Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 2, May 2014, pp. 267–279.

Complete moment convergence of weighted sums for processes under asymptotically almost negatively associated assumptions

JUN AN

College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China
E-mail: scottan@sina.com

MS received 22 January 2013; revised 31 January 2013

Abstract. For weighted sums of sequences of asymptotically almost negatively associated (AANA) random variables, we study the complete moment convergence by using the Rosenthal type moment inequalities. Our results extend the corresponding ones for sequences of independently identically distributed random variables of Chow [4].

Keywords. Asymptotically almost negatively associated; weighted sums; complete moment convergence.

2010 Mathematics Subject Classification. 60F15, 60F99.

1. Introduction

A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be asymptotically almost negatively associated (AANA) if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\begin{aligned}\mathrm{Cov}\,(f(X_n), g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})) &\leqslant q(n)\\ &\times \left[\mathrm{Var}\, f(X_n) \mathrm{Var}\,(g(X_{n+1}, X_{n+2}, \ldots, X_{n+k}))\right]^{1/2}\end{aligned} \tag{1.1}$$

for all $n, k \geqslant 1$ and for all coordinate-wise nondecreasing continuous functions f and g whenever the variances exist. $\{q(n), n \geqslant 1\}$ is called the mixing coefficients of $\{X_n, n \geqslant 1\}$.

Since this concept was introduced by Chandra and Ghosal [6] in 1996, many authors have studied its limit properties. For example, Chandra and Ghosal [6] derived the Kolmogorov type inequality and strong law of large numbers (SLLN). Chandra and Ghosal [7] obtained the almost sure convergence of weighted average. Kim *et al*. [10] established the Hájek–Rényi type inequalities and Marcinkiewicz–Zygmund type SLLN. Cai [2] investigated the complete convergence of weighted sums. Yuan and An [16] got the Rosenthal type inequalities, L_p convergence, complete convergence and Marcinkiewicz–Zygmund type SLLN (see Wang *et al*. [14] who obtained the complete convergence and SLLN). Yang *et al*. [15] derived complete convergence of moving average process for AANA sequence. Recently, An [1] got the following complete convergence of weighted sums for sequences of AANA random variables.

Theorem A [1]. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of identically distributed AANA random variables with mixing coefficients* $\{q(n), n \geqslant 1\}$, $E|X_1|^p < \infty, \alpha p > 1, \alpha > 1/2$. *Suppose that* $EX_1 = 0$ *for* $p > 1$. *Let* $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ *be a sequence of real numbers with* $\sum_{i=1}^n |a_{ni}|^r = O(n), r > (\alpha p - 1)/(\alpha - 1/2)$ *if* $p > 2$; *or* $r = 2$ *if* $0 < p \leqslant 2$. *Take* $\tilde{r} = (1/2^{k-1} - 2/r)r/(r-1)$ *where* k *is a positive integer number satisfying* $2^k < r \leqslant 2^{k+1}$. *If* $\sum_{n=1}^\infty q^{\tilde{r}}(n) < \infty$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) < \infty \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sup_{k\geqslant n}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon\right) < \infty. \tag{1.3}$$

Theorem B [1]. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of mean zero, identically distributed AANA random variables with* $\sum_{n=1}^\infty q^2(n) < \infty$. *Let* $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ *be a sequence of real numbers satisfying* $\sup_{n\geqslant 1}\sum_{i=1}^n a_{ni}^2 < \infty$. *If* $E|X_1|^p < \infty, 0 < p < 2$, then

$$n^{-1/p}\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| \to 0 \textit{ completely.} \tag{1.4}$$

Chow [4] first investigated the complete moment convergence for sequences of independently identically distributed (i.i.d.) random variables and obtained the following result.

Theorem C [4]. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of i.i.d. random variables. If* $0 < p < 2, r > 1, rp \geqslant 1$, *then*

$$E\{|X_1|^{rp} + |X_1|\log(1+|X_1|)\} < \infty \tag{1.5}$$

implies

$$\sum_{n=1}^{\infty} n^{r-1/p-2} E\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i - kEX_1\right| - \varepsilon n^{1/p}\right\}_+ < \infty \tag{1.6}$$

for any $\varepsilon > 0$, *where and in the following* $x_+ = \max\{x, 0\}$.

Recently, Li and Zhang [11] extended Theorem C to moving average processes under dependent assumptions, Chen and Wang [3] extended Theorem C to identically distributed φ-mixing random variables, Zhou and Lin [17] extended Theorem C to moving average processes of φ-mixing random variables and Yang *et al*. [15] extended Theorem C to moving average processes of AANA sequence.

The purpose of this paper is to further study the limit properties of weighted sums for sequences of identically distributed AANA random variables and to obtain complete moment convergence by using the Rosenthal type moment inequality. Our results extend the corresponding one of Chow.

2. Main results

Throughout this paper we use the following notations: $I(\cdot)$ denotes the indicator function, C stands for a positive constant, its value may be different at different places, $\ll$ represents the Vinogradov symbol O, $=:$ means 'defined as'.

Theorem 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed AANA random variables with mean zero and mixing coefficients $\{q(n), n \geqslant 1\}$, $q > 0$, $p > 1$, $\alpha p > 1$, $\alpha > 1/2$. Let $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ be a sequence of real numbers with $\sum_{i=1}^{n} |a_{ni}|^r = O(n)$. $r > (\alpha p - 1)/(\alpha - 1/2)$ if $p \geqslant 2$, or $r = 2$ if $1 < p < 2$. $\sum_{n=1}^{\infty} q^{\tilde{r}}(n) < \infty$ where $\tilde{r} = (1/2^{k-1} - 2/r)r/(r-1)$, k is a positive integer number satisfying $2^k < r \leqslant 2^{k+1}$. If $E|X_1|^p < \infty$ for $q \neq p$, or $E|X_1|^p \log(1 + |X_1|) < \infty$ for $q = p$, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha q - 2} E \left\{ \max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| - \varepsilon n^{\alpha} \right\}_{+}^{q} < \infty \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} E \left\{ \sup_{k \geqslant n} \left| k^{-\alpha} \sum_{i=1}^{k} a_{ni} X_i \right| - \varepsilon \right\}_{+}^{q} < \infty. \tag{2.2}$$

Theorem 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of mean zero, identically distributed AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$ and let $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ be a sequence of real numbers satisfying $\sup_{n \geqslant 1} \sum_{i=1}^{n} a_{ni}^2 < \infty$, $1 \leqslant p < 2$, $q > 0$. If $E|X_1|^p < \infty$ for $q \neq p$, or $E|X_1|^p \log(1 + |X_1|) < \infty$ for $q = p$, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-q/p} E \left\{ \max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| - \varepsilon n^{1/p} \right\}_{+}^{q} < \infty \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} n^{-q/p} E \left\{ \sup_{k \geqslant n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| - \varepsilon \right\}_{+}^{q} < \infty. \tag{2.4}$$

The following lemmas are useful for the proofs of our main results.

Lemma 2.1 [16]. *Let $\{X_n, n \geqslant 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geqslant 1\}$. Let $f_1, f_2, \ldots$ be all nondecreasing (or all nonincreasing) functions, then $\{f_n(X_n), n \geqslant 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{q(n), n \geqslant 1\}$.*

Lemma 2.2 [1,16]. *Let $\{X_n, n \geqslant 1\}$ be a sequence of AANA random variables with mean zero and mixing coefficients $\{q(n), n \geqslant 1\}$. Then there exists a positive constant C_p depending only on p such that*

$$E\left(\max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} X_i \right|^p \right) \leqslant C_p \left\{ \sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n-1} q^{2-2/p}(i) \|X_i\|_p \right)^p \right\} \tag{2.5}$$

for all $n \geqslant 1$ *and* $1 < p \leqslant 2$, *and such that*

$$E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}X_i\right|^p\right) \leqslant C_p \times\left\{\left[1+C\left(\sum_{i=1}^{n-1}q^{\tilde{p}}(i)\right)^{p-1}\right]\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right\} \tag{2.6}$$

for all $n \geqslant 1$ *and* $2^k < p \leqslant 2^{k+1}$ *where integer number* $k \geqslant 1$, $\tilde{p} = (1/2^{k-1} - 2/p)p/(p-1)$.

In particular, if $\sum_{n=1}^{\infty} q^2(n) < \infty$, *then*

$$E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}X_i\right|^p\right) \leqslant C_p\sum_{i=1}^{n}E|X_i|^p \tag{2.7}$$

for all $n \geqslant 1$ *and* $1 < p \leqslant 2$, *and if* $\sum_{n=1}^{\infty} q^{1/(p-1)}(n) < \infty$, *then*

$$E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}X_i\right|^p\right) \leqslant C_p\left\{\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right\} \tag{2.8}$$

for all $n \geqslant 1$ *and* $3\cdot 2^k < p \leqslant 4\cdot 2^{k-1}$ *where integer number* $k \geqslant 1$.

Remark 2.1. Since $2^k < p \leqslant 2^{k+1}$ we know $1/2^{k-1} - 2/p \leqslant 2/p$. By $q(n) \to 0 (n \to \infty)$ and Hölder inequalities one can easily get (2.6) from (2.2) of [16] (see Corollary 2.1 of [1]).

Proof of Theorem 2.1. Without loss of generality, we assume $a_{ni} \geqslant 0$ for all $n \geqslant 1$, $i \geqslant 1$. Let

$$Y_{xi} = X_i I(|X_i| \leqslant x^{1/q}) \quad \text{and} \quad Y'_{xi} = X_i I(|X_i| > x^{1/q}).$$

From Lemma 2.1 it is easy to see that for any fixed $x > 0$, $\{Y_{xi}, i \geqslant 1\}$ and $\{a_{ni}Y_{xi}, i \geqslant 1\}$ are AANA for all $n \geqslant 1$. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}E\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}_+^q \\
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_0^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_i\right| > \varepsilon n^{\alpha} + x^{1/q}\right)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_0^{n^{\alpha q}} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_i\right| > \varepsilon n^{\alpha} + x^{1/q}\right)\mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_i\right| > \varepsilon n^{\alpha} + x^{1/q}\right)\mathrm{d}x
\end{aligned}$$

$$\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right)$$
$$+\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > x^{1/q}\right)\,\mathrm{d}x$$
$$=: I_1 + I_2. \tag{2.9}$$

To prove (2.1) it suffices to prove $I_1 < \infty$ and $I_2 < \infty$.

From Theorem A we have $I_1 < \infty$.

$$I_2 = \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > x^{1/q}\right)\,\mathrm{d}x$$
$$\leqslant \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}Y'_{xi}\right| > x^{1/q}/2\right)\,\mathrm{d}x$$
$$+\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}Y_{xi}\right| > x^{1/q}/2\right)\,\mathrm{d}x$$
$$=: I_{21} + I_{22}. \tag{2.10}$$

No matter $1 < p < 2$ or $p \geqslant 2$, $r \geqslant 2$ holds constantly. By C_r inequality,

$$\sum_{i=1}^{n} a_{ni} = \left[\left(\sum_{i=1}^{n} a_{ni}\right)^{r}\right]^{1/r} \leqslant \left(n^{r-1}\sum_{i=1}^{n} a_{ni}^{r}\right)^{1/r} \ll n.$$

For the first part of (2.10) we get

$$I_{21} \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} x^{-1/q} E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}Y'_{xi}\right|\right)\,\mathrm{d}x$$
$$\leqslant \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} x^{-1/q}\sum_{i=1}^{n} a_{ni}E|X_i|I(|X_i| > x^{1/q})\,\mathrm{d}x$$
$$\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\int_{n^{\alpha q}}^{\infty} x^{-1/q}E|X_1|I(|X_1| > x^{1/q})\,\mathrm{d}x$$
$$= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\sum_{j=n}^{\infty}\int_{j^{\alpha q}}^{(j+1)^{\alpha q}} x^{-1/q}E|X_1|I(|X_1| > x^{1/q})\,\mathrm{d}x$$
$$\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\sum_{j=n}^{\infty} j^{\alpha q-\alpha-1}E|X_1|I(|X_1| > j^{\alpha})$$
$$= \sum_{j=1}^{\infty} j^{\alpha q-\alpha-1}E|X_1|I(|X_1| > j^{\alpha})\sum_{n=1}^{j} n^{\alpha p-\alpha q-1} \tag{2.11}$$

If $q \neq p$, then

$$
\begin{aligned}
I_{21} &\ll \sum_{j=1}^{\infty} j^{\alpha p-\alpha-1} E|X_1| I(|X_1| > j^{\alpha}) \\
&= \sum_{j=1}^{\infty} j^{\alpha p-\alpha-1} \sum_{l=j}^{\infty} E|X_1| I(l^{\alpha} < |X_1| \leqslant (l+1)^{\alpha}) \\
&= \sum_{l=1}^{\infty} E|X_1| I(l^{\alpha} < |X_1| \leqslant (l+1)^{\alpha}) \sum_{j=1}^{l} j^{\alpha p-\alpha-1} \\
&\ll \sum_{l=1}^{\infty} l^{\alpha p-\alpha} E|X_1| I(l^{\alpha} < |X_1| \leqslant (l+1)^{\alpha}) (\text{since } p > 1) \\
&\leqslant \sum_{l=1}^{\infty} E|X_1|^p I(l^{\alpha} < |X_1| \leqslant (l+1)^{\alpha}) \\
&= E|X_1|^p < \infty. \qquad (2.12)
\end{aligned}
$$

If $q = p$, then

$$
\begin{aligned}
I_{21} &\ll \sum_{j=1}^{\infty} j^{\alpha p-\alpha-1} \log j E|X_1| I(|X_1| > j^{\alpha}) \\
&= \sum_{j=1}^{\infty} j^{\alpha p-\alpha-1} \log j \sum_{l=j}^{\infty} E|X_1| I(l^{\alpha} < |X_1| \leqslant (l+1)^{\alpha}) \\
&= \sum_{l=1}^{\infty} E|X_1| I(l^{\alpha} < |X_1| \leqslant (l+1)^{\alpha}) \sum_{j=1}^{l} j^{\alpha p-\alpha-1} \log j \\
&\ll \sum_{l=1}^{\infty} l^{\alpha p-\alpha} \log l E|X_1| I(l^{\alpha} < |X_1| \leqslant (l+1)^{\alpha}) \ (\text{since} p > 1) \\
&\ll \sum_{l=1}^{\infty} E|X_1|^p \log(1+|X_1|) I(l^{\alpha} < |X_1| \leqslant (l+1)^{\alpha}) \\
&= E|X_1|^p \log(1+|X_1|) < \infty. \qquad (2.13)
\end{aligned}
$$

Thus $I_{21} < \infty$ holds. It is easy to see that

$$
\max_{1 \leqslant k \leqslant n} x^{-1/q} \left| \sum_{i=1}^{k} a_{ni} E Y_{xi} \right| \to 0 \quad (\text{as } x \to \infty). \qquad (2.14)
$$

So by (2.14) and (2.6) of Lemma 2.2 we have

$$
\begin{aligned}
I_{22} &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} a_{ni} Y_{xi} \right| > x^{1/q}/2 \right) \mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} a_{ni} (Y_{xi} - EY_{xi}) \right| > x^{1/q}/4 \right) \mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} x^{-r/q} E\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}(Y_{xi}-EY_{xi})\right|^r\right\} \mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} x^{-r/q}\left[1+C\left(\sum_{i=1}^{n} q^{\tilde{r}}(i)\right)^{r-1}\right]\sum_{i=1}^{n} E|a_{ni}Y_{xi}|^r \,\mathrm{d}x \\
&\quad +\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} x^{-r/q}\left(\sum_{i=1}^{n} E(a_{ni}Y_{xi})^2\right)^{r/2} \mathrm{d}x \\
&=: I_{221}+I_{222}. \qquad (2.15)
\end{aligned}$$

Since $\sum_{n=1}^{\infty} q^{\tilde{r}}(n) < \infty$ and $\sum_{i=1}^{n} a_{ni}^r = O(n)$, we get

$$\begin{aligned}
I_{221} &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\int_{n^{\alpha q}}^{\infty} x^{-r/q}\sum_{i=1}^{n} a_{ni}^r E|Y_{xi}|^r \,\mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\int_{n^{\alpha q}}^{\infty} x^{-r/q} E|X_1|^r I(|X_1|\leqslant x^{1/q}) \,\mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\sum_{j=n}^{\infty}\int_{j^{\alpha q}}^{(j+1)^{\alpha q}} x^{-r/q} E|X_1|^r I(|X_1|\leqslant x^{1/q}) \,\mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\sum_{j=n}^{\infty} j^{-\alpha r+\alpha q-1} E|X_1|^r I(|X_1|\leqslant (j+1)^{\alpha}) \\
&= \sum_{j=1}^{\infty} j^{-\alpha r+\alpha q-1} E|X_1|^r I(|X_1|\leqslant (j+1)^{\alpha})\sum_{n=1}^{j} n^{\alpha p-\alpha q-1} \qquad (2.16)
\end{aligned}$$

If $q \neq p$, then

$$\begin{aligned}
I_{221} &\ll \sum_{j=1}^{\infty} j^{\alpha p-\alpha r-1} E|X_1|^r I(|X_1|\leqslant (j+1)^{\alpha}) \\
&= \sum_{j=1}^{\infty} j^{\alpha p-\alpha r-1}\sum_{l=1}^{j+1} E|X_1|^r I((l-1)^{\alpha} < |X_1|\leqslant l^{\alpha}) \\
&= \sum_{l=1}^{\infty} E|X_1|^r I((l-1)^{\alpha} < |X_1|\leqslant l^{\alpha})\sum_{j=l-1}^{\infty} j^{\alpha p-\alpha r-1} \\
&\ll \sum_{l=1}^{\infty} l^{\alpha p-\alpha r} E|X_1|^r I((l-1)^{\alpha} < |X_1|\leqslant l^{\alpha}) \text{ (since } r > p) \\
&\leqslant \sum_{l=1}^{\infty} E|X_1|^p I((l-1)^{\alpha} < |X_1|\leqslant l^{\alpha}) \\
&= E|X_1|^p < \infty. \qquad (2.17)
\end{aligned}$$

If $q = p$, then

$$\begin{aligned}
I_{221} &\ll \sum_{j=1}^{\infty} j^{\alpha p-\alpha r-1} \log j E|X_1|^r I(|X_1| \leqslant (j+1)^{\alpha}) \\
&= \sum_{j=1}^{\infty} j^{\alpha p-\alpha r-1} \log j \sum_{l=1}^{j+1} E|X_1|^r I((l-1)^{\alpha} < |X_1| \leqslant l^{\alpha}) \\
&\leqslant \sum_{l=1}^{\infty} E|X_1|^r I((l-1)^{\alpha} < |X_1| \leqslant l^{\alpha}) \sum_{j=l-1}^{\infty} j^{\alpha p-\alpha r-1} \log(1+j) \\
&\ll \sum_{l=1}^{\infty} l^{\alpha p-\alpha r} \log(1+l) E|X_1|^r I((l-1)^{\alpha} < |X_1| \leqslant l^{\alpha}) \\
&\leqslant \sum_{l=1}^{\infty} E|X_1|^p \log(1+|X_1|) I((l-1)^{\alpha} < |X_1| \leqslant l^{\alpha}) \\
&= E|X_1|^p \log(1+|X_1|) < \infty. \qquad (2.18)
\end{aligned}$$

By C_r inequality and $r \geqslant 2$,

$$\left(\sum_{i=1}^{n} a_{ni}^2\right)^{r/2} \leqslant n^{r/2-1} \sum_{i=1}^{n} a_{ni}^r \ll n^{r/2}.$$

The second part of (2.15) can be dominated by

$$\begin{aligned}
I_{222} &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \int_{n^{\alpha q}}^{\infty} x^{-r/q} \left(\sum_{i=1}^{n} a_{ni}^2 E X_1^2 I(|X_1| \leqslant x^{1/q})\right)^{r/2} \mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q+r/2-2} \int_{n^{\alpha q}}^{\infty} x^{-r/q} \left(E X_1^2 I(|X_1| \leqslant x^{1/q})\right)^{r/2} \mathrm{d}x. \qquad (2.19)
\end{aligned}$$

We consider the following two situations.

(i) If $p \geqslant 2$, then $E X_1^2 I(|X_1| \leqslant x^{1/q}) \leqslant E X_1^2 < \infty$. Since $r > (\alpha p - 1)/(\alpha - 1/2)$, so for $q \neq r$,

$$\begin{aligned}
I_{222} &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q+r/2-2} \int_{n^{\alpha q}}^{\infty} x^{-r/q} \,\mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r+r/2-2} < \infty, \qquad (2.20)
\end{aligned}$$

and for $q = r$,

$$I_{222} \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r+r/2-2} \log n < \infty. \qquad (2.21)$$

(ii) If $1 < p < 2$, then $r = 2$.

$$\begin{aligned} I_{222} &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} \int_{n^{\alpha q}}^{\infty} x^{-2/q} E X_1^2 I(|X_1| \leqslant x^{1/q})\, \mathrm{d}x \\ &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} \sum_{j=n}^{\infty} \int_{j^{\alpha q}}^{(j+1)^{\alpha q}} x^{-2/q} E X_1^2 I(|X_1| \leqslant x^{1/q})\, \mathrm{d}x \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} \sum_{j=n}^{\infty} j^{\alpha q-2\alpha-1} E X_1^2 I(|X_1| \leqslant (j+1)^{\alpha}). \end{aligned} \tag{2.22}$$

Similar to (2.16), (2.17) and (2.18) we have $I_{222} < \infty$. So $I_{22} < \infty$ and $I_2 < \infty$. The first part of Theorem 2.1 is proved.

As for the second part of Theorem 2.1, one can get

$$\begin{aligned} &\sum_{n=1} \infty n^{\alpha p-2} \int_0^{\infty} P\left(\sup_{k\geqslant n}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon + x^{1/q}\right) \mathrm{d}x \\ &= \sum_{j=1}^{\infty} \sum_{n=2^{j-1}}^{2^j-1} n^{\alpha p-2} \int_0^{\infty} P\left(\sup_{k\geqslant n}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon + x^{1/q}\right) \mathrm{d}x \\ &\ll \sum_{j=1}^{\infty} \int_0^{\infty} P\left(\sup_{k\geqslant 2^{j-1}}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon + x^{1/q}\right) \mathrm{d}x \sum_{n=2^{j-1}}^{2^j-1} 2^{j(\alpha p-2)} \\ &\ll \sum_{j=1}^{\infty} 2^{j(\alpha p-1)} \int_0^{\infty} P\left(\sup_{k\geqslant 2^{j-1}}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon + x^{1/q}\right) \mathrm{d}x \\ &= \sum_{j=1}^{\infty} 2^{j(\alpha p-1)} \int_0^{\infty} P\left(\sup_{l\geqslant j}\max_{2^{l-1}\leqslant k<2^l}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon + x^{1/q}\right) \mathrm{d}x \\ &\leqslant \sum_{j=1}^{\infty} 2^{j(\alpha p-1)} \sum_{l=j}^{\infty} \int_0^{\infty} P\left(\max_{2^{l-1}\leqslant k<2^l}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon + x^{1/q}\right) \mathrm{d}x \\ &= \sum_{l=1}^{\infty} \int_0^{\infty} P\left(\max_{2^{l-1}\leqslant k<2^l}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon + x^{1/q}\right) \mathrm{d}x \sum_{j=1}^{l} 2^{j(\alpha p-1)} \\ &\ll \sum_{l=1}^{\infty} 2^{l(\alpha p-1)} \int_0^{\infty} P\left(\max_{2^{l-1}\leqslant k<2^l}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > (\varepsilon + x^{1/q})2^{(l-1)\alpha}\right) \mathrm{d}x \\ &\ll \sum_{l=1}^{\infty} 2^{l(\alpha p-\alpha q-1)} \int_0^{\infty} P\left(\max_{2^{l-1}\leqslant k<2^l}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > 2^{(l-1)\alpha}\varepsilon + y^{1/q}\right) \mathrm{d}y \\ &\qquad\qquad (\text{letting } y = 2^{(l-1)\alpha q}x) \\ &\leqslant \sum_{l=1}^{\infty} 2^{l(\alpha p-\alpha q-1)} \int_0^{\infty} P\left(\max_{1\leqslant k<2^l}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > 2^{(l-1)\alpha}\varepsilon + y^{1/q}\right) \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&\ll \sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} n^{\alpha p-\alpha q-2} \int_0^{\infty} \\
&\quad \times P\left(\max_{1\leqslant k<2^l}\left|\sum_{i=1}^k a_{ni} X_i\right| > 2^{(l+1)\alpha}\cdot 2^{-2\alpha}\varepsilon + y^{1/q}\right) \mathrm{d}y \\
&\leqslant \sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} n^{\alpha p-\alpha q-2} \int_0^{\infty} P\left(\max_{1\leqslant k<n}\left|\sum_{i=1}^k a_{ni} X_i\right| > \varepsilon_0 n^{\alpha} + y^{1/q}\right) \mathrm{d}y \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(\text{letting } \varepsilon_0 = 2^{-2\alpha}\varepsilon\right) \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \int_0^{\infty} P\left(\max_{1\leqslant k<n}\left|\sum_{i=1}^k a_{ni} X_i\right| > \varepsilon_0 n^{\alpha} + y^{1/q}\right) \mathrm{d}y \\
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k a_{ni} X_i\right| - \varepsilon_0 n^{\alpha}\right\}_+^q < \infty. \qquad (2.23)
\end{aligned}
$$

The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. We agree on Y_{xi} and a_{ni} such as the proof of Theorem 2.1. Similarly for the proof of (2.1), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-q/p} E\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k a_{ni} X_i\right| - \varepsilon n^{1/p}\right\}_+^q \\
&= \sum_{n=1}^{\infty} n^{-q/p} \int_0^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k a_{ni} X_i\right| > \varepsilon n^{1/p} + x^{1/q}\right) \mathrm{d}x \\
&\leqslant \sum_{n=1}^{\infty} n^{-q/p} \int_0^{n^{q/p}} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k a_{ni} X_i\right| > \varepsilon n^{1/p}\right) \mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k a_{ni} X_i\right| > x^{1/q}\right) \mathrm{d}x \\
&= \sum_{n=1}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k a_{ni} X_i\right| > \varepsilon n^{1/p}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k a_{ni} X_i\right| > x^{1/q}\right) \mathrm{d}x \\
&=: I_1' + I_2' \qquad (2.24)
\end{aligned}
$$

By Theorem B, we know $I_1' < \infty$. From $\sup_{n\geqslant 1}\sum_{i=1}^n a_{ni}^2 < \infty$ and C_r inequality we get $\sum_{i=1}^n a_{ni} \ll n^{1/2}$. Since $EX_1 = 0$, it is easy to see that

$$
\begin{aligned}
\max_{1\leqslant k\leqslant n} x^{-1/q}\left|\sum_{i=1}^k a_{ni} EY_{xi}\right| &= \max_{1\leqslant k\leqslant n} x^{-1/q}\left|\sum_{i=1}^k a_{ni} EY'_{xi}\right| \\
&\to 0 \quad (\text{as } x \to \infty). \qquad (2.25)
\end{aligned}
$$

So the second part of (2.24) can be dominated by

$$
\begin{aligned}
I_2' \leqslant\ & \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}(Y_{xi}' - EY_{ni}')\right| > x^{1/q}/4\right) \mathrm{d}x \\
& + \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}(Y_{xi} - EY_{xi})\right| > x^{1/q}/4\right) \mathrm{d}x \\
\ll\ & \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} x^{-2/q} E \max_{1\leqslant k\leqslant n}\left(\sum_{i=1}^{k} a_{ni}(Y_{xi}' - Y_{ni}')\right)^2 \mathrm{d}x \\
& + \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} x^{-2/q} E \max_{1\leqslant k\leqslant n}\left(\sum_{i=1}^{k} a_{ni}(Y_{xi} - Y_{ni})\right)^2 \mathrm{d}x \\
\ll\ & \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} x^{-2/q} \sum_{i=1}^{n} a_{ni}^2 EX_1^2 I(|X_1| > x^{1/q})\, \mathrm{d}x \\
& + \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} x^{-2/q} \sum_{i=1}^{n} a_{ni}^2 EX_1^2 I(|X_1| \leqslant x^{1/q})\, \mathrm{d}x \quad \text{(using (2.7))} \\
\ll\ & \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} x^{-2/q} EX_1^2 I(|X_1| > x^{1/q})\, \mathrm{d}x \\
& + \sum_{n=1}^{\infty} n^{-q/p} \int_{n^{q/p}}^{\infty} x^{-2/q} EX_1^2 I(|X_1| \leqslant x^{1/q})\, \mathrm{d}x \\
=\ & \sum_{n=1}^{\infty} n^{-q/p} \sum_{j=n}^{\infty} \int_{j^{q/p}}^{(j+1)^{q/p}} x^{-2/q} EX_1^2 I(|X_1| > x^{1/p})\, \mathrm{d}x \\
& + \sum_{n=1}^{\infty} n^{-q/p} \sum_{j=n}^{\infty} \int_{j^{q/p}}^{(j+1)^{q/p}} x^{-2/q} EX_1^2 I(|X_1| \leqslant x^{1/p})\, \mathrm{d}x \\
\ll\ & \sum_{n=1}^{\infty} n^{-q/p} \sum_{j=n}^{\infty} j^{q/p-2/p-1} EX_1^2 I(|X_1| > j^{1/p}) \\
& + \sum_{n=1}^{\infty} n^{-q/p} \sum_{j=n}^{\infty} j^{q/p-2/p-1} EX_1^2 I(|X_1| \leqslant (j+1)^{1/p}) \\
=\ & \sum_{j=1}^{\infty} j^{q/p-2/p-1} EX_1^2 I(|X_1| > j^{1/p}) \sum_{n=1}^{j} n^{-q/p} \\
& + \sum_{j=1}^{\infty} j^{q/p-2/p-1} EX_1^2 I(|X_1| \leqslant (j+1)^{1/p}) \sum_{n=1}^{j} n^{-q/p}. \qquad (2.26)
\end{aligned}
$$

We consider the following two situations.

(i) If $q \neq p$, then

$$
\begin{aligned}
I_2' &\ll \sum_{j=1}^{\infty} j^{-2/p} E X_1^2 I(|X_1| > j^{1/p}) \\
&\quad + \sum_{j=1}^{\infty} j^{-2/p} E X_1^2 I(|X_1| \leqslant (j+1)^{1/p}) \\
&\ll \sum_{j=1}^{\infty} j^{-2/p} \sum_{j=l}^{\infty} E X_1^2 I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p}) \\
&\quad + \sum_{j=1}^{\infty} j^{-2/p} \sum_{l=1}^{j+1} E X_1^2 I((l-1)^{1/p} < |X_1| \leqslant l^{1/p}) \\
&= \sum_{l=1}^{\infty} E X_1^2 I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p}) \sum_{j=1}^{l} j^{-2/p} \\
&\quad + \sum_{l=1}^{\infty} E X_1^2 I((l-1)^{1/p} < |X_1| \leqslant l^{1/p}) \sum_{j=l-1}^{\infty} j^{-2/p} \\
&\ll \sum_{l=1}^{\infty} l^{-2/p+1} E X_1^2 I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p}) \\
&\quad + \sum_{l=1}^{\infty} l^{-2/p+1} E X_1^2 I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p}) \\
&\ll \sum_{l=1}^{\infty} E|X_1|^p I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p}) \\
&\leqslant E|X_1|^p < \infty. \qquad (2.27)
\end{aligned}
$$

(ii) If $q = p$, then

$$
\begin{aligned}
I_2' &\ll \sum_{j=1}^{\infty} j^{-2/p} \log j \sum_{l=j}^{\infty} E X_1^2 I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p}) \\
&\quad + \sum_{j=1}^{\infty} j^{-2/p} \log j \sum_{l=1}^{j+1} E X_1^2 I((l-1)^{1/p} < |X_1| \leqslant l^{1/p}) \\
&\leqslant \sum_{l=1}^{\infty} E X_1^2 I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p}) \sum_{j=1}^{l} j^{-2/p} \log j \\
&\quad + \sum_{l=1}^{\infty} E X_1^2 I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p}) \sum_{j=l-1}^{\infty} j^{-2/p} \log(1+j) \\
&\ll \sum_{l=1}^{\infty} l^{-2/q+1} \log(1+l) E X_1^2 I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p})
\end{aligned}
$$

$$\ll \sum_{l=1}^{\infty} E|X_1|^p \log(1+|X_1|) I(l^{1/p} < |X_1| \leqslant (l+1)^{1/p})$$
$$\leqslant E|X_1|^p \log(1+|X_1|) < \infty. \tag{2.28}$$

As for (2.4), we can prove it similar to (2.22). We omit its details. The proof of Theorem 2.2 is completed. □

Acknowledgments

The author is very grateful to the anonymous referee for helpful suggestions on the original manuscript. This work is supported by the Projects of Science and Technology Research of Chongqing City Education Committee.

References

[1] An J, Complete convergence of weighted sums for sequences of AANA radom variables (submitted)

[2] Cai G H, Complete convergence for weighted sums of sequences of AANA random variables, *Glasgow Math. J.* **17(1)** (2004) 165–181

[3] Chen P Y and Wang D C, Complete moment convergence for sequence of identically distributed φ-mixing random variables, *Acta Math. Sin. Engl. Ser.* **26(4)** (2010) 679–690

[4] Chow Y S, On the rate of moment complete convergence of sample sums and extremes, *Bull. Inst. Math. Acad. Sin.* **16** (1988) 177–201

[5] Chow Y S and Teicher H, Probability theory: independent, interchangeability, Martingales, third edition (1997) (New York: Springer)

[6] Chandra T K and Ghosal S, Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables, *Acta Math. Hung.* **71(4)** (1996a) 327–336

[7] Chandra T K and Ghosal S, The strong law of large numbers for weighted averages under dependence assumptions, *J. Theor. Probab.* **9(3)** (1996b) 797–809

[8] Ghosal S and Chandra T K, Complete convergence of martingale arrays, *J. Theor. Probab.* **11** (1998) 621–631

[9] Gut A, Probability theory: a graduate course (2005) (New York: Springer)

[10] Kim T S, Ko M H and Lee I H, On the strong laws for asymptotically almost negatively associated random variables, *Rocky Mt. J. Math.* **34(3)** (2004) 979–989

[11] Li Y X and Zhang L X, Complete moment convergence of moving average under dependent assumptions, *Stat. Probab. Lett.* **70** (2004) 191–197

[12] Li D L, Rao B, Jiang T F and Wang X C, Complete convergence and almost sure convergence of weighted sums of random variables, *J. Theor. Probab.* **8** (1995) 49–76

[13] Petrov V V, Limit theorems of probability: sequences of independent random variables (1995) (Oxford: Clarendon Press)

[14] Wang X J, Hu S H and Yang W Z, Complete convergence for arrays of rowwise asymptotically almost negatively associated random variables, *Discret. Dyn. Nat. Soc.*, Article ID 717126 (2011) 11 page. doi:10.1155/2012/717126

[15] Yang W Z, Wang X J, Ling N X and Hu S H, On complete convergence of moving average process for AANA sequence, *Discret. Dyn. Nat. Soc.*, Article ID 86391 (2012) 24 pages. doi:10.1155/2012/863931

[16] Yuan D M and An J, Rosenthal type inqualities for asymptotically almost negatively associated random variables and applications, *Sci. China Ser. A* **52(9)** (2009) 1887–1904

[17] Zhou X C and Lin J G, Complete q-moment convergence of moving average process under φ-mixing assumption, *J. Math. Res. Expo.* **31(4)** (2011) 687–697