Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 3, August 2014, pp. 281–289.

Kirchhoff index of graphs and some graph operations

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MS received 9 May 2013; revised 31 July 2013

Abstract. Let T be a rooted tree, G a connected graph, $x, y \in \mathrm{V}(G)$ be fixed and G_i's be $|\mathrm{V}(T)|$ disjoint copies of G with x_i and y_i denoting the corresponding copies of x and y in G_i, respectively. We define the T-repetition of G to be the graph obtained by joining y_i to x_j for each $i \in \mathrm{V}(T)$ and each child j of i. In this paper, we compute the Kirchhoff index of the T-repetition of G in terms of parameters of T and G. Also we study how $\mathrm{Kf}(G)$ behaves under some graph operations such as joining vertices or subdividing edges.

Keywords. Kirchhoff index; resistance distance.

2010 Mathematic Subject Classification. 05C12, 05C50.

1. Introduction

In this paper, all graphs are supposed to be simple and we assume that G is a connected graph with vertex set $\mathrm{V}(G)$ and edge set $\mathrm{E}(G)$. Also by $\mathrm{N}(x)$ we mean the set of neighbours of a vertex x. The concept of resistance distance was first introduced by Klein and Randić [4]. Recently this concept has got a wide attention from different authors especially those interested in applications in quantum chemistry, see for example [1–3,5,7–9].

If we view G as an electrical network and replace each edge e of G with a resistance ρ_e, the *resistance distance* between any two vertices a and b, denoted by $r_{ab}(G)$ is defined to be the effective resistance between a and b as computed by Ohm's and Kirchhoff's laws. To describe it more concretely, let $I(x, y) = I_{xy}$ be a positive real valued function defined on all pairs of adjacent vertices of G and $P(x)$ be a real valued function defined on $\mathrm{V}(G)$. Fix two vertices a and b of G and assume that I and P satisfy the following conditions:

(i) $I_{xy} = -I_{yx}$,
(ii) $\sum_{y\in \mathrm{N}(x)} I_{xy} = (\delta_{ax} - \delta_{bx})I_0$ for all vertices x, where I_0 is a constant,
(iii) $I_{xy}\rho_{xy} = P(x) - P(y)$ for all pairs of adjacent vertices x, y.

Then we say that I is a *flow function* from a to b (or an (a, b)-flow function) with *total flow* I_0 and P a *potential function* for I. In this case we say $r_{ab} = \frac{P(a)-P(b)}{I_0}$ is the resistance distance between a and b (see [4]). Also we set $r_{aa} = 0$.

Note that we can replace (iii) above with

(iii′) $\sum_{1\le i<k} I_{x_i x_{i+1}}\rho_{x_i x_{i+1}} = 0$ for all cycles $x_1, x_2, \ldots, x_{k-1}, x_1$ in G.

It is easy to see that if (iii′) holds, then a potential function satisfying (iii) exists and is unique up to a constant summand. If a, b are two vertices lying in different components of a graph H, we define $r_{ab}(H) = \infty$. Now the *Kirchhoff index* of G is defined as

$$\mathrm{Kf}(G) = \frac{1}{2}\sum_{a,b\in \mathrm{V}(G)} r_{ab}.$$

If we define the $n \times n$ weighted adjacency matrix $A(G) = (a_{ij})$ of G as

$$a_{ij} = \begin{cases} \frac{1}{\rho_{ij}}, & i \sim j,\\ 0, & \text{otherwise},\end{cases}$$

and the diagonal matrix of vertex degrees $D(G)$ as $D(G) = \mathrm{diag}\left(\sum_{i=1}^n a_{1i}, \ldots, \sum_{i=1}^n a_{ni}\right)$, then the *Laplacian matrix* of G is defined to be $L(G) = D(G) - A(G)$. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of $L(G)$, called the *Laplacian eigenvalues* of G, then $\lambda_1 = 0$ and for $k = 2, \ldots, n$, λ_k are nonzero real numbers (since G is connected and $L(G)$ is symmetric, see [6]). Theorem F of [4] implies the following.

Theorem 1.1. *For any connected n-vertex graph G ($n \ge 2$):*

$$\mathrm{Kf}(G) = n\sum_{k=2}^{n} \frac{1}{\lambda_k}.$$

Let T be a rooted tree and G an arbitrary graph. In §2, we define a composition of T and G (calling it the T-repetition of G) and compute its Kirchhoff index in terms of parameters of T and G. In §3, we study how $\mathrm{Kf}(G)$ behaves under some graph operations such as joining vertices and deleting or subdividing edges. In [9], the Kirchhoff index of some composite graphs, including the join of two arbitrary graphs is computed. But in contrast to that paper, here all ρ_e's are assumed to be units. It is clear that in many real applications such an assumption is restrictive.

2. Kirchhoff index of the T-repetition of G

DEFINITION 2.1

Let T be a weighted rooted tree on vertices $\{1, 2, \ldots, n(T)\}$, $x, y \in \mathrm{V}(G)$ be fixed and G_i's be $n(T)$ disjoint copies of G with x_i and y_i denoting the corresponding copies of x and y in G_i, respectively. By $T(G, x, y)$ (or $T(G)$ if there is no ambiguity), we mean the graph obtained by joining y_i to x_j for each $i \in \mathrm{V}(T)$ and each child j of i. The weight of the edge $y_i x_j$ would be the weight of the edge ij in T.

Figure 1 illustrates the T-repetition of G. To give a formula for $\mathrm{Kf}(T(G))$ we need the following lemmas.

Lemma 2.2 (*Lemma E of* [4]). *Let u be a cut vertex of a graph, and let x and y be vertices belonging to different components $G - u$, say H_1 and H_2, respectively. Then $r_{xy}(G) = r_{xu}(H_1) + r_{uy}(H_2)$.*

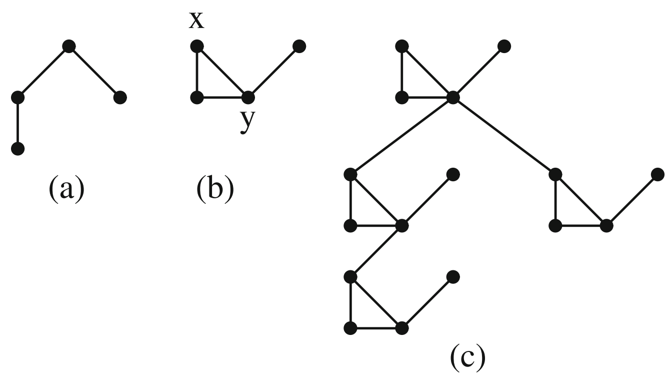

Figure 1. (**a**) A rooted tree T; (**b**) a graph G; (**c**) $T(G)$.

The next example shows that we cannot state a result similar to (2.2) in a graph G with a vertex cut S of size 2 or more. Indeed, it shows that if u, v are in different S-components of G (that is, the subgraph induced by the union of S and a component of $G - S$), say H_1 and H_2 respectively, then $r_{uv}(G)$ is not a function of $r_{ux}(H_1)$'s and $r_{vx}(H_2)$'s, where $x \in S$.

Example 2.3. Assume that G is any of the graphs shown in figure 2, considered with unit resistances at all edges. Then $S = \{a, b\}$ is a vertex cut and if H and K are the S-components of G containing y and x, respectively, then $r_{ya}(H) = r_{yb}(H) = 2$ and $r_{xa}(K) = r_{xb}(K) = 1$. Despite this, in (a) we have $r_{xy}(G) = \frac{3}{2}$ but in (b) we have $r_{xy}(G) = 2$.

Assume that $S \subseteq \mathrm{V}(G)$ and H_1 and H_2 are subgraphs of G. We say that (H_1, H_2) is an S-separation of G, when $G = H_1 \cup H_2$ and $H_1 \cap H_2 = G[S]$. The proof of the following lemma is easy and we leave it to reader.

Lemma 2.4. *Let u be a cut vertex of G and (H_1, H_2) be a u-separation of G. If $x, y \in \mathrm{V}(H_1)$, then $r_{xy}(G) = r_{xy}(H_1)$.*

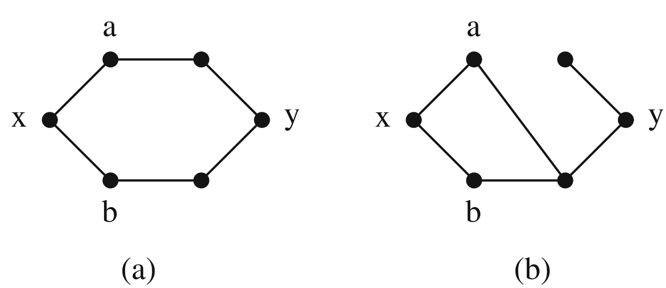

Figure 2(a, b). Graphs of Example 2.3.

Also for a vertex cut of size two we have as follows.

PROPOSITION 2.5

Let G be a graph with a 2-vertex cut S. Assume that (H_1, H_2) is an S-separation of G and $u, v \in \mathrm{V}(H_1)$. Let G' be H_1 in which we set $\rho_{ab}(G') = r_{ab}(H_2)$ for $a, b \in S$ (if $ab \notin \mathrm{E}(H_1)$ we add this edge to G'), then $r_{uv}(G') = r_{uv}(G)$.

Proof. Let I be an (u, v)-flow of G, P be a potential function for I and set $I' = I|_{H_2}$. Since $\sum_{y\in\mathrm{N}(x)} I_{xy} = 0$ for all $a, b \neq x \in \mathrm{V}(H_2)$, we see that I' is an (a, b)-flow or a (b, a)-flow of H_2. Define an (u, v)-flow I'' of G' by $I''_{xy} = I_{xy}$ for edges $ab \neq xy \in \mathrm{E}(G)$, $I''_{ab} = \sum_{y\in\mathrm{N}_{H_2}(a)} I'_{ay}$ and $I''_{ba} = -I''_{ab}$. Now it is easy to see that I'' and $P|_{\mathrm{V}(G')}$ satisfy conditions (i)–(iii) of introduction and hence the result follows. □

The next example shows that we cannot generalize Proposition 2.5 to vertex cuts of size >2.

Example 2.6. Let G be the graph in figure 3(a), $\rho_e = 1$ for all $e \in \mathrm{E}(G)$ and $S = \{a, b, c\}$, which is a vertex cut of G. Let H_1 and H_2 be the two S-components of G and assume that $x, y \in H_1$. Using the well-known serial and parallel rules for computing effective resistances, it is easy to see that $r_{yx}(G) = \frac{3}{4}$.

Similar to Proposition 2.5, we define G' to be the graph obtained from H_1 by adding edges uv with $\rho_{uv}(G') = r_{uv}(H_2) = 2$ for each $u, v \in S$ (figure 3(b)). Then $r_{xy}(G') = \frac{5}{7} \neq r_{xy}(G)$. To see this, note that by symmetry they have the same potential for any (x, y)-flow I of G' between c and b, and hence $I_{cb} = 0$. So again we can apply the serial and parallel rules.

Recall that the *Wiener index* of a graph G is $\mathrm{W}(G) = \frac{1}{2}\sum_{x,y\in\mathrm{V}(G)} d_{xy}$, where d_{xy} denotes the distance of vertices x and y. In what follows, for each $x \in \mathrm{V}(G)$, we set $\mathrm{Kf}_x(G) = \sum_{x\neq y\in\mathrm{V}(G)} r_{xy}$. Obviously $\mathrm{Kf}(G)$ can be written as $\mathrm{Kf}(G) = \frac{1}{2}\sum_{x\in\mathrm{V}(G)} \mathrm{Kf}_x(G)$. We use similar notations for the Wiener index.

Theorem 2.7. *Let T be a tree rooted at the vertex r, G_1 a connected graph and $x, y \in \mathrm{V}(G_1)$. Set $G = T(G_1, x, y)$, $n = |\mathrm{V}(T)|$, $m = |\mathrm{V}(G)|$ and $\omega = \mathrm{W}_r(T)$. Then*

$$\begin{aligned}\mathrm{Kf}(G) &= n\mathrm{Kf}(G_1) + \omega m\mathrm{Kf}_y(G_1) + (n(n-1) - \omega)m\mathrm{Kf}_x(G_1)\\ &\quad + (\mathrm{W}(T) - n(n-1) + \omega)m^2 r_{xy}(G_1) + m^2\mathrm{Kf}(T).\end{aligned}$$

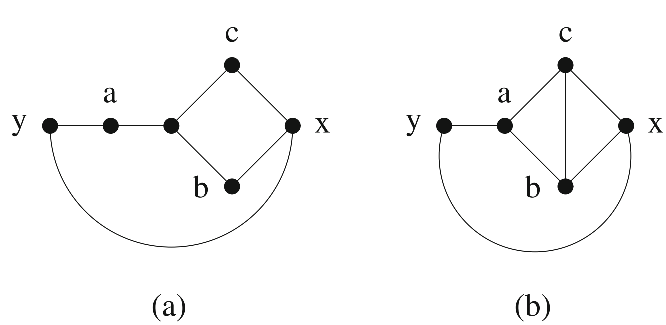

Figure 3. The graphs (**a**) G and (**b**) G' of Example 2.6.

Proof. By definition, $\mathrm{Kf}(G) = \sum\{r_{uv} | \{u, v\} \subseteq \mathrm{V}(G),\ u \neq v\}$. According to Lemmas 2.4 and 2.2, the contribution to $\mathrm{Kf}(G)$ by the pairs u and v of vertices of G belonging to the same copy of G_1 is $A = n\mathrm{Kf}(G_1)$. If the vertices u and v belong to different copies of G_1, say the copies corresponding to vertices i and j of T, respectively, we consider two cases:

Case (*i*). i and j are not ancestors of each other. Then by Lemma 2.2

$$r_{uv} = r_{ux_i}(G_1) + r_{x_i x_j} + r_{x_j v}(G_1).$$

Suppose $u_1, \ldots, u_m$ and $v_1, \ldots v_n$ are the vertices of the i-th and j-th copies of G_1, respectively. Fixing i and j in the above equation and summing over u and v we get

$$\begin{aligned} B_1 = \sum_{k=1}^{m}\sum_{t=1}^{m} r_{u_k v_t} &= \sum_{k=1}^{m}\sum_{t=1}^{m} (r_{u_k x_i}(G_1) + r_{x_i x_j} + r_{x_j v_t}(G_1)) \\ &= m\mathrm{Kf}_x(G_1) + m^2 r_{x_i x_j} + m\mathrm{Kf}_x(G_1) \\ &= 2m\mathrm{Kf}_x(G_1) + m^2 r_{x_i x_j}. \end{aligned}$$

To compute $r_{x_i x_j}$, let d_{ij} be the distance of the two vertices i and j and k be the last common ancestor of i and j in T. Then paths from x_i to x_j in G do not go through k-th copy of G_1 and just meet y_k in this copy. Thus using Lemmas 2.4 and 2.2, it is easy to see that $r_{x_i x_j} = (d_{ij} - 2) r_{xy}(G_1) + r_{ij}(T)$. So

$$B_1 = 2m\mathrm{Kf}_x(G_1) + m^2\big((d_{ij} - 2) r_{xy}(G_1) + r_{ij}(T)\big).$$

Case (*ii*). j is an ancestor of i. In this case $r_{uv} = r_{ux_i}(G_1) + r_{x_i y_j} + r_{y_j v}(G_1)$. Therefore, similar to the previous case,

$$B_2 = \sum_{k=1}^{m}\sum_{t=1}^{m} r_{u_k v_t} = m\mathrm{Kf}_x(G_1) + m\mathrm{Kf}_y(G_1) + m^2 r_{x_i y_j}.$$

Also $r_{x_i y_j} = (d_{ij} - 1) r_{xy}(G_1) + r_{ij}(T)$. Hence we have

$$B_2 = m\mathrm{Kf}_x(G_1) + m\mathrm{Kf}_y(G_1) + m^2((d_{ij} - 1) r_{xy}(G_1) + r_{ij}(T)).$$

Note that the summation of d_{ij}'s over all i's and j's is the Wiener index of T. Also the summation of all $r_{ij}(T)$ is $\mathrm{Kf}(T)$. Furthermore, vertex i of T has d_{ir} ancestors. So Case (ii) occurs for $\omega = \sum_{i=1}^{n} d_{ir}$ pairs of i, j and Case (i) for $\binom{n}{2} - \omega$ times. Consequently, summing B_1 over all pairs of distinct vertices i, j of T, neither of which is an ancestor of the other, and summing B_2 over all pairs of distinct vertices i, j of T with j an ancestor of i, we get

$$\begin{aligned} B &= \omega m\mathrm{Kf}_y(G_1) + (n(n-1) - \omega) m\mathrm{Kf}_x(G_1) + m^2\mathrm{Kf}(T) \\ &\quad + (\mathrm{W}(T) - n(n-1) + \omega) m^2 r_{xy}(G_1). \end{aligned}$$

The proof is completed by summing A and B. □

In Theorem 2.7, if $x = y$, then $G = T(G_1)$ is the cluster $T\{G_1\}$ as defined in [9] and it can be easily checked that the formula for $\mathrm{Kf}(G)$ in the above theorem and in Theorem 3.11 of [9] coincide. Note that in [9] all resistances of edges are assumed to be unit.

3. Kirchhoff index and some other graph operations

In this section, we pay attention to edge deletions and subdivisions and also joining vertices. Using Proposition 2.5 we can deduce as follows.

COROLLARY 3.1

Let $e = xy \in \mathrm{E}(G)$ and set $G' = G - e$. Assume that K is obtained form G by subdividing e and name the new vertex z. Assign resistances $\rho_{xz}(K) = a$ and $\rho_{yz}(K) = b$ to the new edges in K. If $r' = r_{xy}(G')$, then

(i) $r' = \dfrac{\rho_{xy}(G) r_{xy}(G)}{\rho_{xy}(G) - r_{xy}(G)}$,

(ii) $r_{xy}(K) = \dfrac{r'(a+b)}{r'+a+b}$,

(iii) $r_{xz}(K) = \dfrac{a(r'+b)}{r'+a+b}$ *and*

(iv) $r_{yz}(K) = \dfrac{b(r'+a)}{r'+a+b}$.

Proof. Let H be the complete graph on x, y, z with $\rho_{xy}(H) = r'$, $\rho_{xz}(H) = a$ and $\rho_{yz}(H) = b$. In K, $\{x, y\}$ is a vertex cut, so we can apply (2.5) to see that $r_{ij}(K) = r_{ij}(H)$, for each $i, j \in \{x, y, z\}$. From this, the last three formulas follow.

To prove the first one, consider the special case $a = b = \frac{1}{2}\rho_{xy}(G)$. Obviously, in this case $r_{ij}(K) = r_{ij}(G)$ for any $i, j \in \mathrm{V}(G)$ including $i = x$, $j = y$. Therefore, by the above argument $r_{xy}(G) = r_{xy}(K) = \frac{r'(a+b)}{r'+a+b} = \frac{r'\rho_{xy}(G)}{r'+\rho_{xy}(G)}$. Solving this equation for r', we get the first formula of the statement. □

Note that in the notations of the above result, as was mentioned in the proof, if $a + b = \rho_{xy}$, then $r_{ij}(G'') = r_{ij}(G)$ for any $i, j \in \mathrm{V}(G)$. Consequently, in this case we have $\mathrm{Kf}(G'') = \mathrm{Kf}(G) + \mathrm{Kf}_z(G)$.

Theorem 3.2. *Let K_n be the complete graph with vertices $\{1, 2, \ldots, n\}$, $3 \le n$, and G be obtained by subdividing one of its edges. If we assume unit resistances for all edges of G, then*

$$\mathrm{Kf}(G) = \frac{3n^3 - 2n^2 - 3n + 6}{2n^2 - 2n}.$$

Proof. Without loss of generality, we assume that the edge $\{n, n-1\}$ of K_n is subdivided and let $n + 1$ be the new vertex. Let $B = \lambda I - L(G)$. By adding to the first row the sum of all other rows and factorizing λ, we see that $\det B = \lambda \det C$, where

$$C = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 1 & \lambda - n + 1 & 1 & \ldots & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & \lambda - n + 1 & 0 & 1 \\ 1 & 1 & 1 & \ldots & 0 & \lambda - n + 1 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 1 & \lambda - 2 \end{bmatrix}.$$

Now by subtracting the first row of C from all other rows except the last one and then dividing the rows $2, \ldots, n-2$ by $\lambda - n$, we see that $\det B = \lambda(\lambda - n)^{n-3} \det D$, where

$$D = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 0 & \frac{-1}{\lambda - n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda - n & -1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & \lambda - n & 0 \\ 0 & 0 & 0 & \ldots & 1 & 1 & \lambda - 2 \end{bmatrix}.$$

Next, subtract the n-th row from the $(n-1)$-th row and divide it by $\lambda - n + 1$. Then add the $(n-1)$-th row to the n-th row and divide it by $\lambda - n - 1$ to see that $\det B = \lambda(\lambda - n)^{n-3}(\lambda - n + 1)(\lambda - n - 1) \det E$, where E is the same as D except on its $(n-1)$-th and n-th row. The $(n-1)$-th and n-th rows of E are $0, \ldots, 0, 1, -1, 0$, and $0, \ldots, 0, 1, 0$, respectively. Finally, subtract the $(n-1)$-th row of E and then 2 times its n-th row from its $(n+1)$-th row to see that $\det B = \lambda(\lambda - n)^{n-3}(\lambda - n + 1)(\lambda - n - 1)(\lambda - 2)$. Now the result follows from Theorem 1.1. □

Next we consider the Kirchhoff index (or equivalently, the Laplacian eigenvalues) of some specially weighted complete graphs obtained by joining vertices.

Theorem 3.3. *Let H be a complete graph with $\mathrm{V}(H) = \{2, \ldots, n\}$ and $\rho : \mathrm{E}(H) \to \mathbb{R}^+$ be a resistance function. Assume that $a \in \mathbb{R}^+$, with $a > \rho_{i,j}$ for all $1 < i, j \leq n$ and G is the graph obtained by joining the vertex 1 to H with resistances $\rho_{1i} = a$ for all $1 < i \leq n$. Moreover, suppose that H' is the graph H considered with edge resistance $\rho'_{ij} = \frac{a\rho_{ij}}{a - \rho_{ij}}$. Then the Laplacian eigenvalues of G are 0, $\lambda'_2 + \frac{n}{a}, \lambda'_3 + \frac{n}{a}, \ldots, \lambda'_n + \frac{n}{a}$, where λ'_i's are the Laplacian eigenvalues of H'.*

Proof. Let $B = \lambda I_{n \times n} - L(G)$ and $b = \frac{1}{a}$. By adding to the first row the sum of all other rows and factorizing λ, we see that $\det B = \lambda \det C$, where C is the same as B except in the first row, in which C has every entry 1. Now by subtracting b times the first row of C from each of its other rows, we deduce that

$$\det B = \lambda \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & \lambda - 2b - d_2 & a_{23} - b & \ldots & a_{2n} - b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} - b & a_{n3} - b & \ldots & \lambda - 2b - d_n \end{vmatrix},$$

where (a_{ij}) is the adjacency matrix of H and $d_i = \sum_{j=2}^{n} a_{ij}$. If D is the matrix obtained by deleting the first row and the first column of the above matrix, then $\det B = \lambda \det D$. Set $\lambda' = \lambda - nb$. Thus

$$D = \begin{bmatrix} \lambda' - (d_2 - (n-2)b) & a_{23} - b & \ldots & a_{2n} - b \\ a_{21} - b & \lambda' - (d_3 - (n-2)b) & \ldots & a_{2n} - b \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} - b & a_{n3} - b & \ldots & \lambda' - (d_n - (n-2)b) \end{bmatrix}.$$

Now let $A' = (a'_{ij})$ be the Laplacian matrix of H'. Then for $1 < i \neq j \leq n$, we have

$$a'_{ij} = \frac{1}{\rho'_{ij}} = \frac{a - \rho_{ij}}{a\rho_{ij}} = \frac{1}{\rho_{ij}} - \frac{1}{a} = a_{ij} - b.$$

Using this, one can easily check that $D = \lambda' I_{(n-1)\times(n-1)} - L(H')$. Whence the roots of $\det D$ are Laplacian eigenvalues of H' and $\det D$ vanished on $\lambda' = \lambda'_2, \ldots, \lambda' = \lambda'_n$. But this is equivalent to $\lambda = \lambda'_2 + nb, \ldots, \lambda = \lambda'_n + nb$, from which the result follows. □

Example 3.4.

(i) Let $a = 4$ and H be a triangle with all edge resistances $= 4/5$. Then H' of the above theorem is a triangle with unit edge resistances and hence has Laplacian eigenvalues $(0, 3, 3)$ and $\mathrm{Kf}(H') = 2$. Also by (3.3), Laplacian eigenvalues of G are $(0, 1, 4, 4)$ and according to Theorem 1.1, $\mathrm{Kf}(G) = 6$.

(ii) Again, let $a = 4$ and H be a triangle but this time with edge resistances $4/5$, $4/3$ and $4/9$. Then H' is a triangle with edge resistance 1, 2 and $1/2$ and Laplacian eigenvalues $\left(0, \frac{7\pm\sqrt{7}}{2}\right)$ and hence $\mathrm{Kf}(H') = 2$. By the previous theorem, Laplacian eigenvalues of G are $\left(0, 1, \frac{9\pm\sqrt{7}}{2}\right)$ and according to Theorem 1.1, $\mathrm{Kf}(G) = 220/37$. Note that although $\mathrm{Kf}(H')$ and a of the two parts of this example are the same, $\mathrm{Kf}(G)$ differs in these two cases.

Example 3.4 shows that if G and H' are as in Theorem 3.3, $\mathrm{Kf}(G)$ is not a function of $\mathrm{Kf}(H')$ and we need the complete set of Laplacian eigenvalues of H' for computing $\mathrm{Kf}(G)$. Despite this we have as follows.

COROLLARY 3.5

In the notations of the above theorem, assume that $n > 2$ and λ' and λ'' are the minimum and maximum nonzero Laplacian eigenvalues of H', respectively. Let $c_1 = \frac{\lambda'}{\lambda' + \frac{n}{a}}$ and $c_2 = \frac{\lambda''}{\lambda'' + \frac{n}{a}}$. Then $a + \frac{c_1 n}{n-1}\mathrm{Kf}(H') \leq \mathrm{Kf}(G) \leq a + \frac{c_2 n}{n-1}\mathrm{Kf}(H')$. In particular, $a < \mathrm{Kf}(G) < a + \frac{n}{n-1}\mathrm{Kf}(H')$.

Proof. Note that for each $3 \leq i \leq n$, we have $c_1 \frac{1}{\lambda'_i} \leq \frac{1}{\lambda'_i + \frac{n}{a}} \leq c_2 \frac{1}{\lambda'_i}$. Thus the result follows from Theorems 3.3 and 1.1. For the final assertion, note that $0 < c_1, c_2 < 1$. □

The following example shows that these bounds are tight.

Example 3.6. In the notations of Corollary 3.5, assume that $\rho_{ij} = \rho \in \mathbb{R}^+$ for all $1 < i \neq j \leq n$. Then for all such i, j, we have $\rho'_{ij} = \rho' = \frac{a\rho}{a-\rho}$. It is well-known that all of the nonzero Laplacian eigenvalues of a complete graph on n vertices with unit edge resistances are n. From this it easily follows that all nonzero Laplacian eigenvalues of H' are $\lambda' = \frac{n(H')}{\rho'} = \frac{n-1}{\rho'}$. Thus $c_1 = c_2$, so $\mathrm{Kf}(G) = a + \frac{c_1 n}{n-1}\mathrm{Kf}(H')$ and in both sides of the first inequality of Corollary 3.5, equality holds. Also note that if $\rho \to 0$ [$\rho \to a$], then $\lambda' \to \infty$ [$\lambda' \to 0$] and $c_1 \to 1$ [$c_1 \to 0$]. This shows that in the strict inequality of

Corollary 3.5, $\mathrm{Kf}(G)$ can get arbitrarily close to each side, by choosing appropriate edge resistances.

The following theorem shows how Theorem 3.3 can be utilized.

Theorem 3.7. *Assume that* $2 \le n \in \mathbb{N}$ *and* $r_1 > r_2 > \cdots > r_{n-1} > 0$ *are real numbers. Let* $K(r_1, \ldots, r_{n-1})$ *be the complete graph on vertices* $\{1, 2, \ldots, n\}$ *with* $\rho_{ij} = r_i$ *for each* $1 \le i < j \le n$. *The Laplacian eigenvalues of* $K(r_1, \ldots, r_{n-1})$ *are 0 and* $\frac{n-t+1}{r_t} + \sum_{i=1}^{t-1} \frac{1}{r_i}$ *for* $t = 1, 2, \ldots, n-1$.

Proof. The proof is by induction on n with the base step $n = 2$ being trivial. When $n > 2$ we can apply Theorem 3.3 with $G = K(r_1, \ldots, r_{n-1})$, $H = G[2, \ldots, n]$ and $a = r_1$. Clearly H is isomorphic to $K(r_2, r_3, \ldots, r_{n-1})$ and hence H' in Theorem 3.3 is $K\left(\frac{r_1 r_2}{r_1 - r_2}, \frac{r_1 r_3}{r_1 - r_3}, \cdots, \frac{r_1 r_{n-1}}{r_1 - r_{n-1}}\right)$. Therefore, we can apply induction hypothesis on H' and the result follows from Theorem 3.3 by some straightforward calculations. □

COROLLARY 3.8

In the notations of the previous theorem, if we set ${}_t\bar{r}_i = \frac{\prod_{j=1}^{t} r_j}{r_i}$, *then we have*

$$\mathrm{Kf}(K(r_1, \ldots, r_{n-1})) = n \sum_{t=1}^{n-1} \frac{\prod_{i=1}^{t} r_i}{(n-t+1)\, {}_t\bar{r}_t + \sum_{i=1}^{t-1} {}_t\bar{r}_i}.$$

Acknowledgment

The first author (AN) is partially funded by the National Elite Foundation of Iran.

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