Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 2, May 2014, pp. 193–203.

Boundedness for Marcinkiewicz integrals associated with Schrödinger operators

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MS received 18 October 2012

Abstract. Let $L = -\Delta + V$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$, while nonnegative potential V belongs to the reverse Hölder class. In this paper, we will show that Marcinkiewicz integral associated with Schrödinger operator is bounded on BMO_L, and from $H^1_L(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

Keywords. Marcinkiewicz integral; Schrödinger operator.

2000 Mathematics Classification. 42B20, 42B25, 42B35.

1. Introduction

Let $\mathbb{R}^n,\ n \geq 3$ be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with normalized Lebesgue measure $\mathrm{d}\sigma = \mathrm{d}\sigma(x)$. Let $\Omega \in L^s(S^{n-1})$, $s \geq 1$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfy

$$\int_{S^{n-1}} \Omega(x)\,\mathrm{d}\sigma(x) = 0. \tag{1.1}$$

The Marcinkiewicz integral operator μ is defined by

$$\mu f(x) = \left(\int_0^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2}. \tag{1.2}$$

The above operator was introduced by Stein in [8] as an extension of the notion of Marcinkiewicz function from one dimension to higher dimensions. Meanwhile, Stein [8] showed that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ for some $0 < \alpha \leq 1$, then μ is a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p \leq 2$, and a bounded mapping from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. Benedek *et al.* [1] showed that if Ω is continuously differentiable in $x \neq 0$, then (1.2) above holds for $1 < p < \infty$. Recently, Ding *et al.* [2] proved that the Marcinkiewicz function μ is bounded from $H^1(\mathbb{R}^n)$ (Hardy space; see [6]) to $L^1(\mathbb{R}^n)$.

On the other hand, the study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [7] considered L^p estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R_j^L =$

$\frac{\partial}{\partial x_j}L^{-\frac{1}{2}},\ j=1,\dots,n$. Then, Dziubanński and Zienkiewicz [4] introduced the Hardy type space $H^1_L(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$. The dual space of $H^1_L(\mathbb{R}^n)$ is the BMO type space BMO_L investigated by Dziubanński *et al.* [5]. Recently, Dong and Liu [3] proved that the Schrödinger Riesz transforms R^L_j are bounded on $BMO_L(\mathbb{R}^n)$.

Similar to the classical Marcinkiewicz function μ, we define the Marcinkiewicz functions μ^L_j associated with the Schrödinger operator L by

$$\mu^L_j f(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}K^L_j(x,y)f(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2},$$

where $K^L_j(x,y)=\widetilde{K^L_j}(x,y)|x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R^L_j=\frac{\partial}{\partial x_j}L^{-\frac{1}{2}},\ j=1,\ \dots,n$. In particular, when $V=0$, $K^{\triangle}_j(x,y)=\widetilde{K^{\triangle}_j}(x,y)|x-y|=\frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K^{\triangle}_j}(x,y)$ is the kernel of $R_j=\frac{\partial}{\partial x_j}\triangle^{-\frac{1}{2}},\ j=1,\ \dots,n$. In this paper, we write $K_j(x,y)=K^{\triangle}_j(x,y)$ and

$$\mu_j f(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}K_j(x,y)f(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}.$$

Obviously, μ_j are classical Marcinkiewicz functions. Therefore, it will be an interesting thing to study the property of μ^L_j. The main purpose of this paper is to show that Marcinkiewicz integrals associated with Schrödinger operators are bounded on BMO_L, and from $H^1_L(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. To state our results, let us first introduce some notations.

Note that a nonnegative locally L^q integral function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1<q<\infty)$ if there exists $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}V^q(y)\mathrm{d}y\right)^{1/q}\le C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}V(y)\mathrm{d}y\right) \tag{1.3}$$

holds for every $x\in\mathbb{R}^n$ and $0<r<\infty$, where $B(x,r)$ denotes the ball centered at x with radius r. It is worth pointing out that the B_q class is that, if $V\in B_q$ for some $q>1$, then there exists $\epsilon>0$, which depends only on n and the constant C in (1.3), such that $V\in B_{q+\epsilon}$. Throughout this paper, we always assume that $0\not\equiv V\in B_n$.

Let $\rho(x)$ be the auxiliary function defined by

$$\rho(x)=\sup\left\{r>0:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)\,\mathrm{d}y\le 1\right\}.$$

Function f is an element of BMO_L if there exists $C>0$ such that

$$\sup_{x\in\mathbb{R}^n}\sup_{r<\rho(x)}\frac{1}{|B|}\int_{B(x,r)}|f-f_B|\le C \text{ and } \sup_{x\in\mathbb{R}^n}\sup_{r\ge\rho(x)}\frac{1}{|B|}\int_{B(x,r)}|f|\le C,$$

where $f_B=|B|^{-1}\int_B f(y)\,\mathrm{d}y$. Let $\|f\|_{BMO_L(\mathbb{R}^n)}$ be the smallest C in the inequalities above. It is easy to verify that $\|f\|_{BMO(\mathbb{R}^n)}\le 2\|f\|_{BMO_L(\mathbb{R}^n)}$ for all $f\in BMO_L(\mathbb{R}^n)$.

A function $f \in L^1(\mathbb{R}^n)$ is said to be in $H_L^1(\mathbb{R}^n)$ if the semigroup maximal function $M^L f$ belongs to $L^1(\mathbb{R}^n)$. The norm of such a function is defined by

$$\|f\|_{H_L^1(\mathbb{R}^n)} = \|M^L f\|_{L^1(\mathbb{R}^n)},$$

where $M^L f(x) = \sup_{s>0}|T_s^L f(x)|$ and $\{T_s^L\}_{s>0} = \{\mathrm{e}^{-sL}\}_{s>0}$ is a C_0 contraction semigroup generated by Schrödinger operator $L = -\triangle + V$; see [4]. The authors [5] have proven that $BMO_L(\mathbb{R}^n)$ is the dual space of $H_L^1(\mathbb{R}^n)$.

The main results in this paper are stated as follows.

Theorem 1. *The operators μ_j^L are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$.*

Theorem 2. *The operators μ_j^L are bounded on $BMO_L(\mathbb{R}^n)$.*

Theorem 3. *The operators μ_j^L are bounded from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

2. Proof of the main theorems

First, we need the following lemmas.

Lemma 1 [7]. *There exists $l_0 > 0$ such that*

$$\frac{1}{C}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\left(1+\frac{|x-y|}{\rho(x)}\right)^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \sim \rho(y)$ if $|x-y| < C\rho(x)$.

Lemma 2 [7]. *For any $l > 0$, there exists $C_l > 0$ such that*

$$|K_j^L(x,y)| \leq \frac{C_l}{\left(1+\frac{|x-y|}{\rho(x)}\right)^l}\frac{1}{|x-y|^{n-1}}$$

and

$$|K_j^L(x,y) - K_j(x,y)| \leq C\frac{\rho(x)^{-1}}{|x-y|^{n-2}}.$$

Proof of Theorem 1. It suffices to show that

$$\mu_j^L f(x) \leq \mu_j f(x) + CMf(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{2.1}$$

where M denotes the standard Hardy–Littlewood operator.

Fix $x \in \mathbb{R}^n$ and let $r = \rho(x)$. Notice that

$$\begin{aligned}
\mu_j^L f(x) &\le \left(\int_0^r \left|\int_{|x-y|\le t} K_j^L(x,y) f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + \left(\int_r^\infty \left|\int_{|x-y|\le r} K_j^L(x,y) f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + \left(\int_r^\infty \left|\int_{r<|x-y|\le t} K_j^L(x,y) f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le \left(\int_0^r \left|\int_{|x-y|\le t} [K_j^L(x,y) - K_j(x,y)] f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + \left(\int_0^r \left|\int_{|x-y|\le t} K_j(x,y) f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + \left(\int_r^\infty \left|\int_{|x-y|\le r} K_j^L(x,y) f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + \left(\int_r^\infty \left|\int_{r<|x-y|\le t} K_j^L(x,y) f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&:= E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

For E_1, by Lemma 2, we have

$$E_1 \le C\left(\int_0^r \left|\frac{1}{r}\int_{|x-y|\le t} \frac{|f(y)|}{|x-y|^{n-2}}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \le CMf(x).$$

Obviously,

$$E_2 \le \mu_j f(x).$$

For E_3, using Lemma 2 again, we get

$$E_3 \le \left(\int_r^\infty \left|\int_{|x-y|\le r} \frac{|f(y)|}{|x-y|^{n-1}}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \le CMf(x).$$

It remains to estimate E_4. From Lemma 2, we obtain

$$\begin{aligned}
E_4 &\le C\left(\int_r^\infty \left|r\int_{r<|x-y|\le t} \frac{|f(y)|}{|x-y|^n}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le Cr\left(\int_r^\infty \left|\sum_{k=0}^{[\log_2 t/r]+1} (2^k r)^{-n} \int_{|x-y|\le 2^k r} |f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le Cr\left(\int_r^\infty \left|([\log_2 t/r]+1)Mf(x)\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le Cr\left(\int_r^\infty \frac{t}{r} Mf(x)^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \le CMf(x).
\end{aligned}$$

Thus, Theorem 1 is proved. □

Proof of Theorem 2. Let $f \in BMO_L(\mathbb{R}^n)$ and fix a ball $B = B(x_0, r)$. We consider two cases $r \geq \rho(x_0)$ and $r < \rho(x_0)$. If $r \geq \rho(x_0)$, write

$$f = f\chi_{\bar{B}} + f\chi_{(\bar{B})^c} = f_1 + f_2,$$

where $\bar{B} = B(x_0, 2r)$ and χ_S is the characteristic function of set S. By Theorem 1, we know that μ_j^L are bounded on $L^2(\mathbb{R}^n)$. We then have

$$\begin{aligned}\frac{1}{|B|}\int_B |\mu_j^L f_1(x)|\mathrm{d}x &\leq \left(\frac{1}{|B|}\int_{\bar{B}} |\mu_j^L f(x)|^2\mathrm{d}x\right)^{1/2}\\ &\leq C\left(\frac{1}{|\bar{B}|}\int_{\bar{B}} |f(x)|^2\mathrm{d}x\right)^{1/2}\\ &\leq C\|f\|_{BMO_L}.\end{aligned}$$

Let $x \in B$. From Lemma 1, $\rho(x) \leq Cr$. By Lemma 2, we get

$$\begin{aligned}\mu_j^L f_2(x) &= \left(\int_0^\infty \left|\int_{|x-y|\leq t} K_j^L(x,y) f_2(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\ &\leq \left(\int_0^\infty \left|\int_{r\leq|x-y|\leq t} |K_j^L(x,y) f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\ &\leq \left(\int_r^\infty \left|\rho(x)\int_{r\leq|x-y|\leq t} \frac{|f(y)|}{|x-y|^n}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\ &\leq C\rho(x)\left(\int_r^\infty \left|\sum_{k=0}^{[\log_2 t/r]+1} (2^k r)^{-n}\int_{|x-y|\leq 2^k r} |f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\\ &\leq C\rho(x)\left(\int_r^\infty \left|([\log_2 t/r]+1)\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \|f\|_{BMO_L}\\ &\leq C\frac{\rho(x)}{r}\|f\|_{BMO_L} \leq C\|f\|_{BMO_L}.\end{aligned} \tag{2.2}$$

Hence

$$\frac{1}{|B|}\int_B |\mu_j^L f_2(x)|\mathrm{d}x \leq C\|f\|_{BMO_L}.$$

If $r < \rho(x_0)$, we set

$$f = f\chi_{B^*} + f\chi_{(B^*)^c} = f_1^* + f_2^*,$$

where $B^* = B(x_0, 2\rho(x_0))$. Note that $\rho(x) \sim \rho(x_0)$ for any $x \in B$. Similar to (2.2), we have

$$\frac{1}{|B|}\int_B |\mu_j^L f_2^*(x)|\mathrm{d}x \leq C\|f\|_{BMO_L}.$$

It remains to prove that there exists a suitable constant C_B such that

$$\frac{1}{|B|}\int_B |\mu_j^L f_1^*(x) - C_B|\mathrm{d}x \leq C\|f\|_{BMO_L}. \tag{2.3}$$

The left of (2.3) is dominated by

$$\frac{1}{|B|}\int_B |\mu_j^L f_1^*(x) - \mu_j f_1^*(x)|\mathrm{d}x + \frac{1}{|B|}\int_B |\mu_j f_1^*(x) - C_B|\mathrm{d}x.$$

Let $x \in B$ and $B_{x,k} = B(x, 2^{2-k}\rho(x_0)),\ k = 0, 1, \ldots$. Note that $\rho(x) \sim \rho(x_0)$. Let $f(B) = \frac{1}{|B|}\int_B f(y)\mathrm{d}y$ and $\tilde{f}(B) = \frac{1}{|B|}\int_B |f(y)|\mathrm{d}y$. It is easy to see that $|f(B_{x,0})| + |\tilde{f}(B_{x,0})| \le C\|f\|_{BMO_L}$. Note that

$$|\tilde{f}(B_{x,k}) - \tilde{f}(B_{x,k-1})| + |f(B_{x,k}) - f(B_{x,k-1})| \le C\|f\|_{BMO},$$

so

$$|\tilde{f}(B_{x,k})| + |\tilde{f}(B_{x,k})| \le C(k+1)\|f\|_{BMO_L}.$$

From this, we have

$$\int_{B_{x,k}} |f(x)|\mathrm{d}x \le C(k+1)|B_{x,k}|\|f\|_{BMO_L}. \tag{2.4}$$

By Lemma 2 and (2.4), we obtain

$$\begin{aligned}
&|\mu_j^L f_1^*(x) - \mu_j f_1^*(x)| \\
&\le \left(\int_0^\infty \left|\int_{|x-y|<\min\{t,4\rho(x_0)\}} |K_j^L(x,y) - K_j(x,y)||f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le C\left(\int_0^{4\rho(x_0)} \left|\int_{|x-y|<t} |K_j^L(x,y) - K_j(x,y)||f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + C\left(\int_{4\rho(x_0)}^\infty \left|\int_{|x-y|<2\rho(x_0)} |K_j^L(x,y) - K_j(x,y)||f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le C\left(\int_0^{4\rho(x_0)} \left|\rho(x_0)^{-1}\int_{|x-y|<t} \frac{|f(y)|}{|x-y|^{n-2}}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + C\left(\int_{4\rho(x_0)}^\infty \left|\int_{|x-y|<2\rho(x_0)} \frac{|f(y)|}{|x-y|^{n-1}}\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le C\left(\int_0^{4\rho(x_0)} \left|\rho(x_0)^{-1}\sum_{k=-\infty}^{2}(2^k t)^{2-n}\int_{|x-y|<2^k t} |f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\quad + C\left(\int_{4\rho(x_0)}^\infty \left|\sum_{k=-\infty}^{2}(2^k\rho(x_0))^{1-n}\int_{|x-y|<2^k\rho(x_0)} |f(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\
&\le C\|f\|_{BMO_L}.
\end{aligned}$$

Thus

$$\frac{1}{|B|}\int_B |\mu_j^L f_1^*(x) - \mu_j f_1^*(x)|\mathrm{d}x \le C\|f\|_{BMO_L}.$$

We next need only to show

$$\frac{1}{|B|}\int_B |\mu_j f_1^*(x) - C_B|\mathrm{d}x \le C\|f\|_{BMO_L}.$$

Let $B_k^* = B(x_0, 2^{1-k}\rho(x_0)),\ k = 0, 1, \ldots, k_0$, where k_0 satisfies $2^{-k_0-1}\rho(x_0) \le r < 2^{-k_0}\rho(x_0)$. Note that $\mu_j f_1^* = \mu_j(f_1^* - f(B_{k_0}^*))$. Set

$$\begin{aligned} f_1^* - f(B_{k_0}^*) &= (f_1^* - f(B_{k_0}^*))\chi_{B_{k_0}^*} + (f_1^* - f(B_{k_0}^*)\chi_{B_0^*\setminus B_{k_0}^*} - f(B_{k_0}^*)\chi_{(B_0^*)^c} \\ &= f_{1,1} + f_{1,2} + f_{1,3}. \end{aligned}$$

By the $L^2(\mathbb{R}^n)$ boundedness of μ_j, we have

$$\begin{aligned} \frac{1}{|B|}\int_B |\mu_j f_{1,1}(x)|\mathrm{d}x &\le \left(\frac{1}{|B|}\int_B |\mu_j f_{1,1}(x)|^2\mathrm{d}x\right)^{1/2} \\ &\le C\left(\frac{1}{|B_{k_0}^*|}\int_{B_{k_0}^*} |f_1^* - f(B_{k_0}^*)|^2\mathrm{d}x\right)^{1/2} \\ &\le C\|f\|_{BMO}. \end{aligned}$$

Now consider the term $f_{1,2}$. Next , given $x \in B$, we estimate $I := |\mu_j f_{1,2}(x) - \mu_j f_{1,2}(x_0)|$. It is easy to see that

$$\begin{aligned} I &\le \left(\int_0^\infty \left|\int_{|x-y|\le t<|x_0-y|} |K_j(x,y) f_{1,2}(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\ &\quad + \left(\int_0^\infty \left|\int_{|x_0-y|\le t<|x-y|} |K_j(x_0,y) f_{1,2}(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\ &\quad + \left(\int_0^\infty \left|\int_{|x_0-y|\le t,|x-y|\le t} |(K_j(x,y) - K_j(x_0,y)) f_{1,2}(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Since the estimates for I_1 and I_2 follow along similar lines, we only consider I_1. Since $|x_0 - y| \sim |x - y|$, by Minkowski's inequality, we have

$$\begin{aligned} I_1 &\le C\int_{\mathbb{R}^n} \frac{|f_{1,2}(y)|}{|x-y|^{n-1}}\mathrm{d}y \left(\int_{|x_0-y|\le t<|x-y|} \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\ &\le Cr^{1/2}\int_{\mathbb{R}^n} \frac{|f_{1,2}(y)|}{|x-y|^{n+1/2}}\mathrm{d}y \end{aligned}$$

$$\le C\sum_{k=0}^{k_0-1}\frac{2^{(k-k_0)/2}}{|B_k^*|}\int_{B_k^*}(|f(y)-f(B_k^*)|+|f(B_k^*)-f(B_{k_0}^*)|)\mathrm{d}y$$
$$\le C\sum_{k=0}^{k_0-1}(k_0-k+1)2^{(k-k_0)/2}\|f\|_{BMO}\le C\|f\|_{BMO}.$$

For I_3, again by Minkowski's inequality, we get

$$\begin{aligned}
I_3 &\le Cr\int_{\mathbb{R}^n}\frac{|f_{1,2}(y)|}{|x-y|^n}\mathrm{d}y\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\le Cr\int_{\mathbb{R}^n}\frac{|f_{1,2}(y)|}{|x-y|^{n+1}}\mathrm{d}y\\
&\le C\sum_{k=0}^{k_0-1}\frac{2^{(k-k_0)}}{|B_k^*|}\int_{B_k^*}(|f(y)-f(B_k^*)|+|f(B_k^*)-f(B_{k_0}^*)|)\mathrm{d}y\\
&\le C\sum_{k=0}^{k_0-1}(k_0-k+1)2^{(k-k_0)}\|f\|_{BMO}\\
&\le C\|f\|_{BMO}.
\end{aligned}$$

Thus

$$\frac{1}{|B|}\int_B|\mu_j f_{1,2}(x)-\mu_j f_{1,2}(x_0)|\mathrm{d}x\le C\|f\|_{BMO_L}.$$

For the third term, similar to above arguments, by (2.4), we have

$$\begin{aligned}
&|\mu_j f_{1,3}(x)-\mu_j f_{1,3}(x_0)|\\
&\quad\le\ |f(B_{k_0}^*)|\left(\int_0^\infty\left|\int_{|x_0-y|\le t<|x-y|}|K_j(x_0,y)\chi_{(B_0^*)^c}(y)|\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\qquad+|f(B_{k_0}^*)|\left(\int_0^\infty\left|\int_{|x-y|\le t<|x_0-y|}|K_j(x,y)\chi_{(B_0^*)^c}(y)|\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\qquad+|f(B_{k_0}^*)|\left(\int_0^\infty\left|\int_{|x_0-y|\le t,|x-y|\le t}|(K_j(x,y)\right.\right.\\
&\qquad\left.\left.-K_j(x_0,y))\chi_{(B_0^*)^c}(y)\ |\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\quad\le\ C|f(B_{k_0}^*)|\left(r^{1/2}\int_{|x-y|>\rho(x_0)}\frac{1}{|x-y|^{n+1/2}}\mathrm{d}y\right.\\
&\qquad\left.+\ r\int_{|x-y|>\rho(x_0)}\frac{1}{|x-y|^{n+1}}\mathrm{d}y\right)\\
&\quad\le\ C(k_0+1)2^{-k_0/2}\|f\|_{BMO_L}\\
&\quad\le\ C\|f\|_{BMO_L}.
\end{aligned}$$

Therefore

$$\frac{1}{|B|}\int_B |\mu_j f_{1,3}(x) - \mu_j f_{1,3}(x_0)|\mathrm{d}x \le C\|f\|_{BMO_L}.$$

Thus, the proof of Theorem 2 is completed. □

Proof of Theorem 3. From the atom decomposition theory (see [4]), we only need to prove that there exists a constant $C > 0$ such that for any H^1_L atom $a(x)$, $\|\mu^L_j a\|_{L^1(\mathbb{R}^n)} \le C$. A function $a(x)$ is said to be H^1_L atom if it satisfies the following conditions: (i) supp $a \subset B(x_0, r)$, (ii) $\|a\|_2 \le |B|^{-1/2}$ and (iii) if $r < \rho(x_0)$, then $\int_{B(x_0,r)} a(x)\mathrm{d}x = 0$.

Let supp $a \subset B(x_0, r)$ and $B^* = B(x_0, 2r)$. We then have

$$\begin{aligned}\|\mu^L_j a\|_{L^1(\mathbb{R}^n)} &= \int_{B^*} \mu^L_j a(x)\mathrm{d}x + \int_{(B^*)^c} \mu^L_j a(x)\mathrm{d}x\\ &= \mathrm{II}_1 + \mathrm{II}_2.\end{aligned}$$

For II_1, using L^2 boundedness of μ_j, we get

$$\int_{B^*} \mu^L_j a(x)\mathrm{d}x \le \left(\int_{B^*} |\mu^L_j a(x)|^2\mathrm{d}x\right)^{1/2} |B^*|^{1/2} \le C.$$

To estimate II_2, we consider two cases for r.

If $r \ge \rho(x_0)$, note that $r \le |x-y| \sim |x_0 - x|$ for $x \in (B^*)^c$ and $y \in B$. By Lemma 1, we obtain

$$\begin{aligned}\mathrm{II}_2 &\le \int_{(B^*)^c}\left(\int_0^\infty \left|\int_{|x-y|\le t} |K^L_j(x,y)a(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}x\\
&\le \int_{(B^*)^c}\left(\int_0^\infty \left|\rho(x)^{2(l_0+1)}\int_{r\le|x-y|\le t} |x-y|^{-n-2l_0-1}|a(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}x\\
&\le C\int_{(B^*)^c}\left(\int_r^\infty \left|\rho(x_0)^2\int_{|x-y|\le t} |x-x_0|^{-n-2l_0-1+2l_0}|a(y)|\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}x\\
&\le C\int_{(B^*)^c} |x-x_0|^{-n-1}\rho(x_0)^2\left(\int_r^\infty \frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}x\\
&\le C\left(\frac{\rho(x_0)}{r}\right)^2 \le C. \qquad (2.5)\end{aligned}$$

□

If $r < \rho(x_0)$, we have

$$\begin{aligned}\mathrm{II}_2 &= \int_{2r<|x-x_0|<2\rho(x_0)}\left(\int_0^\infty \left|\int_{|x-y|\le t} K^L_j(x,y)a(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}x\\
&\quad + \int_{2\rho(x_0)\le|x-x_0|}\left(\int_0^\infty \left|\int_{|x-y|\le t} K^L_j(x,y)a(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}x\\
&:= \mathrm{II}_{2,1} + \mathrm{II}_{2,2}.\end{aligned}$$

Similar to the proof of (2.5), we have

$$\mathrm{II}_{2,2} \le C.$$

For $\mathrm{II}_{1,1}$, we have

$$\begin{aligned}\mathrm{II}_{2,1} \le & \int_{2r<|x-x_0|<2\rho(x_0)} \left(\int_0^{\rho(x_0)} \left| \int_{|x-y|\le t} (K_j^L(x,y) - K_j(x,y))a(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \mathrm{d}x \\ & + \int_{2r<|x-x_0|<2\rho(x_0)} \left(\int_{\rho(x_0)}^{\infty} \left| \int_{|x-y|\le t} K_j^L(x,y)a(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \mathrm{d}x \\ & + \int_{2r<|x-x_0|<2\rho(x_0)} \left(\int_0^{\rho(x_0)} \left| \int_{|x-y|\le t} K_j(x,y)a(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \mathrm{d}x \\ & := \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3.\end{aligned}$$

For III_1, since $\rho(x) \sim \rho(x_0)$, we obtain

$$\begin{aligned}\mathrm{III}_1 & \le C \int_{2r<|x-x_0|<2\rho(x_0)} \left(\int_0^{\rho(x_0)} \left| \rho(x_0)^{-1} \int_{|x-y|\le t} \frac{|a(y)|}{|x-y|^{n-2}}\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \mathrm{d}x \\ & \le C \int_{2r<|x-x_0|<2\rho(x_0)} \left(\int_0^{\rho(x_0)} \left| \rho(x_0)^{-1} t^{3/2} \int_{|x-y|\le t} \frac{|a(y)|}{|x-x_0|^{n-\frac{1}{2}}}\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \mathrm{d}x \\ & \le C\rho(x_0)^{-1/2} \int_{2r<|x-x_0|<2\rho(x_0)} |x-x_0|^{-(n-\frac{1}{2})}\mathrm{d}x \le C.\end{aligned}$$

For III_2, we get

$$\begin{aligned}\mathrm{III}_2 & \le \int_{2r<|x-x_0|<2\rho(x_0)} \left(\int_{\rho(x_0)}^{\infty} \left| \int_{|x-y|\le t} |K_j^L(x,y)||a(y)|\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \mathrm{d}x \\ & \le C \int_{2r<|x-x_0|<2\rho(x_0)} \left(\int_{\rho(x_0)}^{\infty} \left| \int_{|x-y|\le t} \frac{|a(y)|}{|x-x_0|^{n-1}}\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \mathrm{d}x \\ & \le C\rho(x_0)^{-1} \int_{2r<|x-x_0|<2\rho(x_0)} |x-x_0|^{-(n-1)}\mathrm{d}x \le C.\end{aligned}$$

From the arguments in pages 178–179 in [2], we have

$$\mathrm{III}_3 \le \int_{2r<|x-x_0|} \left(\int_0^{\infty} \left| \int_{|x-y|\le t} K_j(x,y)a(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \mathrm{d}x \le C.$$

Thus, Theorem 3 is proved.

Acknowledgement

This research was supported by the NNSF (111271024) of China.

References

[1] Benedek A, Calderón A P and Panzone R, Convolution operators on Banach space valued functions, *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962) 356–365
[2] Ding Y, Lu S and Xue Q, Marcinkiewicz integral on Hardy spaces, *Integr. Equ. Oper. Theory.* **42** (2002) 174–182
[3] Dong J and Liu H, The BMO_L space and Riesz transforms associated with Schrödinger operators, *Acta Math. Sin. (Engl. Ser.)* **26** (2010) 659–668
[4] Dziubański J and Zienkiewicz J, Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality, *Rev. Mat. Iber.* **15** (1999) 279–296
[5] Dziubański J, Garrigós G, Torrea J and Zienkiewicz J, *BMO* spaces related to Shrödinger operators with potentials satisfying a reverse Hölder inequality, *Math. Z.* **249** (2005) 249–356
[6] Fefferman C and Stein E, H^p-space of serval variables, *Acta Math* **46** (1972) 137–193
[7] Shen Z, L^p estimates for Schrödinger operators with certain potentials, *Ann. Inst. Fourier. Grenoble* **45** (1995) 513–546
[8] Stein E, On the functions of Littlewood–Paley, Lusin, and Marcinkiewicz, *Trans. Amer. Math. Soc.* **88** (1958) 430–466
[9] Torchinsky A and Wang S, A note on Marcinkiewicz integral, *Colloq. Math.* **47** (1990) 235–243