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On *IA*-automorphisms that fix the centre element-wise

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Abstract. Let G be a group. An automorphism of G is called an *IA*-automorphism if it induces the identity mapping on $G/\gamma_2(G)$, where $\gamma_2(G)$ is the commutator subgroup of G. Let $IA_z(G)$ be the group of those *IA*-automorphisms, which fix the centre element-wise and let $\mathrm{Autcent}(G)$ be the group of central automorphisms, the automorphisms that induce the identity mapping on the central quotient. It can be observed that $\mathrm{Autcent}(G) = C_{\mathrm{Aut}(G)}(IA_z(G))$. We prove that $IA_z(G)$ and $IA_z(H)$ are isomorphic for any two finite isoclinic groups G and H. Also, for a finite p-group G, we give a necessary and sufficient condition to ensure that $IA_z(G) = \mathrm{Autcent}(G)$.

Keywords. *IA*-automorphism; class-preserving automorphism; isoclinism; central automorphism.

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1. Introduction

Let G be any group. By $\gamma_2(G)$ and $Z(G)$, we denote the commutator subgroup and the center of G respectively. For $x, y \in G$, x^y denotes the conjugate of x by y i.e., $y^{-1}xy$. Let H be a subgroup of G, then $C_G(H)$ denotes the centralizer of H in G. Let A be an abelian group, then $\mathrm{Hom}(G, A)$ denotes the group of all homomorphisms from G to A. By $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$, we denote the group of all automorphisms and the group of all inner automorphisms respectively. Let σ be an automorphism of G. Following Bachmuth [2], we call σ an *IA*-automorphism if $x^{-1}\sigma(x) \in \gamma_2(G)$ for each $x \in G$. He explained that 'the letters *I* and *A* are used to remind the reader that these automorphisms are those which induce the identity automorphism in the abelianized group'. If $x^{-1}\sigma(x) \in Z(G)$ for each $x \in G$, then we say that σ is a central automorphism. If σ preserves all the conjugacy classes of G, then it is called a class-preserving automorphism. The set of all *IA*-automorphisms, the central automorphisms and the class-preserving automorphisms of G form normal subgroups of $\mathrm{Aut}(G)$ and are denoted by $IA(G)$, $\mathrm{Autcent}(G)$ and $\mathrm{Aut}_c(G)$ respectively. To avoid confusion, we want to mention here that some authors use the notation $\mathrm{Aut}_c(G)$ for the group of all central automorphisms of G (e.g. [4,5]). The set of all *IA*-automorphisms that fix the centre element-wise forms a normal subgroup of $IA(G)$ and is denoted by $IA_z(G)$. It is a well-known fact that $\mathrm{Autcent}(G) = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$. We note that $\mathrm{Autcent}(G)$ centralizes not only $\mathrm{Inn}(G)$, but also $IA_z(G)$. Thus $\mathrm{Autcent}(G) = C_{\mathrm{Aut}(G)}(IA_z(G))$.

Yadav [9] proved that if two finite groups G and H are isoclinic (see §2 for definition), then $\mathrm{Aut}_c(G) \cong \mathrm{Aut}_c(H)$. Notice that any class-preserving automorphism is an IA-automorphism which fixes the centre element-wise and hence $\mathrm{Aut}_c(G) \le IA_z(G)$. The motivation behind this article is to extend the result of Yadav to the group of IA-automorphisms that fix the centre element-wise. Our main result is as follows.

Theorem A. *Let G and H be two finite isoclinic groups. Then there exists an isomorphism $\alpha : IA_z(G) \to IA_z(H)$ such that $\alpha(\mathrm{Aut}_c(G)) = \mathrm{Aut}_c(H)$.*

We apply Theorem A to prove the following result:

Theorem B. *Let G be a finite p-group.*

(1) *Then $IA_z(G) = \mathrm{Autcent}(G)$ if and only if $\gamma_2(G) = Z(G)$.*
(2) *If the nilpotency class of G is 2, then $IA_z(G) = \mathrm{Inn}(G)$ if and only if $\gamma_2(G)$ is cyclic.*

We note that Cheng [3] proved that if $\gamma_2(G)$ is cyclic, then $IA_z(G) = \mathrm{Inn}(G)$ for any finite p-group G with p an odd prime.

The following corollary is a result of Curran and McCaughan [4], and is a consequence of Theorem B.

COROLLARY 1.1

Let G be a finite p-group. Then $\mathrm{Inn}(G) = \mathrm{Autcent}(G)$ if and only if $\gamma_2(G) = Z(G)$ and $\gamma_2(G)$ is cyclic.

2. Preliminary definitions and results

The concept of isoclinism was first introduced by Hall [7]. We say that two groups G and H are isoclinic if there exist isomorphisms ϕ of $G/Z(G)$ onto $H/Z(H)$ and θ of $\gamma_2(G)$ onto $\gamma_2(H)$, such that the following diagram commutes:

$$\begin{array}{ccc} G/Z(G) \times G/Z(G) & \xrightarrow{\phi\times\phi} & H/Z(H) \times H/Z(H) \\ {\scriptstyle a_G}\downarrow & & {\scriptstyle a_H}\downarrow \\ \gamma_2(G) & \xrightarrow{\theta} & \gamma_2(H), \end{array}$$

where $a_G(xZ(G), yZ(G)) = [x, y]$ for $x, y \in G$ and $a_H(kZ(H), lZ(H)) = [k, l]$ for $k, l \in H$. The pair (ϕ, θ) is called an isoclinism of G onto H.

The following lemma is due to [8].

Lemma 2.1 (*Lemma* 1.5 *of* [8]). *Let G and H be isoclinic groups and (ϕ, θ) be an isoclinism of G onto H. Then,*

(1) *$\phi(gZ(G)) = \theta(g)Z(H)$ for $g \in \gamma_2(G)$, and*
(2) *$\theta(\gamma_2(G) \cap Z(G)) = \gamma_2(H) \cap Z(H)$.*

Using Lemma 2.1, we prove the following lemma, which will be used in the proof of Theorem A.

Lemma 2.2 *Let G and H be isoclinic groups and (ϕ, θ) be an isoclinism of G onto H. Then for $b \in \gamma_2(G)$ and $a \in G$ such that $\phi(aZ(G)) = kZ(H)$, we have $\theta(b^a)^{k^{-1}} = \theta(b)$.*

Proof. Let $\phi(bZ(G)) = lZ(H)$. Then

$$\theta(b^a)^{k^{-1}} = k\theta([a, b^{-1}])\theta(b)k^{-1} = k[k, l^{-1}]\theta(b)k^{-1} = lkl^{-1}\theta(b)k^{-1}.$$

From Lemma 2.1, $l^{-1}\theta(b) \in Z(H)$. Hence by the preceding equation, $\theta(b^a)^{k^{-1}} = \theta(b)$. □

The following theorem was proved by Hall [7].

Theorem 2.3. *In every family of isoclinic groups there exists a group G such that $Z(G) \leq \gamma_2(G)$.*

We say that a group G is purely non-abelian if it does not have any abelian direct factor. The following well-known result was proved by Adney and Yen [1].

Theorem 2.4. *Let G be a purely non-abelian group. Then there is a one-to-one correspondence between* Autcent(G) *and* Hom$(G/\gamma_2(G), Z(G))$.

3. Proofs of theorems A and B

Proof of Theorem A. Let (ϕ, θ) be an isoclinism of G onto H. Let $\sigma \in IA_z(G)$. Define a map τ_σ by $\tau_\sigma(h) = h\theta(g^{-1}\sigma(g))$ for $h \in H$, where g is given by $\phi^{-1}(hZ(H)) = gZ(G)$. Notice that τ_σ is well-defined because σ fixes $Z(G)$ elementwise. Next we show that $\tau_\sigma \in IA_z(H)$. Let $h_1, h_2 \in H$ and $\phi^{-1}(h_i Z(H)) = g_i Z(G)$ for $i = 1, 2$. Then,

$$\begin{aligned} \tau_\sigma(h_1h_2) &= h_1h_2\theta(g_2^{-1}g_1^{-1}\sigma(g_1g_2)) \\ &= h_1h_2\theta(g_2^{-1}g_1^{-1}\sigma(g_1)g_2g_2^{-1}\sigma(g_2)) \\ &= h_1h_2\theta(g_2^{-1}g_1^{-1}\sigma(g_1)g_2)h_2^{-1}h_2\theta(g_2^{-1}\sigma(g_2)). \end{aligned}$$

Applying Lemma 2.2, we obtain

$$\begin{aligned} \tau_\sigma(h_1h_2) &= h_1\theta(g_1^{-1}\sigma(g_1))h_2\theta(g_2^{-1}\sigma(g_2)) \\ &= \tau_\sigma(h_1)\tau_\sigma(h_2). \end{aligned}$$

Thus τ_σ is a homomorphism.

Now we show that τ_σ is a bijection. Since H is finite, it is enough to show that it is an injection. Let $\tau_\sigma(h) = 1$ and $\phi^{-1}(hZ(H)) = gZ(G)$. Then $h\theta(g^{-1}\sigma(g)) = 1$. This implies that $h \in \gamma_2(H)$ and hence $g \in \gamma_2(G)Z(G)$. Without loss of generality we can assume that $g \in \gamma_2(G)$. Then we have $h\theta(g^{-1})\theta(\sigma(g)) = 1$. However, by Lemma 2.1, $h\theta(g^{-1}) \in Z(H)$, so that $\theta(\sigma(g)) \in Z(H) \cap \gamma_2(H)$. Using Lemma 2.1 again, $\sigma(g) \in Z(G)$, which implies $g \in Z(G)$. Hence $\sigma(g) = g$, as σ fixes the centre element-wise. Thus $h = 1$ and hence τ_σ is a bijection. It is easy to see that τ_σ fixes $Z(H)$ element-wise, so that $\tau_\sigma \in IA_z(H)$.

Define α as $\alpha(\sigma) = \tau_\sigma$. We show that α is an isomorphism. Let $\sigma_1, \sigma_2 \in IA_z(G)$ and let $h \in H$ with $\phi^{-1}(hZ(H)) = gZ(G)$. Then, $\tau_{\sigma_1}\tau_{\sigma_2}(h) = \tau_{\sigma_1}(h\theta(g^{-1}\sigma_2(g)))$. Using Lemma 2.1, we have $\phi^{-1}(h\theta(g^{-1}\sigma_2(g))Z(H)) = \sigma_2(g)Z(G)$. Therefore $\tau_{\sigma_1}\tau_{\sigma_2}(h) = h\theta(g^{-1}\sigma_2(g))\theta(\sigma_2(g^{-1})\sigma_1\sigma_2(g))$, which equals $h\theta(g^{-1}\sigma_1\sigma_2(g))$, which is $\tau_{\sigma_1\sigma_2}(h)$. Since h is arbitrary, we have $\tau_{\sigma_1\sigma_2} = \tau_{\sigma_1}\tau_{\sigma_2}$ and hence α is a homomorphism.

To show that α is a bijection, notice that (ϕ^{-1}, θ^{-1}) is an isoclinism of H onto G. So given a $\tau \in IA_z(H)$, in a similar manner as above, $\sigma_\tau \in IA_z(G)$ can be defined. Define $\beta\colon IA_z(H) \to IA_z(G)$ as $\beta(\tau) = \sigma_\tau$. Let $g \in G$ and $\phi(gZ(G)) = hZ(H)$. Then

$$\begin{aligned}\beta\alpha(\sigma)(g) &= \beta(\tau_\sigma)(g)\\ &= g\theta^{-1}(h^{-1}\tau_\sigma(h))\\ &= g\theta^{-1}(h^{-1}h\theta(g^{-1}\sigma(g)))\\ &= \sigma(g).\end{aligned}$$

Since g is arbitrary, this shows that $\beta\alpha(\sigma) = \sigma$. However, as σ is arbitrary, $\beta\alpha = 1$. Similarly $\alpha\beta = 1$. Thus α is an isomorphism between $IA_z(G)$ and $IA_z(H)$.

It now remains to show that $\alpha(\mathrm{Aut}_c(G)) = \mathrm{Aut}_c(H)$. Let $\sigma \in \mathrm{Aut}_c(G)$. To show that $\tau_\sigma \in \mathrm{Aut}_c(H)$, let $h \in H$ with $\phi^{-1}(hZ(H)) = gZ(G)$. Then $\tau_\sigma(h) = h\theta(g^{-1}\sigma(g))$. However, σ is a class-preserving automorphism and so there exists an $a \in G$ such that $\sigma(g) = a^{-1}ga$. Hence, we have $\tau_\sigma(h) = h\theta([g, a])$. Suppose $\phi(aZ(G)) = bZ(H)$. Then by commutativity of the diagram in the definition of isoclinism we get $\tau_\sigma(h) = h[h, b]$, i.e., $\tau_\sigma(h) = b^{-1}hb$. This shows that $\tau_\sigma \in \mathrm{Aut}_c(H)$. So we have $\alpha(\mathrm{Aut}_c(G)) \leq \mathrm{Aut}_c(H)$. Similarly we can show that $\beta(\mathrm{Aut}_c(H)) \leq \mathrm{Aut}_c(G)$. We have already shown that $\alpha\beta = \beta\alpha = 1$, therefore $\alpha(\mathrm{Aut}_c(G)) = \mathrm{Aut}_c(H)$.

Proof of Theorem B.

(1) First suppose that $\gamma_2(G) = Z(G)$, then clearly $IA(G) = \mathrm{Autcent}(G)$. Now each central automorphism fixes $\gamma_2(G) = Z(G)$ element-wise, therefore each IA-automorphism fixes $Z(G)$ element-wise. This shows that $IA_z(G) = \mathrm{Autcent}(G)$.

Conversely, suppose $IA_z(G) = \mathrm{Autcent}(G)$. Then, since $\mathrm{Inn}(G) \leq IA_z(G)$, G must be a group of class 2. Now, we prove that G is purely non-abelian. For this we rely on the technique used by Yadav of (Proposition 3.4 of [11]). Suppose on the contrary, $G = A \times N$, where A is non-trivial and abelian and N is purely non-abelian. Clearly $IA_z(G) \cong IA_z(N)$. Next, $|\mathrm{Autcent}(G)| > |\mathrm{Autcent}(A)||\mathrm{Autcent}(N)| \geq |\mathrm{Autcent}(N)|$. Therefore,

$$|IA_z(G)| = |IA_z(N)| \leq |\mathrm{Autcent}(N)| < |\mathrm{Autcent}(G)|,$$

which is a contradiction to the assumption that $IA_z(G) = \mathrm{Autcent}(G)$. Hence G is purely non-abelian. Now by Theorem 2.3, there exists a group H isoclinic to G such that $Z(H) \leq \gamma_2(H)$. Since G is of class 2, H is also of class 2 and therefore $\gamma_2(H) = Z(H)$. By what we just proved, $IA_z(H) = \mathrm{Autcent}(H)$. Now, since $Z(H) \leq \gamma_2(H)$, H is purely non-abelian, and hence by Theorem 2.4,

$$\begin{aligned}|IA_z(H)| &= |\mathrm{Hom}(H/\gamma_2(H), Z(H))|\\ &= |\mathrm{Hom}(H/Z(H), \gamma_2(H))| = |\mathrm{Hom}(G/Z(G), \gamma_2(G))|.\end{aligned}$$

However, by Theorem A, $|IA_z(H)| = |IA_z(G)|$, which, by the hypothesis and Theorem 2.4 equals $|\mathrm{Hom}(G/\gamma_2(G), Z(G))|$. Therefore, we have $|\mathrm{Hom}(G/Z(G), \gamma_2(G))| = |\mathrm{Hom}(G/\gamma_2(G), Z(G))|$. It now follows from Lemma 2.8 of [5] that $\gamma_2(G) = Z(G)$.

(2) By Theorem 2.3, there exists a group H isoclinic to G such that $Z(H) \leq \gamma_2(H)$. Since G is of class 2, H is also of class 2 and $\gamma_2(H) = Z(H)$. Thus $IA_z(H) = \mathrm{Autcent}(H)$ and hence by Theorem 2.4,

$$|IA_z(H)| = |\mathrm{Hom}(H/Z(H), \gamma_2(H))|,$$

which in turn equals $|\mathrm{Hom}(G/Z(G), \gamma_2(G))|$ using the definition of isoclinism. Applying Theorem A, we also have that $|IA_z(H)| = |IA_z(G)|$.

Now suppose that $IA_z(G) = \mathrm{Inn}(G)$. Then

$$|IA_z(G) = |G/Z(G)| = |\mathrm{Hom}(G/Z(G), \gamma_2(G))|.$$

However, since G is of class 2, the exponents of $G/Z(G)$ and $\gamma_2(G)$ are the same. Hence the preceding equality gives that $\gamma_2(G)$ is cyclic.

Conversely, suppose that $\gamma_2(G)$ is cyclic. Then

$$|\mathrm{Hom}(H/Z(H), \gamma_2(H))| = |H/Z(H)| = |G/Z(G)| = |\mathrm{Inn}(G)|,$$

since $\gamma_2(G) \cong \gamma_2(H)$. This gives that $IA_z(G) = \mathrm{Inn}(G)$.

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