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Homogeneous bilateral block shifts

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Abstract. A new 3-parameter family of homogeneous 2-by-2 block shifts is described. These are the first examples of irreducible homogeneous bilateral block shifts of block size larger than 1.

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1. Introduction

We write $\mathbb{D}$ for the complex unit disc and G for the Möbius group, the group of holomorphic self-maps of $\mathbb{D}$. A bounded operator T on a Hilbert space $\mathcal{H}$ is said to be homogeneous if its spectrum is contained in $\bar{\mathbb{D}}$ and for every g in G there exists a unitary operator $U(g)$ such that

$$g(T) = U(g)^{-1} T U(g).$$

As shown in [1], if T is irreducible, $U(g)$ can be taken to be a representation of $\tilde{G}$, the universal covering group of G. It is called the associated representation of T. It is also shown in [1] that $\mathcal{H}$ is the orthogonal direct sum of subspaces $\mathcal{H}(n)$ $(n \in I)$ such that $T\mathcal{H}(n) \subset \mathcal{H}(n+1)$. The set I can be $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$; accordingly, T is said to be a bilateral or unilateral block shift.

In the case where $\dim \mathcal{H}(n) \leq 1$ for all n (scalar shifts) a complete classification was given by Bagchi and Misra [1]. The homogeneous operators in the Cowen–Douglas class were classified in [3]; they are unilateral block shifts of arbitrary block size (i.e. $\dim \mathcal{H}(n)$ can be anything). However, no examples of irreducible homogeneous bilateral block shifts of block size larger than 1 were known until now.

In §2 of this article, we construct a 3-parameter family of such operators with $\dim \mathcal{H}(n) = 2$ for all $n \in \mathbb{Z}$.

In §3, we describe all the homogeneous operators whose associated representation is the direct sum of two copies of a representation of $\tilde{G}$ from the principal series or the complementary series ([5], cf. also [1]). In the case of the complementary series these turn out to be exactly the operators constructed in §2, where the questions of their irreducibility and inequivalence are fully discussed. In the principal series case we also get a non-trivial 3-parameter family, but we do not give such a complete discussion.

The infinitesimal method used in §3 is a further development of a method used in [4]. As we show, it incidentally also gives a considerable simplification of some of the proofs in [1]. It does not seem unlikely that it could be used to get a complete classification of all 2-by-2 homogeneous block shifts.

2. A class of operators

The following is a generalization of Theorem 5 in [2].

Lemma 2.1. *Assume that T_1 and T_2 are homogeneous operators on $\mathcal{H}$ and have the same associated representation $U(g)$. Then, for any $\alpha \in \mathbb{C}$, the operator*

$$\tilde{T} = \begin{pmatrix} T_1 & \alpha(T_1 - T_2) \\ 0 & T_2 \end{pmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$ is homogeneous with associated representation $U(g) \oplus U(g)$. The unitary equivalence class of $\tilde{T}$ depends only on $|\alpha|$. The operator $\tilde{T}'$ gotten by interchanging the roles of T_1 and T_2 is unitarily equivalent to $\tilde{T}$.

Proof. The lemma is trivial for $\alpha = 0$, i.e. for $T_1 \oplus T_2$. For general α, let

$$S = \begin{pmatrix} I & \alpha I \\ 0 & I \end{pmatrix}.$$

Then $\tilde{T} = S^{-1}(T_1 \oplus T_2)S$, and since S commutes with $U(g) \oplus U(g)$ we have

$$\begin{aligned} g(\tilde{T}) &= S^{-1}(g(T_1) \oplus g(T_2))S \\ &= S^{-1}(U(g) \oplus U(g))^{-1}(T_1 \oplus T_2)(U(g) \oplus U(g))S \\ &= (U(g) \oplus U(g))^{-1}\tilde{T}(U(g) \oplus U(g)), \end{aligned}$$

proving the first statement. For the second, we note that conjugating $\tilde{T}$ by the unitary operator $(e^{i\theta} I) \oplus I$ has the effect of multiplying α by $e^{i\theta}$.

Finally, setting

$$U = \frac{1}{\sqrt{\alpha^2 + 1}} \begin{pmatrix} -\alpha I & I \\ I & \alpha I \end{pmatrix},$$

we can verify that $U\tilde{T}' = \tilde{T}U$, proving the last statement. □

For fixed $a, b \in (0, 1)$ with $a \neq b$, we set

$$t_n = t_n(a, b) = \left(\frac{n + a}{n + b}\right)^{1/2}, \quad (n \in \mathbb{Z}). \tag{2.1}$$

Writing $\{e_n\}_{n \in \mathbb{Z}}$ for the natural basis of $\ell^2(\mathbb{Z})$, the operator $T = T(a, b)$ defined by $Te_n = t_n e_{n+1}$ is homogeneous. By [1], these are exactly the operators whose associated representation belongs to the complementary series. In the matrix of T the $(n+1, n)$-entry is t_n, all other entries are 0.

Given a, b as above, and $\alpha > 0$ we define

$$\tilde{T} = \tilde{T}(a, b, \alpha) = \begin{pmatrix} T(a, b) & \alpha(T(a, b) - T(b, a)) \\ 0 & T(b, a) \end{pmatrix} \tag{2.2}$$

acting on $\mathcal{H}\oplus\mathcal{H}$. $T(a,b)$ and $T(b,a)$ have the same associated representation, so Lemma 2.1 applies to $\tilde{T}$.

We call $\mathcal{H}(n)$ the subspace of $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ spanned by a copy of e_n from each summand. So $T\mathcal{H}(n) \subset \mathcal{H}(n+1)$, and rearranging the basis we can write the matrix of $\tilde{T}$ as a block matrix with 0 everywhere except the block in the $(n+1,n)$-position, which is

$$\tilde{T}_n = \tilde{T}_n(a,b,\alpha) = \begin{pmatrix} t_n & \alpha\left(t_n - t_n^{-1}\right) \\ 0 & t_n^{-1} \end{pmatrix}. \tag{2.3}$$

In the following we think of $\tilde{T}$ as this block matrix. For later use we observe that $A = \tilde{T}^*\tilde{T}$ and $B = \tilde{T}\tilde{T}^*$ are block diagonal; writing A_n, B_n for the blocks in the (n,n)-position we have

$$\begin{aligned} A_n &= T_n^* T_n = \frac{1}{(n+a)(n+b)} \\ &\quad \times \begin{pmatrix} (n+a)^2 & \alpha(a-b)(n+a) \\ \alpha(a-b)(n+a) & \alpha^2(a-b)^2 + (n+b)^2 \end{pmatrix}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} B_n &= T_{n-1} T_{n-1}^* = \frac{1}{(n+a-1)(n+b-1)} \\ &\quad \times \begin{pmatrix} (n+a-1)^2 + \alpha^2(a-b)^2 & \alpha(a-b)(n+b-1) \\ \alpha(a-b)(n+b-1) & (n+b-1)^2 \end{pmatrix}. \end{aligned} \tag{2.5}$$

By Lemma 2.1, $\tilde{T}(a,b,\alpha)$ and $\tilde{T}(b,a,\alpha)$ are unitarily equivalent. So we can restrict ourselves to the case $a < b$.

Theorem 2.2. *For any choice of the parameters* $0 < a < b < 1, \alpha > 0$, *the operators* $\tilde{T} = \tilde{T}(a,b,\alpha)$ *are irreducible and mutually unitarily inequivalent.*

Proof. The singular values of $\tilde{T}$ are the singular values of all the T_n taken together. Since $\det T_n = 1$, they come in pairs $\lambda_n > 1$, $\dfrac{1}{\lambda_n}$. The sum of their squares is the trace of $T_n^* T_n$:

$$\lambda_n^2 + \frac{1}{\lambda_n^2} = \frac{(1+\alpha^2)(a-b)^2}{(n+a)(n+b)} + 2$$

whence

$$\left(\lambda_n - \frac{1}{\lambda_n}\right)^2 = \frac{(1+\alpha^2)(a-b)^2}{(n+a)(n+b)}. \tag{2.6}$$

A consequence of (2.6) is that when $a + b \neq 1$, every singular value has multiplicity one: This is immediate from the identity

$$(n+a)(n+b) = \left(n + \frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2. \tag{2.7}$$

When $a + b = 1$, the same identity shows

$$\lambda_n = \lambda_{-n-1} \tag{2.8}$$

for all n; in this case the singular values all have multiplicity 2.

We write $\{v_n^{(1)}, v_n^{(2)}\}$ for an orthonormal basis of $\mathcal{H}(n)$ diagonalizing A_n, $\{u_n^{(1)}, u_n^{(2)}\}$ for a basis diagonalizing B_n, and set $\lambda_n^{(1)} = \lambda_n$, $\lambda_n^{(2)} = \dfrac{1}{\lambda_n}$. So we have

$$T_n v_n^{(j)} = \lambda_n^{(j)} u_{n+1}^{(j)}, \quad (n \in \mathbb{Z};\ j = 1, 2).$$

We claim that there is some $n = m$ such that the two bases of $\mathcal{H}(m)$ are different (not equal even up to scalar factors). To prove this, it suffices to show that $A_n B_n = B_n A_n$ cannot hold for all $n \in \mathbb{Z}$. We write $\hat{A}_n$, $\hat{B}_n$ for (2.4) and (2.5) with the scalar factors in front of the matrices omitted. The (1,2)-entry of the matrix $\hat{B}_n \hat{A}_n - \hat{A}_n \hat{B}_n$ is a cubic polynomial in n. It will be zero for all n only if all coefficients of the polynomial are zero. Computing the coefficient of the first degree term we find that it is

$$2\alpha(1 + \alpha^2)(a - b)^3$$

proving the claim.

To prove irreducibility of $\tilde{T}$, let $\mathcal{K}$ be a non-zero reducing subspace. Then $\mathcal{K}$ is also invariant for $A = \tilde{T}^*\tilde{T}$ and $B = \tilde{T}\tilde{T}^*$. By elementary spectral theory, if $f \in \mathcal{K}$, then the projection of f onto any eigenspace of A or B is also in $\mathcal{K}$.

We claim that it suffices to prove that $\mathcal{K} \cap \mathcal{H}(n) \neq 0$ for some n. In fact, then applying $\tilde{T}^k$ and $\tilde{T}^{*k}$ it follows that $\mathcal{K} \cap \mathcal{H}(n \pm k) \neq 0$. Taking a non-zero f in $\mathcal{K} \cap \mathcal{H}(m)$ we have $f = \alpha v_m^{(1)} + \beta v_m^{(2)} = \gamma u_m^{(1)} + \delta u_m^{(2)}$ with at least one of $\alpha\beta$ and $\gamma\delta$ non-zero. Then either $v_m^{(1)}$ and $v_m^{(2)}$ or $u_m^{(1)}$ and $u_m^{(2)}$ are in $\mathcal{K}$. In any case, $\mathcal{H}(m) \subset \mathcal{K}$. Applying powers of $\tilde{T}$ and $\tilde{T}^*$, we see that $\mathcal{H}(n) \subset \mathcal{K}$ for all n, proving the claim.

To prove that $\mathcal{K} \cap \mathcal{H}(n) \neq 0$ for some n, we distinguish cases. If $a + b \neq 1$, the eigenspaces of A are $\mathbb{C}v_n^{(j)}$ $(n \in \mathbb{Z};\ j = 1, 2)$. A non-zero $f \in \mathcal{K}$ has a non-zero projection onto one of these. This gives a non-zero $\mathcal{K} \cap \mathcal{H}(n)$.

If $a + b = 1$, we know from (2.8) that the eigenspaces of A are $\mathcal{V}_n^{(j)} = \mathbb{C}v_n^{(j)} + \mathbb{C}v_{-n-1}^{(j)}$ $(n \geq 0;\ j = 1, 2)$, and those of B are $\mathcal{U}_n^{(j)} = \mathbb{C}u_{n+1}^{(j)} + \mathbb{C}u_{-n}^{(j)}$. For some n, j there is a non-zero element $f = \sigma v_n^{(j)} + \tau v_{-n-1}^{(j)}$ in $\mathcal{K}$. We write $v_n^{(j)}$ in terms of the other basis of $\mathcal{H}(n)$, likewise with $v_{-n-1}^{(j)}$. Then

$$f = \sigma p u_n^{(1)} + \sigma q u_n^{(2)} + \tau r u_{-n-1}^{(1)} + \tau s u_{-n-1}^{(2)}.$$

The four terms of this sum are in four different eigenspaces of B, namely $\mathcal{U}_{n-1}^{(1)}$, $\mathcal{U}_{n-1}^{(2)}$, $\mathcal{U}_{n+1}^{(1)}$, $\mathcal{U}_{n+1}^{(2)}$, so each one is in $\mathcal{K}$. At least one of them is non-zero, so $\mathcal{K} \cap \mathcal{H}(n)$ or $\mathcal{K} \cap \mathcal{H}(-n-1)$ is non-zero, finishing the proof of irreducibility.

To prove the inequivalence statement we look at the singular values $\lambda_n, \lambda_n^{-1}$ of $\tilde{T}$. The operators $\tilde{T}(a, b, \alpha)$ and $\tilde{T}(1 - b, 1 - a, \alpha)$ have the same set of singular values, but coincide if $a + b = 1$, and are unitarily inequivalent if $a + b \neq 1$, because they have inequivalent associated representations by [1]. It follows that it will be sufficient to prove that the set of singular values uniquely determines a, b, α under the extra condition $a + b \leq 1$. (Note that $a < b$ is still assumed.)

Instead of the set of singular values we can work with the set of all $(\lambda_n - \frac{1}{\lambda_n})^2$, or with the set S of its reciprocals, which by (2.6) is

$$S = \left\{ \frac{(n+a)(n+b)}{(1+\alpha^2)(a-b)^2} \Big| n \in \mathbb{Z} \right\}.$$

Now the number $A = (1+\alpha^2)(a-b)^2$ is determined by S, since the number of elements x in S such that $|x| < N^2$ is asymptotically $2A^{1/2}N$ as $N \to \infty$. So it suffices to show that the set

$$S' = \{(n+a)(n+b) | n \in \mathbb{Z}\}$$

determines a, b. From (2.7) it is clear that the smallest and second smallest elements of S' are given by $n = 0$, respectively $n = -1$. These then determine a, b, finishing the proof. □

3. Operators with prescribed associated representation

In this section we show that the irreducible homogeneous operators described in Theorem 2.2 are the only ones whose associated representation is the sum of two copies of a complementary series representation. Our arguments apply equally well to the principal series, so with the same effort we will also get a list of all possible homogeneous operators corresponding to sums of two copies of principal series representations.

Suppose that $g \mapsto U(g)$ is a unitary representation of $\tilde{G}$ which is a discrete direct sum of irreducibles, and U_* is the corresponding representation of the Lie algebra of $\mathfrak{g} = \mathfrak{su}(1,1)$ extended by linearity to the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$. The matrices

$$h = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

form a basis of $g^{\mathbb{C}}$. To follow the notation of Pukánszky [5] we write

$$H^0 = U_*(h), \quad H^+ = U_*(f), \quad H^- = U_*(e).$$

As shown in [5] these three operators always have a common dense domain, and they determine the representation U.

If T is a homogeneous operator with U associated to it, in the identity $g(T) = U(g)^{-1}TU(g)$ we set $g = \exp tx$, $(x \in \mathfrak{g})$ and differentiate at $t = 0$. This gives for any $x \in \mathfrak{g}$ and by linearity also for any $x \in \mathfrak{g}^{\mathbb{C}}$,

$$x(T) = -[U_*(x), T], \tag{3.1}$$

where the meaning of the left-hand side is

$$x(T) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (\exp tx)(T).$$

To see that this derivative exists, it is enough to compute it for a basis of $\mathfrak{g}$. Since $SL(2,\mathbb{C})$ has a local action on $\mathbb{D}$, we can use the complex basis h, f, e and obtain

$$\begin{aligned} h(T) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \mathrm{e}^{-t}T = -T, \\ f(T) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 T(-tT+I)^{-1} = T^2, \\ e(T) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (T+tI) = I. \end{aligned}$$

It follows that (3.1) is equivalent to the three equations

$$[H^0, T] = T, \tag{3.2}$$

$$[H^+, T] = -T^2, \tag{3.3}$$

$$[H^-, T] = -I. \tag{3.4}$$

We will approach the problem of finding all T with a given corresponding U by trying to solve these three equations. In this article, we only deal with multiples of irreducible representations from the continuous series. These irreducibles are characterized [5] by two parameters λ, q where

$$-\frac{1}{2} < \lambda \leq \frac{1}{2},$$

$$|\lambda|(1-|\lambda|) < q$$

(we changed Pukánszky's τ to λ, which equals τ for $\tau \leq \frac{1}{2}$ and $\tau - 1$ for $\frac{1}{2} < \tau < 1$. This λ coincides with the $\frac{\lambda}{2}$ of [1].) The 'principal series' is given by $q \geq \frac{1}{4}$ (the case $q = \frac{1}{4}, \lambda = \frac{1}{2}$ excluded, since reducible), the 'complementary series' by $q < \frac{1}{4}$. In both cases we can use the parameter $\sigma = (\frac{1}{4} - q)^{1/2}$ instead of q (non-negative imaginary, respectively positive).

The eigenvalues of H^0 are then $\lambda + n$ ($n \in \mathbb{Z}$), with corresponding one-dimensional eigenspaces $\mathcal{H}(n)$. There are isometries $V_n : \mathcal{H}(n) \to \mathcal{H}(n+1)$ such that, denoting the restrictions of H^+ and H^- to $\mathcal{H}(n)$ respectively $\mathcal{H}(n+1)$ by H_n^+, respectively H_n^-, we have

$$H_n^+ = r_n V_n,$$
$$H_n^- = r_n V_n^{-1}$$

with

$$\begin{aligned} r_n &= a_n b_n, \\ a_n &= \left(\lambda + \frac{1}{2} + n + \sigma\right)^{1/2}, \\ b_n &= \left(\lambda + \frac{1}{2} + n - \sigma\right)^{1/2}, \end{aligned}$$

the square roots chosen so that $r_n > 0$.

Equation (3.2) expresses the fact that T maps $\mathcal{H}(n)$ into $\mathcal{H}(n+1)$. We denote by T_n this restriction of T. In terms of T_n, (3.3) and (3.4) become

$$H^+_{n+1}T_n - T_{n+1}H^+_n = -T_{n+1}T_n \quad (n \in \mathbb{Z}), \tag{3.5}$$

$$H^-_{n+1}T_{n+1} - T_nH^-_n = -I_{n+1} \quad (n \in \mathbb{Z}). \tag{3.6}$$

At the moment we are dealing with an irreducible representation, so $\dim \mathcal{H}(n) = 1$, and T_n is determined by a scalar: $T_n = t_nV_n$. So (3.5) and (3.6) amount to an infinite system of scalar equations

$$r_{n+1}t_n - r_nt_{n+1} = -t_{n+1}t_n, \tag{3.7}$$

$$r_{n+1}t_{n+1} - r_nt_n = -1. \tag{3.8}$$

It is not hard to verify that the only solutions of this system are

$$t_n = -\frac{a_n}{b_n} \quad (n \in \mathbb{Z})$$

and

$$t_n = -\frac{b_n}{a_n} \quad (n \in \mathbb{Z}).$$

This, incidentally, gives us all the homogeneous operators associated to irreducible representations of the continuous series, giving another proof of results of Bagchi and Misra [1]. In the complementary series case, we can use $\lambda + \frac{1}{2} \pm \sigma$ as parameters and call them a, b; up to signs, i.e. up to simple unitary equivalence this gives (2.1). In the principal series case, $|t_n| = 1$ for all n, so all solutions give unitarily equivalent operators; however, in the 2-by-2 block case the differences between the solutions become important.

We now proceed to our actual goal, the determination of all homogeneous operators whose associated representation is the direct sum of two copies of a continuous series representation. From here on, U will stand for such a direct sum. We continue to use the notations H^0, $H^\pm$, $H^\pm_n$ with this U. Each $\mathcal{H}(n)$ is now two-dimensional, still the $(\lambda+n)$-eigenspace of H^0. A homogeneous operator T is still determined by its restrictions T_n : $\mathcal{H}(n) \to \mathcal{H}(n+1)$ which satisfy (3.5) and (3.6) for all $n \in \mathbb{Z}$.

To solve this system we use an orthonormal basis of $\mathcal{H}$ put together from bases of the $\mathcal{H}(n)$ that are mapped onto each other by the isometries V_n. We consider (3.5) and (3.6) as matrix equations. The matrix of $H^\pm_n$ is now the 2-by-2 scalar matrix r_nI_2.

We can choose the basis so that, say, T_0 is upper triangular. This implies by (3.6) that every T_n is triangular; we denote

$$T_n = \begin{pmatrix} u_n & v_n \\ 0 & z_n \end{pmatrix}.$$

Looking at the (1,1)-entry in equations (3.5) and (3.6) we get the scalar equations (3.7), (3.8) for u_n in place of t_n. The situation is the same for the (2,2)-entry. So, by our preceding result we know that for both u_n and z_n the only possibilities are $-\frac{a_n}{b_n}$ and $-\frac{b_n}{a_n}$.

The (2,1)-entry of (3.6) gives

$$r_{n+1}v_{n+1} - r_n v_n = 0,$$

hence $v_n = c/r_n$ for all n with some constant c. Since we have

$$\frac{a_n}{b_n} - \frac{b_n}{a_n} = \frac{a_n^2 - b_n^2}{a_n b_n} = \frac{2\sigma}{r_n},$$

we can also write this in the form

$$v_n = \alpha \left(\frac{a_n}{b_n} - \frac{b_n}{a_n} \right) \tag{3.9}$$

with another constant α.

Finally, the (2,1)-entry of (3.5) gives the condition

$$r_{n+1}v_n - v_{n+1}r_n = u_{n+1}v_n + v_{n+1}z_n. \tag{3.10}$$

If $\sigma = 0$ is our representation, then this is trivially satisfied, every T_n is diagonal, and T is the direct sum of two scalar shifts. If $\sigma \neq 0$, it is easy to check that (3.10) is satisfied if and only if

$$u_n = -\frac{a_n}{b_n}, \quad z_n = -\frac{b_n}{a_n}, \tag{3.11}$$

or

$$u_n = -\frac{b_n}{a_n}, \quad z_n = -\frac{a_n}{b_n} \tag{3.12}$$

for all n in $\mathbb{Z}$.

In the case of the complementary series, these formulas show that (up to sign changes which amount to trivial unitary equivalences) the only corresponding operators are the ones constructed in §2.

In the principal series case, it also follows that all corresponding operators arise by the construction of Lemma 2.1 from the scalar shift operator with weights $-\frac{a_n}{b_n}$. (Note that, by Lemma 2.1, (3.11) and (3.12) give unitarily equivalent operators.) One could make a study of their irreducibility and inequivalence similar to Theorem 2.2.

Finally we observe that the method of the present section is also adaptable to the discrete series of representations.

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