* **12** (2009)
[4] Hilton P, Pedersen J and Somer L, On Lucasian numbers, *Fibonacci Quart.* **35** (1997) 43–47
[5] Keskin R and Demirtürk B, Fibonacci and Lucas congruences and their applications, *Acta Math. Sin.* **27(4)** (2011) 725–736
[6] Keskin R and Yosma Z, On Fibonacci and Lucas numbers of the form cx^2, *J. Integer Seq.* **14** (2011) article 11.9.3
[7] Koshy T, Fibonacci and Lucas numbers with applications (2001) (New York-Toronto: John Wiley and Sons)
[8] Vajda S, Fibonacci and Lucas numbers and the golden section (1989) (England: Ellis Horwood Limited Publ.)