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# A note on automorphisms of the sphere complex

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**Abstract.** In this note, we shall give another proof of a theorem of Aramayona and Souto, namely the group of simplicial automorphisms of the sphere complex $\mathbb{S}(M)$ associated to the manifold $M = \sharp_n S^2 \times S^1$ is isomorphic to the group $\mathrm{Out}(F_n)$ of outer automorphisms of the free group $F_n$ of rank $n \geq 3$.

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## 1. Introduction

In [1], Aramayona and Souto have shown that the group $\mathrm{Aut}(\mathbb{S}(M))$ of simplicial automorphisms of the sphere complex $\mathbb{S}(M)$ associated to the manifold $M = \sharp_n S^2 \times S^1$ is isomorphic to the group $\mathrm{Out}(F_n)$ of outer automorphisms of the free group $F_n$ of rank $n \geq 3$. The idea of the proof is as follows: the sphere complex contains an embedded copy of the spine $K_n$ of the reduced outer pace $cv_n$ [6]. In [1], it is shown that $K_n$ is invariant under $\mathrm{Aut}(\mathbb{S}(M))$ and that the restriction homomorphism $\mathrm{Aut}(\mathbb{S}(M))$ to $\mathrm{Aut}(K_n)$ is injective. The result follows from a result of Bridson–Vogtmann [2] which says that $\mathrm{Out}(F_n)$ is the full automorphism group of $K_n$.

In [11], it is shown that the group $\mathrm{Aut}(NS(M))$ of simplicial automorphisms of the complex $NS(M)$ of non-separating embedded spheres associated to $M$ is isomorphic to the group $\mathrm{Out}(F_n)$. As in [1], it is shown that the spine $K_n$ is invariant under the group $\mathrm{Aut}(NS(M))$ by showing that given any simplicial automorphism $\phi$ of $NS(M)$, it maps simplices in $NS(M)$ corresponding to *reduced* sphere systems in $M$ to simplices in $NS(M)$ corresponding to reduced sphere systems in $M$. The idea is to look at the links of such simplices in $NS(M)$. The key observation is that the link of a simplex in $NS(M)$ corresponding to a reduced sphere system in $M$ is spanned by finitely many vertices of $NS(M)$ and the link of an $(n-1)$-simplex which corresponds to a non-reduced system of non-separating embedded spheres in $M$ is spanned by infinitely many vertices of $NS(M)$. A *simple simplex* in $NS(M)$ is a simplex which corresponds to a simple sphere system in $M$. Then, one can see that every simplicial automorphism $\phi$ of $NS(M)$ maps a simple simplex in $NS(M)$ to a simple simplex in $NS(M)$. From this it follows that the reduced outer space $cv_n$ is invariant under $\phi$ and then the rest of the proof follows as in [1].

In this note, we observe that the same idea (in [11]) of looking at the links of vertices of the sphere complex works for automorphisms of the sphere complex. We shall see a characterization of a separating as well as a non-separating embedded sphere in $M$ in terms of links of their isotopy classes as vertices in $\mathbb{S}(M)$. We observe that a simplicial automorphism $\phi$ of $\mathbb{S}(M)$ preserves the combinatorics of the vertices of $\mathbb{S}(M)$. Similar observations are made for automorphisms of disk complex in [8]. This will imply that the reduced outer space $cv_n$ is invariant under $\phi$. From this it follows that the group of simplicial automorphisms $\mathrm{Aut}(\mathbb{S}(M))$ is isomorphic to the group $\mathrm{Out}(F_n)$ as in [1].

## 2. Preliminaries

### 2.1 *The model* 3-*manifold*

Consider the 3-manifold $M = \sharp_n S^2 \times S^1$, i.e., the connected sum of $n$ copies of $S^2 \times S^1$. A description of $M$ can be given as follows: Consider the 3-sphere $S^3$ and let $A_i, B_i, 1 \leq i \leq n$, be a collection of $2n$ disjointly embedded 3-balls in $S^3$. Let $S_i$ (respectively, $T_i$) denote the boundary of $A_i$ (respectively, $B_i$). Let $P$ be the complement of the union of the interiors of all the balls $A_i$ and $B_i$. Then, $M$ is obtained from $P$ by glueing together $S_i$ and $T_i$ with an orientation reversing diffeomorphism $\varphi_i$ for each $i, 1 \leq i \leq n$. The image of $S_i$ (hence $T_i$) in $M$ will be denoted by $\Sigma_i^0$. Clearly each $\Sigma_i^0$ is a non-separating embedded sphere in $M$, i.e., its complement in $M$ is connected.

Fix a base point $x_0$ away from $\Sigma_i^0$. For each $1 \leq i \leq n$, consider the element $\alpha_i \in \pi_1(M)$ represented by a closed path $\gamma_i$ in $M$ starting from $x_0$, going to $A_i$, piercing $\Sigma_i^0$ and returning to the base point from $B_i$. We choose this closed path $\gamma_i$ such that it does not intersect any $\Sigma_j^0$, $j \neq i$. Then, as the complement of $\Sigma_i^0$ in $M$ is simply-connected, the collection $\{\alpha_1, \ldots, \alpha_n\}$ forms a free basis of $G = \pi_1(M)$. So, we have $G = \langle \alpha_1, \ldots, \alpha_n \rangle$. Any directed closed path in $M$ hitting the $\Sigma_i^0$ transversely represents a word in $\{\alpha_1, \ldots, \alpha_n\}$ by the way it pierces each $\Sigma_i^0$ and the order in which it does so. Without a base point chosen, such a closed path represents a conjugacy class or equivalently the cyclic word in $\pi_1(M)$. We identify the free group $\mathbb{F}_n$ of rank $n$ with $\pi_1(M)$ up to conjugacy of the base point.

A smooth embedded 2-sphere $S$ in $M$ is said to be *essential* if it does not bound a 3-ball in $M$. We shall always consider essential smooth embedded 2-spheres in $M$ throughout this paper. So by an embedded sphere, we mean an essential embedded 2-sphere in $M$. Two embedded 2-spheres $S$ and $S'$ in $M$ are *parallel* if they are isotopic, i.e., they bound a manifold of the form $S^2 \times (0, 1)$. By Laudenbach's work (Theorem II of [10]), the embedded spheres $S$ and $S'$ in $M$ are parallel if and only if they are homotopic to each other.

*A system of* 2-*spheres* in $M$ is defined to be a collection of disjointly embedded essential 2-spheres $S_i \subset M$ such that no two spheres in this collection are parallel. It is easy to see that a system of 2-spheres in $M$ is a finite collection of embedded spheres in $M$. A system of spheres in $M$ is *maximal* if it is not properly contained in another system of spheres in $M$.

By $M_{n,r}$ we shall denote an $r$-punctured $\sharp_n S^2 \times S^1$, i.e., the manifold $M$ with interiors of $r$ disjointly embedded 3-balls removed.

### 2.2 *Sphere complex*

We shall recall the following definition:

DEFINITION 2.1

The *sphere complex* $\mathbb{S}(M)$ associated to $M$ is the simplicial complex whose vertices are isotopy classes of essential embedded spheres in $M$. A finite collection of isotopy classes of embedded spheres in $M$ is deemed to span a simplex in $\mathbb{S}(M)$ if they can be realized disjointly (up to isotopy) in $M$.

The maximal simplices of $\mathbb{S}(M)$ all have the same dimension, namely $3n - 4$, as one sees by Euler characteristic considerations using the fact that the complementary regions of a maximal system of 2-spheres are all 3-punctured 3-spheres. In [6], Hatcher has proved that the sphere complex $\mathbb{S}(M)$ is contractible.

For $M_{n,r}$, we shall define the complex $\mathbb{S}(M_{n,r})$ as a simplicial complex whose vertices are isotopy classes of essential embedded spheres in $M_{n,r}$ which are not parallel to any boundary component and simplices correspond to isotopy classes of sphere system in $M_{n,r}$ with no component parallel to any boundary component. By Theorem 3.3 of [4], one can see that these complexes are flag, i.e., in dimensions two and higher, a simplex is present if its face are. Also for $n \geq 2$, diameter of the 1-skeleton of each of these complexes is infinite [7].

In [11], the complex $NS(M)$ of non-separating embedded spheres in $M$ is considered. This is a subcomplex of the sphere complex spanned by the vertices which are isotopy classes of non-separating embedded spheres in $M$. One can see that the inclusion map of the 1-skeleton of $NS(M)$ into the sphere graph, i.e. the 1-skeleton of $\mathbb{S}(M)$ is an isometric embedding. Moreover, every vertex in the sphere graph corresponding to an isotopy class of a separating embedded sphere in $M$ is at a distance one from a vertex in the sphere graph corresponding to an isotopy class of a non-separating embedded sphere in $M$, i.e., it is at a distance one from $NS(M)$. So, the 1-skeleton of $NS(M)$ is clearly quasi-isometric to the sphere graph. As the sphere graph is Gromov hyperbolic [5], so is the 1-skeleton of $NS(M)$.

### 2.3 *The mapping class group of M and* Out($\mathbb{F}_n$)

DEFINITION 2.2

The *mapping class group* $\mathcal{M}\text{ap}(M)$ of $M$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $M$.

It is due to Laudenbach (see Theorem 4.3, part 2 of [9]) that we can replace diffeomorphisms by homeomorphisms in the definition of the mapping class group of $M$. As we have identified $\mathbb{F}_n$ with $\pi_1(M)$, we have a natural homomorphism

$$\Phi : \mathcal{M}\text{ap}(M) \to \text{Out}(\mathbb{F}_n),$$

where $\text{Out}(\mathbb{F}_n)$ is the group of outer automorphisms of $\mathbb{F}_n$. By Laudenbach's work (Theorem 4.3 of [9]), the above homomorphism $\Phi$ is surjective and has finite kernel generated by Dehn-twists along essential embedded 2-spheres in $M$. A Dehn-twist along an embedded $S^2$ in $M$ is a diffeomorphism supported in a tubular neighborhood $S^2 \times [0, 1]$ taking the slice $S^2 \times \{t\}$ to itself by a rotation through angle $2\pi t$ with respect to some chosen axis of $S^2$. Laudenbach showed that kernel of $\Phi$ is isomorphic to the product of $n$ copies of $\mathbb{Z}_2$.

The mapping class group $\mathcal{M}\text{ap}(M)$ of $M$ acts simplicially on the sphere complex $\mathbb{S}(M)$. This yields a homomorphism

$$\Phi' : \mathcal{M}\text{ap}(M) \to \text{Aut}(\mathbb{S}(M)),$$

where $\mathrm{Aut}(\mathbb{S}(M))$ is the group of simplicial automorphisms of the sphere complex $\mathbb{S}(M)$. It follows from the work of Laudenbach [9] that kernel of $\Phi$ and $\Phi'$ are equal. In particular, the action of $\mathcal{M}\mathrm{ap}(M)$ on the sphere complex $\mathbb{S}(M)$ induces a simplicial action of $\mathrm{Out}(\mathbb{F}_n)$ on the sphere complex. This yields a homomorphism

$$\Phi'' : \mathrm{Out}(\mathbb{F}_n) \to \mathrm{Aut}(\mathbb{S}(M)).$$

In [1], Aramayona and Souto have shown that this homomorphism is an isomorphism.

**Theorem 2.3.** *The group of simplicial automorphisms of the sphere complex* $\mathbb{S}(M)$ *is isomorphic to the group* $\mathrm{Out}(F_n)$ *for* $n \geq 3$.

We shall see another proof of the above theorem in this note. Throughout this note, we shall consider $n \geq 3$.

## 3. Embedded spheres and simplicial automorphisms of $\mathbb{S}(M)$

In this section, we shall show that a simplicial automorphisms $\phi$ of $\mathbb{S}(M)$ preserves the combinatorics of $\mathbb{S}(M)$.

We recall the following definitions:

DEFINITION 3.1

The *link* of a simplex $\sigma$ in a simplicial complex $\mathbb{K}$ is a subcomplex of $\mathbb{K}$ consisting of the simplices $\tau$ that are disjoint from $\sigma$ and such that both $\sigma$ and $\tau$ are faces of some higher-dimensional simplex in $\mathbb{K}$.

So if $\sigma$ is a simplex in $\mathbb{S}(M)$, then $\sigma$ corresponds to a system $S = \cup_i S_i$ of embedded spheres in $M$. The $\mathrm{link}(\sigma)$ is the subcomplex of $\mathbb{S}(M)$ spanned by the isotopy classes $[T]$ of embedded spheres $T$ in $M$ such that all $T$'s are disjoint from all and distinct from all $S_i$ up to isotopy. In particular, for a vertex $v$ in $\mathbb{S}(M)$ corresponding to an isotopy class of an embedded sphere $S$ in $M$, the $\mathrm{link}(v)$ is spanned by all the isotopy classes of embedded spheres which are distinct and disjoint from $S$.

DEFINITION 3.2

If $\mathbb{K}$ and $\mathbb{L}$ are non-empty simplicial complexes with disjoint vertex sets, the *join* $\mathbb{K} \vee \mathbb{L}$ is the complex which is the union of $\mathbb{K} \cup \mathbb{L}$ with the collection of all simplices $\sigma \vee \tau$, where $\sigma$ and $\tau$ are simplices in $\mathbb{K}$ and $\mathbb{L}$ respectively and $\sigma \vee \tau$ denote a simplex with $\sigma$ and $\tau$ as faces of $\sigma \vee \tau$ and the vertex set of $\sigma \vee \tau$ is the union of the vertex sets of $\sigma$ and $\tau$.

Note that the diameter of the 1-skeleton of a join of simplicial complexes is either 1 or 2. From this, it follows that the complex $\mathbb{S}(M_{n,r})$ is not a join as the diameter of the 1-skeleton of $\mathbb{S}(M_{n,r})$ is infinite for $n \geq 2$. If we cut $M$ along a non-separating embedded sphere then we get a manifold homeomorphic to $M_{n-1,2}$. So we have the following lemma:

*Lemma* 3.3. *If an embedded sphere* $S$ *in* $M$ *is non-separating*, *then the* $\mathrm{link}([S])$ *is isomorphic to* $\mathbb{S}(M_{n-1,2})$ *and hence it is not a join.*

Let $S$ be a separating embedded sphere in $M$ with $M \backslash \mathrm{neigh}(S) = X \cup Y$. If $X$ or $Y$ is homeomorphic to a 1-punctured $S^2 \times S^1$, i.e., $S^2 \times S^1$ with interior one embedded 3-ball removed, then we call $S$ as a *handle sphere*. In this case, the puncture corresponds to the embedded sphere $S$.

*Lemma* 3.4. *An embedded sphere $S$ in $M$ is separating if and only if the* link([$S$]) *is a join. Furthermore, in this case* link([$S$]) *is realized as a join in exactly one way, up to permuting factors.*

*Proof.* Suppose $M \setminus \text{neigh}(S) = X \cup Y$. Then, $X$ is a 1-punctured $\sharp_k S^2 \times S^1$ for some $k \leq n-1$ and $Y$ is also a 1- punctured $\sharp_{k'} S^2 \times S^1$ for some $k' \leq n-1$. In this case both $\mathbb{S}(X)$ and $\mathbb{S}(Y)$ are non-empty. It follows that $\text{link}([S]) = \mathbb{S}(X) \vee \mathbb{S}(Y)$ and neither factor is empty. Further this join is realized uniquely because $\mathbb{S}(X)$ is not itself a join, $\mathbb{S}(X)$ is flag and join is associative. Converse is obvious from Lemma 3.3. □

DEFINITION 3.5

A *cone* is the join of a point with a non-empty simplicial complex.

*Lemma* 3.6. *A separating embedded sphere $S$ in $M$ is a handle sphere if and only if* link([$S$]) *in* $\mathbb{S}(M)$ *is a cone.*

*Proof.* By cutting $M$ along $S$, we get two components, say $X$ and $Y$. Suppose $X$ is 1-punctured $S^2 \times S^1$. Then $X$ contains a unique isotopy class of embedded sphere which is not boundary parallel. Therefore, $\mathbb{S}(X)$ is just a one point. Clearly, in this case $\text{link}([S]) = \mathbb{S}(X) \vee \mathbb{S}(Y)$. As $\mathbb{S}(X)$ is a point, link([$S$]) is a cone.

Now suppose that the link([$S$]) is a cone with apex vertex, say [$S'$] in $\mathbb{S}(M)$. Since a cone is the join of the apex with the base, by Lemma 3.4, $S$ is a separating embedded sphere in $M$. If we cut $M$ along $S$, then we get two components, say $X$ and $Y$. Thus $\text{link}(S) = \mathbb{S}(X) \vee \mathbb{S}(Y)$. By uniqueness of decomposition of a join in Lemma 3.4, either $\mathbb{S}(X)$ or $\mathbb{S}(Y)$ is equal to $\{[S']\}$. Suppose $\mathbb{S}(X) = \{[S']\}$. Then $X$ must be homeomorphic to $M_{1,1}$. Hence, $S$ is a handle sphere. □

From the above lemmas, we get as follows:

*Lemma* 3.7. *An embedded sphere $S$ in $M$ is non-separating if and only if there is an embedded sphere $S'$ so that the* link([$S'$]) *is a cone with apex* [$S$].

If an automorphism of $\mathbb{S}(M)$ maps a vertex $v$ of $\mathbb{S}(M)$ to another vertex $v'$ of $\mathbb{S}(M)$, then it maps link($v$) to link($v'$) isomorphically. From the above lemmas, it is clear that $\phi$ maps a vertex in $S(M)$ corresponding to a separating embedded sphere in $M$ to a vertex which corresponds to a separating embedded sphere in $M$. Moreover, it maps a vertex in $\mathbb{S}(M)$ corresponding to a handle sphere in $M$ to a vertex in $\mathbb{S}(M)$ corresponding to a handle sphere in $M$. The same is true for vertices corresponding to non-separating embedded spheres.

DEFINITION 3.8

A *reduced* sphere system is a sphere system in $M$ with connected, simply-connected complement, i.e., it cuts the manifold $M$ into a single simply-connected component.

Note that a reduced sphere system in $M = \sharp_n S^2 \times S^1$ contains exactly $n$ non-separating sphere components. Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ be a reduced sphere system in $M$. We choose a transverse orientation for each sphere $\Sigma_i$ so we may speak of a positive and a negative side of $\Sigma_i$. Let $p \in M$ be a base point in the complement of $S$.

A basis dual to $\Sigma$ is a set of homotopy classes of loops $\gamma_1, \ldots, \gamma_n$ in $M$ based at $p$ such that the loop $\gamma_i$ is disjoint from $\Sigma_j$ for all $j \neq i$ and intersects $\Sigma_i$ in a single point. We orient $\gamma_i$ such that it approaches $\Sigma_i$ from the positive side. Since the complement of $\Sigma$ is simply connected, the homotopy classes of loops $\gamma_i$ define a basis of $\pi_1(M, p)$. Thus, a reduced sphere system gives a basis of the fundamental group of $M$. Note that the sphere system $\Sigma_i^0$ described in subsection 2.1 is a reduced sphere system.

We recall the following:

DEFINITION 3.9

A *simple* sphere system $\Sigma = \cup_i \Sigma_i$ in $M$ is a sphere system in $M$ such that each complementary component of $\Sigma$ in $M$ is simply-connected.

A maximal sphere system in $M$ is simple. Splitting $M$ along a maximal sphere system produces a finite collection of 3-punctured 3-spheres. Here, a 3-punctured 3-sphere is the complement of the interiors of three disjointly embedded 3-balls in a 3-sphere. A reduced sphere system is also simple.

We call a simplex in $\mathbb{S}(M)$ *simple* (respectively, *reduced*) if it corresponds to a simple (respectively, reduced) sphere system in $M$.

We shall look at the links of reduced simplices and non-reduced simplices in $\mathbb{S}(M)$.

### 3.1 *Link of a reduced simplex*

Consider a reduced simplex $\sigma$ in $\mathbb{S}(M)$ corresponding to a reduced sphere system $\Sigma = \cup_i \Sigma_i$ in $M$. Cutting $M$ along $\Sigma$ then gives a simply-connected 3-manifold $M'$ with $2n$ boundary components $\Sigma_i^+, \Sigma_i^-$, for $1 \leq i \leq n$, where $\Sigma_i^+$ and $\Sigma_i^-$ correspond to the sphere $\Sigma_i$ for each $i$. Note that $M'$ is homeomorphic to a 3-sphere with interiors of $2n$ disjointly embedded 3-balls removed. Now, every non-boundary parallel embedded sphere in $M'$ gives a partition of the boundary spheres of $M'$ into two sets, say $X_S$ and $Y_S$. Moreover, its isotopy class in $M'$ is determined by such a partition of the boundary spheres of $M'$. Clearly, there are only finitely many isotopy classes of such embedded spheres in $M'$. The isotopy classes of images in $M$ of these embedded spheres in $M'$ span the link of the simplex $\sigma$ in $\mathbb{S}(M)$. Therefore, one can see that the link$(\sigma)$ has a finite number of vertices.

As each sphere in a reduced sphere system is non-separating, any simplicial automorphism $\phi$ of $\mathbb{S}(M)$ maps a reduced simplex to an $(n-1)$ simplex in $\mathbb{S}(M)$ which corresponds to a sphere system of non-separating embedded spheres in $M$.

### 3.2 *Link of an $(n-1)$-simplex which is not reduced*

Consider an $(n-1)$-simplex $\sigma'$ in $\mathbb{S}(M)$ such that it corresponds to a system $\Sigma' = \cup_i \Sigma_i'$ of non-separating spheres in $M$ which is not reduced.

*Lemma* 3.10. *The sphere system $\Sigma'$ is not a simple sphere system.*

*Proof.* If $\Sigma'$ is simple, then it must have more than one complementary component as $\Sigma'$ is not a reduced sphere system. Consider the graph of groups $\mathcal{G}$ with underlying graph $\Gamma$ associated to $\Sigma'$. The edges of $\Gamma$ correspond to the components of $\Sigma'$ and vertices correspond to the complementary components of $\Sigma'$ in $M$. Then, $\Gamma$ has more than one

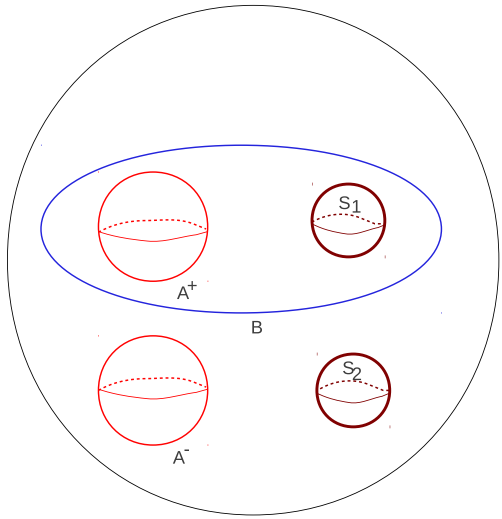


**Figure 1.** Sphere $B$ in $P$ = 2-punctured $S^2 \times S^1$.

vertex with all the vertex groups and edge groups trivial. In this case, $F_n = \pi_1(M) = \pi_1(\mathcal{G})$ is the fundamental group of $\Gamma$. But $\Gamma$ has more than one vertex and $n$ edges, so $\pi_1(\Gamma)$ can not be a free group on $n$ generators. Therefore, $\Sigma'$ can not be simple. □

Cutting $M$ along $\Sigma'$ then gives a (not necessarily connected) 3-manifold $M'$ with $2n$ boundary components $\Sigma_i'^+, \Sigma_i'^-$ for $1 \leq i \leq n$ such that $\Sigma_i'^+$ and $\Sigma_i'^-$ correspond to the sphere $\Sigma_i'$ for each $i$. Now, as $\Sigma'$ is not reduced and each $\Sigma_i'$ is non-separating, by Lemma 3.10 there is at least one non simply-connected component of $M'$, say $R$, with at least two boundary spheres. Note that there are at least two boundary spheres of $R$ which correspond to two distinct components of $\Sigma'$ and if $M'$ has more that one component, then there are at least two boundary spheres, say $S_1 = \Sigma_i'^\epsilon$ and $S_2 = \Sigma_j'^\eta$, for some $i$ and $j$ and for some signs $\epsilon, \eta \in \{+, -\}$ such that $\Sigma_i'^{-\epsilon}$ and $\Sigma_j'^{-\eta}$ are not boundary spheres of $R$, i.e., they are boundary components of some other component of $M'$. Note that $R$ is a $p$-punctured $\sharp_k S^2 \times S^1$, for appropriate integers $p \geq 2$ and $k \geq 1$. Now, we shall see that for $k = 1$ and $p = 2$, link$(\sigma')$ has infinitely many vertices by constructing infinitely many non-boundary parallel non-separating embedded spheres in $R$. The same construction will work for all the other possibilities of $k$ and $p$ obviously. This construction is also given in [11].

Now, suppose $k = 1$ and $p = 2$. In this case, $R$ is a 2-punctured $S^2 \times S^1$. Let $S_1$ and $S_2$ be the two boundary components of $R$. These boundary spheres correspond to two distinct components of $\Sigma'$. In this case, $R$ has infinitely many embedded spheres which separate the boundary spheres $S_1$ and $S_2$. These spheres in $R$ are non-separating embedded spheres in $M$. We construct such examples as follows: Let $A$ be a non-boundary parallel, non-separating embedded sphere in $R$ disjoint from $S_1$ and $S_2$. If we cut $R$ along $A$, then we get a manifold $R'$ which is a 4-punctured $S^3$ with boundary components $S_1$, $S_2$, $A^+$ and $A^-$, where $A^+$ and $A^-$ correspond to the embedded sphere $A$ in $R$. Consider an embedded sphere $B$ in $R'$ which separates $A^+$ and $S_1$ from $A^-$ and $S_2$ as shown in figure 1.

The (image under identification map from $R'$ to $R$ of) embedded sphere $B$ is non-separating in $R$. Let $S_1'$ be an embedded sphere parallel to $S_1$ in $R$ which is disjoint from all the spheres $S_1$, $S_2$ and $A$. Now, we consider an embedded sphere $T_m$ which is a connected sum of $B$ and $S_1'$ in $R$ along with a tube $t_m$ such that

(1) the tube $t_m$ intersects transversely $m$ times the embedded sphere $A$ as follows: Starting from $B$ it hits $A^-$, piercing $A$, again hitting $A^-$ through $A^+$, it hits $(m-1)$ times the sphere $A$ in this fashion, it ends at $S_1'$ as shown in figure 2, for $m = 3$;
(2) The circles of intersection of $t_m$ with $A$ split $t_m$ into components called pieces such that each piece separates the boundary spheres $S_1$ and $S_2$.

One can see that each sphere $T_m$ is non-separating in $M$ as it separates the boundary spheres of $R$. Each $T_m$ is disjoint from all and distinct from all the spheres $\Sigma_i'$. Thus, link$(\sigma')$ is infinite.

If an automorphism of $\mathbb{S}(M)$ maps a simplex $\tau$ of $\mathbb{S}(M)$ to another simplex $\tau'$ of $\mathbb{S}(M)$, then it maps link$(\tau)$ to link$(\tau')$ isomorphically. As link$(\sigma)$ of a reduced simplex $\sigma$ has finitely many vertices and link$(\sigma')$ of a non-reduced $(n-1)$-simplex whose vertices correspond to non-separating embedded spheres has infinitely many vertices, the automorphism $\phi$ can not map link$(\sigma)$ to link$(\sigma')$ and hence it can not map the reduced simplex $\sigma$ to the non-reduced simplex $\sigma'$. This shows that $\phi$ maps a reduced simplex in $\mathbb{S}(M)$ to a reduced simplex in $\mathbb{S}(M)$.

Thus we have the following lemma:

*Lemma* 3.11. *The automorphism $\phi$ maps a reduced simplex in $\mathbb{S}(M)$ to a reduced simplex in $\mathbb{S}(M)$.*

From the above lemma, we get the following lemma. Arguments here are the same as in [11] given for simplicial automorphisms of complex $NS(M)$ of non-separating embedded spheres. For the sake of completeness, we write the proof here again.

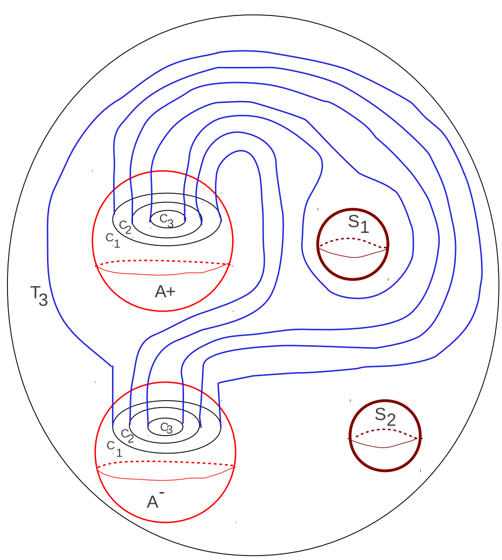


**Figure 2.** Sphere $T_3$ in $P$ = 2-punctured $S^2 \times S^1$.

*Lemma* 3.12. *Every automorphism $\phi$ of $\mathbb{S}(M)$ maps a simple simplex in $\mathbb{S}(M)$ to a simple simplex in $\mathbb{S}(M)$.*

*Proof.* Note that a sphere system in $M$ which contains a reduced sphere system as its subset is simple. Moreover, every simple sphere system $\Sigma = \cup_i \Sigma_i$ in $M$ contains a reduced sphere system in $M$ as its subset. For, consider the graph of groups $\mathcal{G}$ with underlying graph $\Gamma$ associated to $\Sigma$. The edges of $\Gamma$ correspond to the embedded spheres $\Sigma_i$ and the vertices correspond to the complementary components of $\Sigma$ in $M$. As $\Sigma$ is simple, $\Gamma$ has with all the vertex groups and edge groups trivial. In this case, $F_n = \pi_1(M) = \pi_1(\mathcal{G})$ is the fundamental group of $\Gamma$. Consider a maximal tree $\mathbb{T}$ in $\Gamma$. Then, $\Gamma \setminus \mathbb{T}$ has $n$ edges, say $e_1, \ldots, e_n$. Let $\Sigma_{i_1}, \ldots, \Sigma_{i_n}$ be the components of $\Sigma$ corresponding to the edges $e_1, \ldots, e_n$. As the the tree $\mathbb{T}$ is connected and contains all the vertices of $\Gamma$, one can see that the complement of the embedded spheres $\Sigma_{i_k}$ in $M$ is connected. The fundamental group of the complement of these spheres $\Sigma_{i_k}$ in $M$ is the fundamental group of the graph of groups $\mathcal{G}'$ with underlying graph $\mathbb{T}$. As $\mathbb{T}$ is a tree with all vertex groups and edge groups trivial, the fundamental group of $\mathcal{G}'$ is trivial. Hence, the complement of the spheres $\Sigma_{i_k}$ in $M$ is simply-connected. This shows that the $n$ embedded spheres $\Sigma_{i_k}$'s corresponding to these $n$ edges outside the maximal tree $\mathbb{T}$ form a reduced sphere system in $M$.

If $\sigma$ is a simple simplex in $\mathbb{S}(M)$, it has a face $\tau$ which is reduced. Then, $\phi(\sigma)$ is a simplex in $\mathbb{S}(M)$ with face $\phi(\tau)$. By Lemma 3.11, $\phi(\tau)$ is a reduced simplex which implies that $\phi(\sigma)$ is a simple simplex in $\mathbb{S}(M)$. Thus, $\phi$ maps a simple simplex in $\mathbb{S}(M)$ to a simple simplex in $\mathbb{S}(M)$. □

## 4. Culler–Vogtmann space and the reduced outer space

We briefly recall the definition of *Culler–Vogtmann space* $CV_n$ for free group $\mathbb{F}_n$ of rank $n$ [3, 12]. It is also known as the *outer space*. A point in $CV_n$ is an equivalence class of marked metric graphs $(h, X)$ such that

(1) $X$ is a metric graph with $\pi_1(X) = \mathbb{F}_n$ having edge lengths which sum to 1, with all vertices of valence at least 3.
(2) The marking is given by a homotopy equivalence $h : R_n \to X$, where $R_n$ is a graph with one vertex $v$ and $n$ edges. The free group of rank $n$ is identified with $\pi_1(R_n, v)$ where the identification takes generators of the free group to the edges of $R_n$.
(3) Two such marked graphs $(h, X)$ and $(h', X')$ are equivalent if they are isometric via an isometry $g : X \to X'$ such that $g \circ h$ is homotopic to $h'$.

In [6], the outer space $CV_n$ is denoted by $\mathbb{O}_n$. We shall follow the same notation as in [6]. An edge $e$ of a graph $X$ is called a separating edge if $X - e$ is disconnected. There is a natural equivariant deformation retraction of $\mathbb{O}_n$ onto the subspace $cv_n$ consisting of marked metric graphs $(h, X)$ such that $X$ has no separating edges. The deformation proceeds by uniformly collapsing all separating edges in all marked graphs in $\mathbb{O}_n$. The subspace $cv_n$ of $\mathbb{O}_n$ is also known as *reduced outer space*.

The group $\mathrm{Out}(\mathbb{F}_n)$ acts on $\mathbb{O}_n$ on the right by changing the markings: given $\phi \in \mathrm{Out}(\mathbb{F}_n)$, choose a representative $f : R_n \to R_n$ for $\phi$, then $(g, X)\phi = (g \circ f, X)$. This action preserves the reduced outer space $cv_n$ as a subset of $\mathbb{O}_n$. In particular, we have an $\mathrm{Out}(\mathbb{F}_n)$-action on the reduced outer space $cv_n$.

In [6], it was shown that the space $\mathbb{O}_n$ is $\mathrm{Out}(\mathbb{F}_n)$-equivariantly homeomorphic to a subset of sphere complex $\mathbb{S}(M)$ as described as follows: We interpret points in the sphere complex $\mathbb{S}(M)$ as (positively) weighted sphere systems in $M$. Let $\mathbb{S}_\infty$ be the subcomplex of $\mathbb{S}(M)$ consisting of those elements $\sum_i a_i S_i \in \mathbb{S}(M)$ such that $M \setminus \cup_i S_i$ has at least one non-simply connected component. To a point $\sum_i a_i S_i \in \mathbb{S}(M) \setminus \mathbb{S}_\infty$, we associate the dual graph to $\cup_i S_i$ and assign the edge corresponding to $S_i$ to have length $a_i$. This gives a map $\theta : \mathbb{O}_n \to \mathbb{S}(M) \backslash \mathbb{S}_\infty$. In [6], it is shown that $\theta$ is a homeomorphism. Under this homeomorphism the reduced outer space $cv_n$ is also a subset of the sphere complex. The points of the reduced outer space $cv_n$ correspond to weighted simple systems of non-separating embedded spheres in $\mathbb{S}(M)$.

Culler and Vogtmann defined what is called the *spine* $K_n$ of the reduced outer space $cv_n$. Considering $cv_n$ as a subset of $\mathbb{S}(M)$, the spine $K_n$ is the maximal full subcomplex of the first barycentric subdivision of $\mathbb{S}(M)$ which is contained in $cv_n$ and is disjoint from $\mathbb{S}_\infty$. The interior of every simplex contained in $cv_n$ intersects $K_n$. We have an action of $\mathrm{Out}(\mathbb{F}_n)$ on $\mathbb{S}(M)$ which preserves $cv_n$, i.e., we have an action $\mathrm{Out}(\mathbb{F}_n)$ on $cv_n$. This action of $\mathrm{Out}(F_n)$ on $cv_n$ preserves the spine which yields an action of $\mathrm{Out}(\mathbb{F}_n)$ on the spine $K_n$. This gives a homomorphism

$$\Omega : \mathrm{Out}(\mathbb{F}_n) \to \mathrm{Aut}(K_n),$$

where $\mathrm{Aut}(K_n)$ is the group of simplicial automorphisms of $K_n$.

We have the following theorem of Bridson and Vogtmann (see [2]).

**Theorem 4.1.** *The homomorphism $\Omega$ is an isomorphism for $n \geq 3$.*

## 5. Proof of the main theorem

Now, we have the following lemma:

*Lemma* 5.1. *Every automorphism $\phi \in \mathrm{Aut}(\mathbb{S}(M))$ preserves the reduced outer space $cv_n$ and hence the spine $K_n$.*

*Proof.* We have seen that every automorphism $\phi$ of $\mathbb{S}(M)$ maps simple simplices of $\mathbb{S}(M)$ to simple simplices $\mathbb{S}(M)$ (Lemma 3.12). Therefore, it maps a simple simplex corresponding to a system of non-separating spheres to a simple simplex corresponding to a system of non-separating embedded sphere in $M$. From this it follows that every automorphism $\phi \in \mathrm{Aut}(\mathbb{S}(M))$ preserves the reduced outer space $cv_n$ and hence preserves the spine $K_n$ of $cv_n$. □

Thus, every simplicial automorphism of $\mathbb{S}(M)$ induces a simplicial automorphism on the spine $K_n$ of $cv_n$.

Therefore, we have a homomorphism

$$\Lambda : \mathrm{Aut}(\mathbb{S}(M)) \to \mathrm{Aut}(K_n) \cong \mathrm{Out}(\mathbb{F}_n).$$

We shall recall the following lemma from [1].

*Lemma* 5.2. *The identity is the only automorphism of $\mathbb{S}(M)$ acting trivially on the spine $K_n$.*

From the above lemma, it follows that $\Lambda$ is injective. The action of Out$(\mathbb{F}_n)$ on $\mathbb{S}(M)$ which preserves $cv_n$, i.e., we have an action Out$(\mathbb{F}_n)$ on $cv_n$. This action of Out$(F_n)$ on $cv_n$ preserves the spine which yields an action of Out$(\mathbb{F}_n)$ on the spine $K_n$. As this action of Out$(F_n)$ on the spine $K_n$ induces the isomorphism $\Omega$ : Aut$(K_n) \to$ Out$(F_n)$ (Theorem 4.1), it follows that $\Lambda$ is surjective. This proves Theorem 2.3 that the group of simplcial automorphisms of the sphere complex is isomorphic to the group Out$(F_n)$.

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