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Gaussian curvature on hyperelliptic Riemann surfaces

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Abstract. Let C be a compact Riemann surface of genus $g \geq 1$, $\omega_1, \ldots, \omega_g$ be a basis of holomorphic 1-forms on C and let $H = (h_{ij})_{i,j=1}^{g}$ be a positive definite Hermitian matrix. It is well known that the metric defined as $\mathrm{d}s_H^2 = \sum_{i,j=1}^{g} h_{ij}\omega_i \otimes \overline{\omega_j}$ is a Kähler metric on C of non-positive curvature. Let $K_H : C \to \mathbb{R}$ be the Gaussian curvature of this metric. When C is hyperelliptic we show that the hyperelliptic Weierstrass points are non-degenerated critical points of K_H of Morse index $+2$. In the particular case when H is the $g \times g$ identity matrix, we give a criteria to find local minima for K_H and we give examples of hyperelliptic curves where the curvature function K_H is a Morse function.

Keywords. Hyperelliptic curve; Weierstrass points; Gaussian curvature.

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1. Introduction and statement of results

In this paper the phrases compact Riemann surface and smooth algebraic curve or just curve, will be used interchangeably.

Let C be a compact Riemann surface of genus $g \geq 1$ and let $H^0(C, \Omega_C)$ be its space of holomorphic 1-forms. If $\omega_1, \ldots, \omega_g$ is a basis of $H^0(C, \Omega_C)$ and $H = (h_{ij})_{i,j=1}^{g}$ is a positive definite Hermitian matrix, it is well known that the metric defined as

$$\mathrm{d}s_H^2 = \sum_{i,j=1}^{g} h_{ij}\omega_i \otimes \overline{\omega_j}$$

is a Kähler metric on C of non-positive curvature. Let $K_H : C \to \mathbb{R}$ be the Gaussian curvature of this metric. In local coordinates we can give an expression for K_H. Let $U \subset C$ be an open subset of C and let $z : U \subset C \to \mathbb{C}$ be a local holomorphic coordinate on C. Let $\omega_i = f_i \mathrm{d}z$ be the local expression of ω_i in the coordinate z, where $f_i : U \to \mathbb{C}$ is a holomorphic function. Denote $f_i(z) := f_i \circ z^{-1} : z(U) \to \mathbb{C}$. In this local coordinate z we have that $\mathrm{d}s_H^2 = \sum_{i,j=1}^{g} h_{ij}\omega_i \otimes \overline{\omega_j} = \rho^2(z)\mathrm{d}z \otimes \mathrm{d}\bar{z}$, where $\rho^2(z) = \sum_{i,j=1}^{g} h_{ij} f_i(z)\overline{f_j}(z)$ is the conformality factor of this metric. Denote $\mathbf{f}(z) = (f_1(z), \ldots, f_g(z))$, then $\rho^2(z) = \langle \mathbf{f}(z), \mathbf{f}(z) \rangle$, where $\langle , \rangle$ is the hermitian inner product on $\mathbb{C}^g$ induced by the hermitian matrix $H = (h_{ij})$. The Gaussian curvature K_H

of the metric ds^2_H is computed by the formula $K_H(z) = \frac{-(\triangle(\log\rho(z))}{\rho^2(z)}$, where $\triangle$ is the Laplace operator in real coordinate $u = \mathrm{Re}(z)$, $v = \mathrm{Im}(z)$ (see p. 77 of [5]). Thus in the coordinate $z = u + iv$, we have $\triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = 4\partial\bar{\partial}$, where $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, then we have that $K_H(z) = \frac{-4}{\rho^2(z)}\partial\bar{\partial}(\log(\rho(z)))$. Since $\partial\bar{\partial}(\rho^2(z)) = \langle \mathbf{f}'(z), \mathbf{f}'(z)\rangle$, $\frac{\partial}{\partial z}\rho^2(z) = \langle \mathbf{f}'(z), \mathbf{f}(z)\rangle$, $\frac{\partial}{\partial \bar{z}}\rho^2(z) = \langle \mathbf{f}(z), \mathbf{f}'(z)\rangle$, where $\mathbf{f}'(z) = (f_1'(z), \ldots, f_g'(z))$. In the coordinate z, $K_H(z)$ has the following local expression:

$$K_H(z) = \frac{-4}{\rho^4(z)}[\langle \mathbf{f}(z), \mathbf{f}(z)\rangle\langle \mathbf{f}'(z), \mathbf{f}'(z)\rangle - |\langle \mathbf{f}'(z), \mathbf{f}(z)\rangle|^2] \leq 0. \quad (1)$$

The curvature function K_H has zeroes if and only if C is hyperelliptic in which case the zeroes are exactly the hyperelliptic Weierstrass points (see for example [8]) and these points are global maxima.

A hyperelliptic Riemann surface of genus g is biholomorphically equivalent to a complex affine plane curve in $\mathbb{C}^2$ defined by an equation $y^2 = f(x)$, where $f(x)$ is a complex polynomial of degree $2g+1$ or $2g+2$ with distinct roots. In this work we study the nature of the critical points of K_H for hyperelliptic curves. I do not know if this has been done before, so the purpose of this work is make the first steps toward this direction. The first case where we think is interesting in studying the Gaussian curvature K_H is in genus two. The moduli space of curves of genus two, say $\mathcal{M}_2$, has interesting geometric properties. For example, when the Bergman metric is considered on a hyperelliptic curve, the study of extremal properties of the Laplacian in this metric on $\mathcal{M}_2$ is related to a smooth functional $\mathcal{F}$ on $\mathcal{M}_2$, which can be considered as a function on the Siegel upper half-space $\mathcal{H}$ invariant under the action of the symplectic group $Sp(4, \mathbb{Z})$. In [7], the authors studied properties of the determinant of the Laplacian in the Bergman metric on $\mathcal{M}_2$ by considering the critical points of $\mathcal{F}$ as function on $\mathcal{H}$, and these critical points correspond to hyperelliptic Riemann surfaces of genus 2 with largest automorphism group. The study of the asymptotic behavior for families of metrics in Riemann surfaces is also very interesting to understand the limits and behavior of metrics under degenerations of Riemann surfaces (see, for example, [6]). We think that studying the nature of critical points of the Gaussian curvature K_H can help in understanding some problems in connection with the geometry of $\mathcal{M}_2$ (see for example, [1, 3, 10, 11]).

Given any positive definite hermitian matrix $H = (h_{ij})^g_{i,j=1}$ we show that the hyperelliptic Weierstrass points are non-degenerate critical points of K_H of Morse index $+2$. In the particular case when H is the $g \times g$ identity matrix we write $K_H = K$ and in this case we give a criteria to find local minima for K and we give examples of hyperelliptic curves where the curvature function K is a Morse function.

The paper is organized as follows. In first section we show in Lemma 3 that given any positive definite hermitian matrix H the Weierstrass points of a hyperelliptic Riemann surface of genus $g \geq 2$ are non-degenerate critical points of Morse index $+2$ of the corresponding Gaussian curvature function K_H. In the second and third sections we consider the case when H is the $g \times g$ identity matrix. In Lemma 4, we give a criteria to find local minima for K; these points are related to the roots of the derivative of the polynomial $f(x)$ that defines the hyperelliptic curve $y^2 = f(x)$. In the last section we make some numerical computations to study other critical points of the curvature K for some classes of hyperelliptic Riemann surfaces. In particular for Riemann surfaces of genus two with largest automorphism group $\mathbb{Z}_5$, D_6, S_4 we show that K is a Morse function. Also we discuss some of the difficulties in studying the nature of all critical points of K_H for families of hyperelliptic curves depending on 1-complex parameters and when H is a matrix different that the identity matrix.

2. The Hessian of K_H at the Weierstrass points

Let M be a smooth manifold of dimension m and $F : M \to \mathbb{R}$ a real-valued smooth function with a critical point $x \in M$. Consider in a local coordinate system $(x_1, \ldots, x_m)$ around the point x the matrix of second derivatives of F, $\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)$. This matrix defines a bilinear form $\mathrm{Hess}_x(F)$ on the tangent space $T_x M$ called the Hessian of F at x. A critical point x is non-degenerate if $\mathrm{Hess}_x(F)$ is non singular. The Morse index of a non-degenerate critical point x is the number of negative eigenvalues of $\mathrm{Hess}_x(F)$. If a smooth function has only non-degenerate critical points, such function is called a Morse function. A well known result states that if M is compact and $F : M \to \mathbb{R}$ is a Morse function, and $I_q(F)$ is the number of critical points of Morse index q of F and $\chi(M)$ is the Euler characteristic of M, then $\chi(M) = \sum_{q=0}^{m} (-1)^q I_q(F)$ (see p. 192 of [4]).

2.1 *Hyperelliptic Riemann surfaces*

A compact Riemann surface M of genus $g \geq 2$ is said to be hyperelliptic if there exists a meromorphic function $h : M \to \mathbb{P}^1$ of degree 2.

Let $f(x)$ be a monic polynomial with distinct roots where the degree of $f(x)$ is $2g + 1 + \epsilon$, with $\epsilon \in \{0, 1\}$. Let C_0 be the complex affine plane curve

$$C_0 := \{(x, y) \in \mathbb{C}^2 | y^2 = f(x)\}. \tag{2}$$

Note that C_0 is a non-compact smooth plane curve. Let $U = \{(x, y) \in C_0 : x \neq 0\}$; U is an open subset of C_0. Let $\tilde{C}_0 := \{(\tilde{x}, \tilde{y}) \in \mathbb{C}^2 : \tilde{y}^2 = \tilde{x}^{2g+2} f(\tilde{x}^{-1})\}$, note that $\tilde{x}^{2g+2} f(\tilde{x}^{-1})$ is a polynomial in $\tilde{x}$ and has distinct roots. Let $V = \{(\tilde{x}, \tilde{y}) \in \tilde{C}_0 : \tilde{x} \neq 0\}$; V is an open subset of $\tilde{C}_0$. Set $\tilde{x} = \frac{1}{x}$, $\tilde{y} = \frac{y}{x^{g+1}}$, then we have an isomorphism $\phi : U \to V$ given by $\phi(x, y) = \left(\frac{1}{x}, \frac{y}{x^{g+1}}\right)$. Glueing C_0 and $\tilde{C}_0$ along U and V via ϕ we obtain a compact Riemann surface C of genus g. We can think of C as being C_0 with some extra points added at infinity: one point which we will call P_∞, if $\epsilon = 0$; and two extra points, which we will call ∞_1 and ∞_2, if $\epsilon = 1$. The map $\pi : C \to \mathbb{P}^1$ which sends $(x, y) \to x$ and sends the points at infinity to $\infty \in \mathbb{P}^1$ is a meromorphic function of degree 2; hence C is hyperelliptic. It can be shown that every hyperelliptic Riemann surface of genus g is isomorphic to a surface C constructed in this way from an smooth affine plane curve. The involution $\iota : C_0 \to C_0$, $\iota(x, y) = (x, -y)$ extends to C by $\iota(\infty) = \infty$ if $\epsilon = 0$ or by $\iota(\infty_1) = \infty_2$ and $\iota(\infty_1) = \infty_2$, if $\epsilon = 1$. We call $\iota : C \to C$ the hyperelliptic involution. The involution generates the symmetry group $\mathbb{Z}_2$. The fixed points of the involution are the points $(x, 0)$ where x is a root of f (together P_∞ if f is of odd degree). These $2g + 2$ points are the hyperelliptic Weierstrass points of C. It is well known that the holomorphic 1-forms $\left\{\frac{dx}{y}, \frac{x\,dx}{y}, \ldots, \frac{x^{g-1}dx}{y}\right\}$ is a basis for $H^0(C, \Omega_C)$ (see [5]).

Let C be a hyperelliptic Riemann surface defined as in (2) with deg $(f) = 2g + 2$. The Weierstrass point of C are the points $p_k = (a_k, 0)$, $1 \leq k \leq 2g+2$. Note that $f'(a_k) \neq 0$ and by the implicit function theorem, there exists a small neighbourhood U of p_k, such that y is a local coordinate. Let $g_k(x) = \prod_{i=1, i\neq k}^{2g+2}(x-a_i)$ and set $z : C_0 \to \mathbb{C}$, $z(x, y) = \frac{y}{\sqrt{g_k(x)}}$ in a neighbourhood of p_k. From now on, we work on the basis of holomorphic 1-forms $\left\{\frac{\mathrm{d}x}{y}, \frac{x\mathrm{d}x}{y}, \ldots, \frac{x^{g-1}\mathrm{d}x}{y}\right\}$ of C and in particular we will give a local expression of this basis in the branch points p_k for each $1 \leq k \leq 2g+2$. With the above notation, we have the following lemma:

Lemma 1. *The function z is a local coordinate in $p_k = (a_k, 0)$ and in this coordinate the local expression of the 1-holomorphic differentials $(x - a_k)^{j-1}\frac{\mathrm{d}x}{y}$ is given by $\alpha(z)z^{2(j-1)}\mathrm{d}z$ for $j = 1, \ldots, g$, where $\alpha(z)$ is holomorphic and $\alpha(p_k) \neq 0$.*

Proof. Since $\mathrm{d}y = \frac{f'(x)\mathrm{d}x}{2y} \neq 0$ in a neighbourhood of p_k, $g_k(a_k) \neq 0$ and $y(a_k) = 0$. Then computing differentials and evaluating in p_k we have that $\mathrm{d}z(p_k) = \left(\frac{\sqrt{g_k(x)}\mathrm{d}y - y\mathrm{d}\sqrt{g_k(x)}}{g_k(x)}\right)(p_k) = \left(\frac{\mathrm{d}y}{\sqrt{g_k(x)}}\right)(p_k) \neq 0$. Then there exists an open neighbourhood $\mathcal{V}$ of p_k where z is a local coordinate. Note that $\frac{\mathrm{d}x}{y} = \frac{2\mathrm{d}y}{f'(x)}$, then for $j = 1, \ldots, g$ we have that

$$\frac{(x-a_k)^{j-1}\mathrm{d}x}{y} = \frac{2(x-a_k)^{j-1}\mathrm{d}y}{f'(x)} = z^{2(j-1)}\left(\frac{2\mathrm{d}y}{dz}\mathrm{d}z\right)\frac{1}{f'(x)} = \alpha(z)z^{2(j-1)}\mathrm{d}z,$$

where $\alpha(z)$ is a holomorphic function such that $\alpha(p_k) \neq 0$. □

Using the local coordinate of Lemma 1, we have that

$$\mathbf{f}(z) = (f_1(z), \ldots, f_g(z)) = \alpha(z) \cdot F(z),$$

where $F(z) = \left(1, z^2, z^4, \ldots, z^{2(g-1)}\right)$. Denote by $F'(z)$ the complex vector of derivatives of the entries of $F(z)$. Let $H = (h_{ij})_{i,j=1}^{g}$ be any positive definite Hermitian matrix with $\langle, \rangle$ the inner product on $\mathbb{C}^g$ induced by H. Set $\beta(z) := \langle F(z), F(z)\rangle$. We have that $\mathrm{d}s_H^2 = \rho^2(z)\mathrm{d}z\wedge\mathrm{d}\bar{z}$, where $\rho^2(z) = |\alpha(z)|^2\beta(z)$. Define $G(z) := \beta(z)\frac{\partial^2\beta(z)}{\partial z\partial\bar{z}} - \frac{\partial\beta(z)}{\partial z}\cdot\frac{\partial\beta(z)}{\partial\bar{z}}$. The expression for $G(z)$ is given by

$$\begin{aligned} G(z) = &\langle F(z), F(z)\rangle\langle F'(z), F'(z)\rangle \\ &-\langle F'(z), F(z)\rangle\langle F(z), F'(z)\rangle. \end{aligned}$$

Since $\alpha(z)$ is holomorphic, $\frac{\partial^2}{\partial\bar{z}\partial z}\log(\alpha(z)\overline{\alpha(z)}) = 0$, so the local expression for the Gaussian curvature function K_H on C is given by $K_H(z) = \frac{-4G(z)}{\hat{\rho}^2(z)}$, where $\widehat{\rho(z)} := \rho^2(z)\beta(z)$ is a positive function. By the product rule for derivatives we have:

Lemma 2. *The following identities holds*:

$$\begin{aligned} \frac{\partial G}{\partial z} &= \langle F', F\rangle\langle F', F'\rangle + \langle F, F\rangle\langle F'', F'\rangle \\ &\quad -\langle F', F'\rangle\langle F', F\rangle - \langle F, F'\rangle\langle F'', F\rangle, \\ \frac{\partial^2 G}{\partial^2} &= \langle F'', F\rangle\langle F', F'\rangle + \langle F', F\rangle\langle F'', F'\rangle \\ &\quad +\langle F', F\rangle\langle F'', F'\rangle + \langle F, F\rangle\langle F''', F'\rangle \\ &\quad -\langle F'', F'\rangle\langle F', F\rangle - \langle F', F'\rangle\langle F'', F\rangle \\ &\quad -\langle F', F'\rangle\langle F'', F\rangle - \langle F, F'\rangle\langle F''', F\rangle, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial \bar{z}}\left(\frac{\partial G}{\partial z}\right) \;=\; & \frac{\partial^2 G}{\partial \bar{z}\partial z} = \langle F', F'\rangle\langle F', F'\rangle + \langle F', F\rangle\langle F', F''\rangle \\
& +\langle F, F'\rangle\langle F'', F'\rangle + \langle F, F\rangle\langle F'', F''\rangle \\
& -\langle F', F''\rangle\langle F', F\rangle - \langle F', F'\rangle\langle F', F'\rangle \\
& -\langle F, F''\rangle\langle F'', F\rangle - \langle F, F'\rangle\langle F'', F'\rangle.
\end{aligned}
$$

Lemma 3. *For any positive definite hermitian matrix* $H = (h_{ij})_{i,j=1}^{g}$, *the Weirstrass points of a hyperelliptic Riemann surface of genus* $g \geq 2$ *are non-degenerate critical points of Morse index* $+2$ *for the Gaussian curvature function* K_H.

Proof. We have

$$
\begin{aligned}
\langle F(p_k), F(p_k)\rangle &= h_{11}, \\
\langle F''(p_k), F''(p_k)\rangle &= 4h_{22}, \\
\langle F''(p_k), F(p_k)\rangle &= 2h_{12}. \qquad (3)
\end{aligned}
$$

By Lemma 2 and equation (3), and the fact that $F'(p_k) = F'''(p_k) = (0, \ldots, 0)$ we obtain that

$$
\begin{aligned}
&\frac{\partial^2 G}{\partial \bar{z}\partial z}(p_k) = 4(h_{11}h_{22} - h_{12}h_{21}), \\
&\frac{\partial G}{\partial z}(p_k) = \frac{\partial^2 G}{\partial z^2}(p_k) = 0. \qquad (4)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\frac{\partial K_H}{\partial z} &= \frac{-4}{\hat{\rho}^2(z)} \cdot \frac{\partial G}{\partial z} + G \cdot \frac{\partial}{\partial z}\left(\frac{-4}{\hat{\rho}^2(z)}\right), \\
\frac{\partial^2 K_H}{\partial z^2} &= \frac{-4}{\hat{\rho}^2(z)} \cdot \frac{\partial^2 G}{\partial z^2} + \frac{\partial G}{\partial z} \cdot \frac{\partial}{\partial z}\left(\frac{-4}{\hat{\rho}^2(z)}\right) + G \cdot \frac{\partial^2}{\partial z^2}\left(\frac{-4}{\hat{\rho}^2(z)}\right) \\
&\quad + \frac{\partial G}{\partial z} \cdot \frac{\partial}{\partial z}\left(\frac{-4}{\hat{\rho}^2(z)}\right). \qquad (5) \\
\frac{\partial^2 K_H}{\partial \bar{z}\partial z} &= \left(\frac{-4}{\hat{\rho}^2(z)}\right) \cdot \frac{\partial^2 G}{\partial \bar{z}\partial z} + \frac{\partial G}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}\left(\frac{-4}{\hat{\rho}^2(z)}\right) \\
&\quad + G \cdot \frac{\partial^2}{\partial \bar{z}\partial z}\left(\frac{-4}{\hat{\rho}^2(z)}\right) + \frac{\partial}{\partial z}\left(\frac{-4}{\hat{\rho}^2(z)}\right) \cdot \frac{\partial G}{\partial \bar{z}}. \qquad (6)
\end{aligned}
$$

Lemma 2, equations (3), (4), (5) and the fact that $G(p_k) = 0$ implies that

$$
\begin{aligned}
\frac{\partial^2 K_H}{\partial z^2}(p_k) &= 0 = \frac{\partial^2 K_H}{\partial \bar{z}^2}(p_k), \\
\frac{\partial^2 K_H}{\partial \bar{z}\partial z}(p_k) &= \frac{-16(h_{11}h_{22} - h_{12}h_{21})}{\hat{\rho}^2(p_k)}. \qquad (7)
\end{aligned}
$$

Set $r := \frac{-16(h_{11}h_{22}-h_{12}h_{21})}{\hat{\rho}^2(p_k)}$ and note that $r < 0$. We have the complex Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 K_H}{\partial z^2} & \frac{\partial^2 K_H}{\partial z \partial \bar{z}} \\ \frac{\partial^2 K_H}{\partial \bar{z} \partial z} & \frac{\partial^2 K_H}{\partial \bar{z}^2} \end{pmatrix} (p_k) = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}.$$

Let $z = u + iv$ be, we recall that $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$. By chain rule we have in real coordinates the following identities:

$$\begin{aligned} \frac{\partial^2}{\partial^2 u^2} &= 2\frac{\partial^2}{\partial z \partial \bar{z}} + \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2}\right), \\ \frac{\partial^2}{\partial^2 v^2} &= 2\frac{\partial^2}{\partial z \partial \bar{z}} - \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2}\right), \\ \frac{\partial^2}{\partial u \partial v} &= \frac{1}{i}\left(\frac{\partial^2}{\partial \bar{z}^2} - \frac{\partial^2}{\partial z^2}\right). \end{aligned} \tag{8}$$

Thus, the real Hessian of K_H in p_k is the matrix $\mathrm{Hess}_{\mathbb{R}}(K_H)(p_k) = \begin{pmatrix} 2r & 0 \\ 0 & 2r \end{pmatrix}$. Hence for $1 \leq k \leq 2g+2$, the Weierstrass point $p_k = (a_k, 0)$ is a non-degenerated critical point of K of Morse index $+2$. □

3. Criteria to find local minima for the Gaussian curvature

Let C be a hyperelliptic Riemann surface of genus g defined by the equation $y^2 = f(x) = (x - a_1)\cdots(x - a_{2g+2})$ with $a_i \neq a_j, \forall i \neq j$. In this section we consider the $g \times g$ identity matrix $H = \mathrm{Id}$ which induces the usual inner product on $\mathbb{C}^g$, and in this case the metric induced on C is given by the inner product

$$\eta \cdot \mathrm{Id} \cdot \overline{\eta^t} = \rho^2(x)\mathrm{d}x \wedge \mathrm{d}\bar{x},$$

where $\eta = \left(\frac{\mathrm{d}x}{y}, \frac{x\mathrm{d}x}{y}, \ldots, \frac{x^{g-1}\mathrm{d}x}{y}\right)$ and $\rho^2(x) = \frac{\sum_{j=1}^{g} |x|^{2j-2}}{|y^2|}$. For $H = \mathrm{Id}$, we write K for the corresponding Gaussian curvature function on C. With this notation, we have the following lemma.

Lemma 4. *Let $(x_0, y_0) \in C$ be such that $f(x_0) \neq 0$.*

(a) *Suppose that $f'(x_0) = 0$, then (x_0, y_0) is a critical point of K.*
(b) *Suppose that $f'(x_0) = f''(x_0) = 0$, then (x_0, y_0) is a local minima of Morse index zero of K.*

Proof.

(a) Let $(x_0, y_0) \in C$ be such that $f(x_0) \neq 0$, and $f'(x_0) = 0$. We take a small open neighbourhood U_{x_0} of x_0 as the coordinate of (x_0, y_0). We have a local biholomorphism $\tau : U_{x_0} \to \mathbb{C}$, where $\tau(x) = x - x_0$. In this coordinate we transform our basis

$\left\{\frac{x^j \mathrm{d}x}{y}, j=0,\ldots,g-1\right\}$ in the basis $\frac{\tau^k \mathrm{d}\tau}{y} = \frac{(x-x_0)^k \mathrm{d}x}{y}$. Set $g(x) := 1+\sum_{j=2}^{g} |x-x_0|^{2j-2}$. The conformality factor for the metric is given by

$$\rho^2(\tau(x)) = \frac{\sum_{j=1}^{g} |\tau(x)|^{2j-2}}{|y|^2} = \frac{g(x)}{|y|^2}.$$

We have that $\partial\bar{\partial}\log(f\bar{f}) = 0$ and $\partial\bar{\partial}\log\left(\frac{1}{|y^2|}\right) = \frac{1}{2}\partial\bar{\partial}\log\left(\left(\frac{1}{|y^4|}\right)\right) = \frac{1}{2}\partial\bar{\partial}\log\left(\left(\frac{1}{f\bar{f}}\right)\right) = 0$. The curvature function $K : U_{x_0} \to \mathbb{R}$ in the coordinate x is given by

$$K(x) = -4\partial\bar{\partial}\log\left(\frac{g(x)}{|y^2|}\right)\cdot\frac{|y^2|}{g(x)} = -\alpha\frac{|f(x)|}{g^3(x)}, \tag{9}$$

where $\alpha = 4$. Since $f(x_0) \neq 0$, we have that $\partial|f|(x_0) = \frac{\partial|f|}{\partial x}(x_0) = \frac{f'(x_0)\overline{f(x_0)}}{2|f(x_0)|}$, $\bar{\partial}|f|(x_0) = \frac{\partial|f|}{\partial\bar{x}}(x_0) = \frac{\overline{f'(x_0)}f(x_0)}{2|f(x_0)|}$. Note that $(\partial g)(x_0) = \frac{\partial g}{\partial x}(x_0) = 0 = (\bar{\partial}g)(x_0) = \frac{\partial g}{\partial\bar{x}}(x_0)$. We have

$$\begin{aligned}
\frac{\partial K}{\partial x} &= -\alpha\cdot\frac{\left[g^3(x)\frac{\partial|f|}{\partial x} - |f(x)|3g^2(x)\frac{\partial g}{\partial x}\right]}{g^6(x)},\\
\frac{\partial K}{\partial\bar{x}} &= -\alpha\cdot\frac{\left[g^3(x)\frac{\partial|f|}{\partial\bar{x}} - |f(x)|3g^2(x)\frac{\partial g(x)}{\partial\bar{x}}\right]}{g^6(x)}.
\end{aligned} \tag{10}$$

In real coordinates $x = u + iv$, from (8) we have that

$$\begin{aligned}
\frac{\partial K}{\partial u} &= \frac{-\alpha}{g^3(x)}(\partial|f(x)| + \bar{\partial}|f(x)|) - \frac{3\alpha|f(x)|}{g^4(x)}(\partial g - \bar{\partial}g),\\
\frac{\partial K}{\partial v} &= -i\left[\frac{-\alpha}{g^3(x)}(\bar{\partial}|f(x)| - \partial|f(x)|) - \frac{3\alpha|f(x)|}{g^4(x)}(\bar{\partial}g - \partial g)\right].
\end{aligned} \tag{11}$$

By hypothesis $f(x_0) \neq 0$, then if $f'(x_0) = 0$ we have that $(x_0, y_0) \in C$ is a critical point of K.

(b) It is easy to see that $\bar{f}(x)\cdot f''(x) = \frac{\partial^2|f|^2}{\partial x^2} = 2\left(\frac{\partial|f|}{\partial x}\right)^2 + 2|f|\frac{\partial^2|f|}{\partial x^2}$. We have the following partial derivatives:

$$\begin{aligned}
\frac{\partial^2 K}{\partial\bar{x}\partial x} &= \alpha\frac{\left[6g^5(x)\frac{\partial g}{\partial\bar{x}}\cdot\left[g^3(x)\frac{\partial|f|}{\partial x} - |f(x)|3g^2(x)\frac{\partial g}{\partial x}\right]\right.}{g^{12}(x)}\\
&\quad -\alpha\frac{\left[3g^2(x)\frac{\partial|f|}{\partial x}\frac{\partial g}{\partial\bar{x}} + g^3(x)\frac{\partial^2|f|}{\partial\bar{x}\partial x}\right]}{g^6(x)}\\
&\quad -\alpha\frac{\left[-|f(x)|\left(6g(x)\frac{\partial g}{\partial\bar{x}}\cdot\frac{\partial g}{\partial x} + 3g^2(x)\frac{\partial^2 g}{\partial\bar{x}\partial x}\right) - 3g^2(x)\frac{\partial g}{\partial x}\cdot\frac{\partial|f|}{\partial\bar{x}}\right]}{g^6(x)},
\end{aligned}$$

$$\frac{\partial^2 K}{\partial x^2} = -\alpha \frac{\left[3\frac{\partial|f|}{\partial x}g^2(x)\frac{\partial g}{\partial x}+g^3(x)\frac{\partial^2|f|}{\partial x^2}-6|f(x)|g(x)\left(\frac{\partial g}{\partial x}\right)^2-3g^2(x)\left(|f(x)|\frac{\partial^2 g}{\partial x^2}+\frac{\partial g}{\partial x}\frac{\partial|f|}{\partial x}\right)\right]}{g^6(x)}$$
$$+\alpha\frac{6\frac{\partial g}{\partial x}}{g^7(x)}\cdot\left[g^3(x)\frac{\partial|f|}{\partial x}-|f(x)|3g^2(x)\frac{\partial g}{\partial x}\right],$$
$$\frac{\partial^2 K}{\partial \overline{x}^2} = \overline{\frac{\partial^2 K}{\partial x^2}}.$$

By hypothesis $f(x_0) \neq 0$, $f'(x_0) = f''(x_0) = 0$, then $\frac{\partial^2|f|^2}{\partial x^2}(x_0) = 0$ and $\frac{\partial|f|}{\partial x}(x_0) = 0 = \frac{\partial|f|}{\partial \overline{x}}(x_0)$. We have that $\frac{\partial^2|f|}{\partial x^2}(x_0) = 0 = \frac{\partial^2|f|}{\partial\overline{x}\partial x}(x_0)$. Since $\frac{\partial^2 g}{\partial x^2}(x_0) = 0 = \frac{\partial^2 g}{\partial\overline{x}^2}(x_0)$ and $\frac{\partial^2 g}{\partial\overline{x}\partial x}(x_0) = g(x_0) = 1$, and $\alpha = 4$, then by the above computations we have $\lambda := \frac{\partial^2 K}{\partial\overline{x}\partial x}(x_0) = 12|f(x_0)| > 0$ and $\frac{\partial^2 K}{\partial\overline{x}^2}(x_0) = \overline{\frac{\partial^2 K}{\partial x^2}}(x_0) = 0$. From (7) and (8) is easy to see that for $(u_0, v_0) = (\text{Re }(x_0), \text{Im }(x_0))$, $\frac{\partial^2 K}{\partial u\partial v}(u_0, v_0) = 0$ and $\frac{\partial^2 K}{\partial^2 u^2}(u_0, v_0) = 2\lambda = \frac{\partial^2 K}{\partial^2 v^2}(u_0, v_0) > 0$, so the real Hessian of K at z_0 is given by the matrix:

$$\text{Hess}_{\mathbb{R}} K(u_0, v_0) = \begin{pmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{pmatrix}.$$

Let $\pi : C \to \mathbb{P}^1$ be the double cover, since $f(x_0) \neq 0$, then $\pi(x_0, y_0) = x_0$ is not a branch point, so there are small open neighbourhoods U_{x_0} of $x_0 \in \mathbb{C} \subset \mathbb{P}^1$ and W of $(x_0, y_0) \in C$ such that $\pi|_W : W \to U_{x_0}$ is a biholomorphism. This biholomorphism preserves the index of the critical point of K. Then we have that the point $(x_0, y_0) \in C$ is a non-degenerate critical point of K of Morse index zero which is a local minima. Since $\pi : C \to \mathbb{P}^1$ is a double cover, in particular we have two local minima corresponding to the two points that satisfies $y_0^2 = f(x_0)$. □

Remark 5. Consider $2g + 2$ points $a_j \in \mathbb{C}$, for $1 \leq j \leq 2g + 2$ such that $a_i \neq a_j$ for al $i \neq j$. Take the equation $y^2 = f(x) = \prod_{i=1}^{2g+2}(x - a_i) = x^{2g+2}\prod_{i=1}^{2g+2}\left(1 - \frac{a_i}{x}\right)$, as in §1.1. We can take the coordinates $\tilde{x} = \frac{1}{x}$, $\tilde{y} = \frac{y}{x^{g+1}}$ and consider the curve $\tilde{C}_0 = \{(\tilde{x}, \tilde{y}) : \tilde{y}^2 = \prod_{i=1}^{2g+2}(1 - a_i\tilde{x}\}$. For $\tilde{x} = 0$, the polynomial $h(\tilde{x}) = \prod_{i=1}^{2g+2}(1 - a_i\tilde{x})$ satisfies that $h(0) \neq 0$ and $h'(\tilde{x}) = -\sum_{k=1}^{2g+2}(a_k\prod_{j\neq k}^{2g+2}(1 - a_j\tilde{x}))$. The proof of Lemma 4 shows that if $h'(0) = h''(0) = 0$, then the two points $\infty_1, \infty_2 \in C$ lying over $\infty = [1 : 0] \in \mathbb{P}^1$ are local minima of K of Morse index zero.

4. Examples

In this section we consider the $g \times g$ identity matrix $H = \text{Id}$. We will give examples of hyperelliptic curves where the Gaussian curvature K is a Morse function. We recall that the constant α that appear in the local expression for K in the following examples is $\alpha = 4$.

Example 6. Let C be the hyperelliptic Riemann surface of genus $g = 2$ defined by the equation $y^2 = x^6 - 1$. The reduced automorphis group $\text{Aut}/\mathbb{Z}_2$ of C is the dihedral group D_6 and the order of the full symmetry group $D_6 \times \mathbb{Z}_2$ of C is 24. This surface has the automorphism $\varphi : C \to C$, $\varphi(x, y) = (\zeta x, y)$, where $\zeta = e^{\frac{\pi i}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The polynomial $f(x) = x^6 - 1$ satisfies that $f(0) \neq 0$, $f'(0) = 0$, $f''(0) = 0$. From Lemma 4 and Remark 5, we have that the points $(0, \pm i)$ on C lying over $0 = [0 : 1] \in \mathbb{P}^1$ and the two points on C lying over $\infty = [1 : 0] \in \mathbb{P}^1$ are local minima for K of Morse index zero. We take x as the local coordinate. Then the function K is given by

$$K(x) = -\alpha \frac{|x^6 - 1|}{(|x|^2 + 1)^3},$$

Consider the function K^2. By chain rule for derivatives note that $d_x(K^2) = 2K(x) \cdot d_x(K)$, so except at the Weierstrass points, the derivative $d_x(K^2)$ and $d_x(K)$ have the same critical points. In polar coordinates $x = re^{i\theta}$, $r > 0, \theta \in [0, 2\pi)$, we have that $\frac{1}{\alpha^2}K^2(x) = \frac{(x^6-1)\overline{(x^6-1)}}{(|x^2|+1)^6} = \frac{1}{\alpha^2}K^2(r, \theta) = \frac{r^{12}-2r^6cos(6\theta)+1}{(r^2+1)^6}$. We have

$$\begin{aligned}\frac{1}{\alpha^2} \cdot \frac{\partial K^2(r, \theta)}{\partial r} &= \frac{12r^{11} - 12r^5\cos(6\theta)}{(r^2+1)^6} - \frac{12r(r^{12} - 2r^6\cos(6\theta) + 1)}{(r^2+1)^7},\\ \frac{1}{\alpha^2} \cdot \frac{\partial K^2(r, \theta)}{\partial \theta} &= \frac{12r^6\sin(6\theta)}{(r^2+1)^6}.\end{aligned}$$

All critical points of dK^2 are the points corresponding to $0 \in \mathbb{C} \subset \mathbb{P}^1$, the roots of $x^6 - 1 = 0$ and the points $(1, \theta_k)$ where $\theta_k = \frac{(2k+1)\pi}{6}$, $k = 0, 1, \ldots, 5$. The points $(1, \theta_k)$ corresponds to the complex number $\beta_k = \zeta^k \beta_1$, where $\zeta = e^{\frac{i\pi}{3}}$, $\beta_1 = e^{\frac{i\pi}{6}}$, $k = 0, 1, \ldots, 5$. The points β_k are the roots of $x^6 + 1 = 0$. A straightforward computation gives

$$\begin{aligned}\frac{1}{\alpha^2}\frac{\partial K^2}{\partial x} &= \frac{(|x^2|+1)^6 \cdot 6x^5\overline{(x^6-1)} - 12\bar{x}(|x|^2+1)^5(x^6-1)\overline{(x^6-1)}}{(|x^2|+1)^{12}},\\ \frac{1}{\alpha^2} \cdot \frac{\partial^2 K^2}{\partial^2 x^2} &= \frac{30x^4\overline{(x^6-1)}}{(|x|^2+1)^6} - \frac{144|x^2|x^4\overline{(x^6-1)}}{(|x|^2+1)^7} + \frac{168\bar{x}^2|x^6-1|^2}{(|x|^2+1)^8}\\ &= \frac{1}{\alpha^2} \cdot \frac{\partial^2 K^2}{\partial^2 \bar{x}^2},\\ \frac{1}{\alpha^2} \cdot \frac{\partial^2 K^2}{\partial \bar{x} \partial x} &= \frac{36|x|^{10}}{(|x|^2+1)^6} + \frac{72(\bar{x}^6 - x^6) - 12|x^6-1|^2 - 144x^6\overline{(x^6-1)}}{(|x|^2+1)^7}\\ &\quad + \frac{168|x|^2|x^6-1|^2}{(|x|^2+1)^8}.\end{aligned}$$

Since $\beta_1^6 = -1$ then $\beta_1^4 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, $\beta_1^2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\beta_1^4 + \bar{\beta}_1^2 = 0$. This implies that

$$\frac{\partial^2 K^2}{\partial x^2}(\beta_1) = \frac{84(\beta_1^4 + \bar{\beta}_1^2) + 84\bar{\beta}_1^2}{2^6} = \frac{84\bar{\beta}_1^2}{2^6}, \quad \frac{\partial^2 K^2}{\partial \bar{x} \partial x}(\beta_1) = \frac{36}{2^6}.$$

Let $(u_0, v_0) = (\text{Re}\,\beta_1, \text{Im}\,\beta_1)$. From (8) we have in real coordinates

$$\frac{\partial^2 K^2}{\partial u^2}(u_0, v_0) = \frac{36 + 42(\bar{\beta}_1^2 + \beta_1^2)}{2^5},$$

$$\begin{aligned}\frac{\partial^2 K^2}{\partial^2 v^2}(u_0, v_0) &= \frac{36 - 42(\bar{\beta}_1^2 + \beta_1^2)}{2^5},\\ \frac{\partial^2 K^2}{\partial u \partial v}(u_0, v_0) &= \frac{42\sqrt{3}}{2^5}.\end{aligned} \tag{12}$$

Note that $\bar{\beta}_1^2 + \beta_1^2 = 1$, $\beta_1^2 - \bar{\beta}_1^2 = i\sqrt{3}$, so

$$\text{Hess } K(u_0, v_0) = \frac{\alpha^2}{2^5}\begin{pmatrix} 78 & 42\sqrt{3} \\ 42\sqrt{3} & -6 \end{pmatrix}.$$

This implies that β_1 is a saddle point of Morse index $+1$. Under the automorphism φ, note that $\varphi(\beta_k, y) = (\beta_{k+1}, y)$. This implies that each point β_k is a saddle point of Morse index $+1$. The roots of $x^6 + 1 = 0$ are not ramification points of the double cover $\pi : C \to \mathbb{P}^1$. Then we have that the 12 points $\pi^{-1}(\{x^6+1 = 0\}) = \{(x_k, \pm i\sqrt{2}) \in C : x_k^6+1 = 0, k = 0, 1, \ldots, 5\}$ are the saddle points of K of Morse index $+1$. Thus, the curvature function $K : C \to \mathbb{R}$ has $I_0 = 4$ critical points of Morse index zero (local minima), $I_1 = 12$ critical points of Morse index $+1$ (saddle points) and $I_2 = 6$ critical points of Morse index $+2$ (local maxima). The Euler characteristic $\chi(C)$ of C is $2 - 2g = -2 = I_0 - I_1 + I_2$. K is a Morse function on C.

Example 7. Let C be the hyperelliptic Riemann surface of genus $g \geq 2$ given by the equation $y^2 = x^{2g+2} - 1$, and $\pi : C \to \mathbb{P}^1$, $\pi(x, y) = x$ the ramified double cover on $\mathbb{P}^1$. This curve has the automorphism $\phi : C \to C$, $\phi(x, y) = \left(e^{\frac{\pi i}{g+1}} x, y\right)$. By similar computations as in Example 6, we have that the Gaussian curvature $K : C \to \mathbb{R}$ is given in a local coordinate by

$$K(x) = -\alpha \frac{|x^{2g+2} - 1|}{\left(\sum_{k=1}^{g} |x|^{2k-2}\right)^3}.$$

The function K has $2g + 2$ critical points of Morse index $+2$ (the Weirstrass points), 4 local minima of Morse index zero (the points lying over $\infty, 0 \in \mathbb{P}^1$). The function K^2 in polar coordinates is given by $K^2(r, \theta) = \alpha^2 \frac{r^{4g+4} - 2r^{2g+2}\cos((2g+2)\theta) + 1}{\left(\sum_{k=1}^{g} r^{2k-2}\right)^6}$. We have

$$\begin{aligned}\frac{1}{\alpha^2} \cdot \frac{\partial K^2(r, \theta)}{\partial r} &= \frac{(4g+4)r^{4g+3} - (2g+2)r^{2g+1}\cos((2g+2)\theta)}{\left(\sum_{k=1}^{g} r^{2k-2}\right)^6}\\ &\quad - \frac{6\left(\sum_{k=1}^{g}(2k-2)r^{2k-3}\right)(r^{4g+4} - 2r^{2g+2}\cos((2g+2)\theta) + 1)}{\left(\sum_{k=1}^{g} r^{2k-2}\right)^7},\\ \frac{1}{\alpha^2} \cdot \frac{\partial K^2(r, \theta)}{\partial \theta} &= \frac{(4g+4)r^{2g+2}\sin((2g+2)\theta)}{\left(\sum_{k=1}^{g} r^{2k-2}\right)^6}.\end{aligned}$$

It is easy to see that the other critical points of K are the points $(1, \theta_n)$ with $\theta_n = \frac{(2n+1)\pi}{2g+2}$, $n = 0, \ldots, 2g+1$ which corresponds to the complex numbers $\beta_n = e^{\frac{n\pi i}{g+1}}\beta_1$, where $\beta_1 = e^{\frac{i\pi}{2g+2}}$. In other words, the points $\pi^{-1}\{x \in \mathbb{P}^1 : x^{2g+2}+1 = 0\} \subset C$ are $4g+4$ critical points of K. By analogous computations for derivatives as in Example 6 we have that these points are saddle points of Morse index +1 of K, and K is a Morse function. Thus for every genus $g \geq 2$ we have a curve C with a Gaussian curvature function which is a Morse function.

Example 8. Consider the hyperelliptic curve given by the equation $y^2 = x^5 - 1$. The reduced automorphism group $\mathrm{Aut}/\mathbb{Z}_2$ of C is $\mathbb{Z}_5$ which is generated by the automorphism $\phi : C \to C, \phi(x, y) = (e^{\frac{2\pi i}{5}}x, y)$. The order of the full symmetry group $\mathbb{Z}_5 \times \mathbb{Z}_2$ is 10. In this case the curvature function $K : C \to \mathbb{R}$ has 6 local maxima of Morse index +2 (the Weierstrass points), and one of these points is the point at infinity P_∞. From Lemma 4 we have that the two points lying over $0 \in \mathbb{P}^1$ are local minima of Morse index zero. It is easy to see that in polar coordinates the partial derivatives for K^2 are

$$\begin{aligned}
\frac{1}{\alpha^2} \cdot \frac{\partial K^2(r,\theta)}{\partial r} &= \frac{10r^9 - 10r^4\cos(5\theta)}{(r^2+1)^6} - \frac{12r(r^{10} - 2r^5\cos(5\theta) + 1)}{(r^2+1)^7},\\
\frac{1}{\alpha^2} \cdot \frac{\partial K^2(r,\theta)}{\partial \theta} &= \frac{10r^5\sin(5\theta)}{(r^2+1)^6}.
\end{aligned}$$

We can see by similar computations as in Example 6 that the other critical points of K are the points $\{(x, y) \in C : x^5 + 1 = 0\}$. Since these points are not ramification points of the double cover $\pi : C \to \mathbb{P}^1$, we have that the 10 points $\pi^{-1}\{x \in \mathbb{P}^1 : x^5 + 1 = 0\} \subset C$ are critical points of K. Straightforward computations of second partial derivatives for K^2 show that these points are saddle points of K of Morse index $+1$, hence $I_0 - I_1 + I_2 = 2 - 2g = -2$, that is, K is a Morse function.

In a similar way is easy to see that the Gaussian curvature function for the hyperelliptic curve $y^2 = x^{2g+1} - 1$ is a Morse function.

Example 9. The full symmetry group of automorphism for the hyperelliptic curve $y^2 = x(x^4 - 1)$ is the group $S_4 \times \mathbb{Z}_2$ and is of order 48. In this case the curvature function K has 6 local maxima of Morse index $+2$, where the point lying $0 \in \mathbb{P}^1$ and the point P_∞ are Weierstrass points of C. In this case the curvature K has 8 saddle points of Morse index +1 which correspond to the preimage by the double cover $\pi : C \to \mathbb{P}^1$ of the roots of the polynomial $x^4 + 1 = 0$. K has no local minima and we have that $\chi(C) = -I_1 + I_2 = 2 - 2g = -2 = \chi(C)$. This is true also for the hyperelliptic curve $y^2 = x(x^{2g} - 1)$.

For genus two case, the examples in 6, 8 and 9 are Riemann surfaces with largest reduced automorphism group $D_6, \mathbb{Z}_5, S_4$ respectively (see [2]). In these examples the saddle points of the respective curvature function K are 'rotations' of the hyperelliptic Weierstrass points by some specific angle in the complex plane. The automorphism groups of these curves reflects that symmetry.

4.1 *Gaussian curvature in a* 1*-complex parameter family of hyperelliptic curves of genus* 2

In Bolza clasification (see [2]) we have that the 1-complex parameter family of hyperelliptic curves C_t given by the equation $y^2 = (x^3 - 1)(x^3 - t^3)$, $t \in \mathbb{C} - \{0, 1\}$, has reduced automorphism group D_3 and the order of the full symmetry group $D_3 \times \mathbb{Z}_2$ is 12. In this case the Gaussian curvature $K_t : C_t \to \mathbb{R}$ has six local maxima of Morse index $+2$ which correspond to the roots $x^3 = 1$ and $x^3 = t^3$. Note also that for $x = 0$, the polynomial $f(x) = (x^3 - 1)(x^3 - t^3) = x^6 - t^3x^3 - x^3 + t^3$ satisfies the hypothesis of Lemma 4, hence in the same way as in the above examples we have that for the 1-parameter family of curves C_t, the corresponding Gaussian curvature function K_t for C_t has four local minima of Morse index zero (the points lying over $0, \infty \in \mathbb{P}^1$). However in this family it is not easy to compute the saddle points (which will depend on the complex parameter t). We expect that K_t has 12 saddle points of Morse index $+1$.

4.2 *Some words about the study of K_H for other metrics*

When we consider on $\mathbb{C}^g$ an hermitian metric given by an hermitian matrix $H = (h_{ij})$ which is positive definite such that $h_{21} = 0$, we have that the criteria of Lemma 4 to find a local minima for K_H is valid in this case. However if $h_{21} \neq 0$ we do not have a criteria as in Lemma 4. When $h_{21} \neq 0$, it is more complicated to find a local minima and saddle points due to the expression of the conformal factor of the induced metric on C. One example of this situation is when we consider on the space $H^0(C, \Omega_C)$ of holomorphic 1-forms on C the inner product $(\cdot, \cdot) : H^0(C, \Omega_C) \times H^0(C, \Omega_C) :\to \mathbb{C}$ defined by $(\omega, \theta) = \frac{i}{2} \int_C \omega \wedge \bar{\theta}$. Let $\{\omega_1, \ldots, \omega_g\}$ be any orthonormal basis with respect to this inner product. By duality we have an inner product on $H^0(C, \Omega)^\vee \equiv \mathbb{C}^g$ and we obtain a metric on C given by $\sum_{j=1}^g \omega_j \otimes \overline{\omega_j}$. Since the basis $\{\omega_j\}_{j=1}^g$ is determined up to unitary transformation, this metric is independent of the choice of basis. It is well known that given a symplectic basis of homology $\{\lambda_k, \lambda_{g+k}, k = 1, \ldots, g\}$ on C there exist a unique orthonormal basis $\omega_1, \ldots, \omega_g$ of $H^0(C, \Omega)$ such that the $2g \times g$ period matrix of C defined as $\left(\int_{\lambda_k} \omega_j, \int_{\lambda_{g+k}} \omega_j\right)$, $k = 1, \ldots, g$, is given (with respect to this choice of basis) by (I, Z) where I is the $g \times g$ identity matrix. From the Riemann bilinear relations (pp. 231–232 of [5]) we have that Z is symmetric and the hermitian inner product form $(,)$ on $H^0(C, \Omega)$ satisfies $\frac{1}{2}(\omega_k, \omega_j) = \mathrm{Im}\left(\int_{\lambda_{g+k}} \omega_j\right) > 0$, that is, $\mathrm{Im}(Z)$ is a positive definite matrix.

In the hyperelliptic case, we consider the homology symplectic basis given by canonical dissections (see [11]). We have that the basis $\left\{\eta_1 = \frac{\mathrm{d}x}{y}, \ldots, \eta_g = \frac{x^{g-1}\mathrm{d}x}{y}\right\}$ is not orthonormal with respect to this choice of homology. However we can make a Gram–Schmidt process to the basis $\{\eta_j, j = 1, \ldots, g\}$ to obtain an orthomormal basis $\omega_k := \sum_j b_{jk}\eta_j,\ k = 1, \ldots, g$. In this case the conformality factor for the metric is given by $\rho^2(x) = \frac{g(x)}{|y^2|}$, where $g(x) = b_{11} + b_{21}x + b_{12}\bar{x} + b_{22}|x|^2 + \cdots$. In genus two, we have by the theory of period matrices that the coefficient b_{21} is related with some relations between the periods of C and $b_{21} \neq 0$. From my point of view, the strategy to study the nature of the critical points for the Gaussian curvature with respect to this metric should be a little bit different.

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