Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 2, May 2014, pp. 141–154.

Repdigits in k-Lucas sequences

JHON J BRAVO[1,*] and FLORIAN LUCA[2,3]

[1]Departamento de Matemáticas Universidad del Cauca,
Calle 5 No 4–70, Popayán, Colombia
[2]Mathematical Institute, UNAM Juriquilla Juriquilla,
76230, Santiago de Querétaro, Querétaro de Arteaga, México
[3]School of Mathematics, University of the Witwatersrand, P. O. Box Wits 2050,
Johannesburg, South Africa
[*]Correspondence author.
E-mail: jbravo@unicauca.edu.co; fluca@matmor.unam.mx

MS received 29 January 2013; revised 6 May 2013

Abstract. For an integer $k \geq 2$, let $(L_n^{(k)})_n$ be the k-Lucas sequence which starts with $0, \ldots, 0, 2, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. In 2000, Luca (*Port. Math.* **57(2)** 2000 243–254) proved that 11 is the largest number with only one distinct digit (the so-called *repdigit*) in the sequence $(L_n^{(2)})_n$. In this paper, we address a similar problem in the family of k-Lucas sequences. We also show that the k-Lucas sequences have similar properties to those of k-Fibonacci sequences and occur in formulae simultaneously with the latter.

Keywords. Generalized Fibonacci and Lucas numbers; lower bounds for nonzero linear forms in logarithms of algebraic numbers; repdigits.

2010 Mathematics Subject Classification. 11B39, 11J86.

1. Introduction

Let $k \geq 2$ be an integer. We consider the linear recurrence sequence of order k $G^{(k)} := (G_n^{(k)})_{n \geq 2-k}$ defined as

$$G_n^{(k)} = G_{n-1}^{(k)} + G_{n-2}^{(k)} + \cdots + G_{n-k}^{(k)} \qquad \text{for all } n \geq 2,$$

with the initial conditions $G_{-(k-2)}^{(k)} = G_{-(k-3)}^{(k)} = \cdots = G_{-1}^{(k)} = 0$, $G_0^{(k)} = a$ and $G_1^{(k)} = b$.

Observe that if $a = 0$ and $b = 1$, then $G^{(k)}$ is nothing more than the k-generalized Fibonacci sequence, or for simplicity, the k-Fibonacci sequence $F^{(k)} := (F_n^{(k)})_{n \geq 2-k}$. In this case, if we choose $k = 2$, we obtain the classical Fibonacci sequence $(F_n)_n$. In the general case, and as a remarkable property, we have that the first $k + 1$ non-zero terms in $F^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1, \qquad F_2^{(k)} = 1, \qquad F_3^{(k)} = 2, \qquad F_4^{(k)} = 4, \; \ldots, \; F_{k+1}^{(k)} = 2^{k-1}, \quad (1)$$

while the next term is $F_{k+2}^{(k)} = 2^k - 1$.

On the other hand, if $a = 2$ and $b = 1$, then $G^{(k)}$ is known as the k-generalized Lucas sequence, or for simplicity, the k-Lucas sequence $L^{(k)} := (L_n^{(k)})_{n\geq 2-k}$. In the special case of $k = 2$, the usual Lucas sequence is obtained, namely

$$L_0 = 2, \quad L_1 = 1 \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2.$$

$$(L_n)_{n\geq 0} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \ldots\}.$$

For example, if $k = 3$, then the 3-Lucas sequence is

$$\left(L_n^{(3)}\right)_{n\geq -1} = \{0, 2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, 1350, 2483, 4567, \ldots\}.$$

If $k = 4$, we get the 4-Lucas sequence

$$\left(L_n^{(4)}\right)_{n\geq -2} = \{0, 0, 2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, \ldots\}.$$

The above sequences are among the several generalizations of the Fibonacci numbers which have been studied in literature. Other generalizations are also known (see, for example, [3, 7, 13]).

Recall that a positive integer is called a *repdigit* if it has only one distinct digit in its decimal expansion. In particular, such number has the form $d(10^m - 1)/9$ for some $m \geq 1$ and $1 \leq d \leq 9$. Several authors have worked on problems involving generalized Fibonacci sequences. For instance, Luca [8] and Marques [9] proved that 55 and 44 are the largest repdigits in the sequences $F^{(2)}$ and $F^{(3)}$, respectively. Moreover, Marques conjectured that there are no repdigits, with at least two digits, belonging to $F^{(k)}$, for $k > 3$. In the recent work [2], we confirmed this conjecture. In addition, the diophantine equation $F_n^{(k)} = 2^m$ was studied in [1].

It was also proved in [8] that 11 is the largest repdigit which appears in the usual Lucas sequence $(L_n)_n$. Here we look at a similar problem for the terms of the k-Lucas sequence; i.e., we determine all the solutions of the diophantine equation

$$L_n^{(k)} = d \cdot \left(\frac{10^\ell - 1}{9}\right), \tag{2}$$

in positive integers n, k, d, ℓ with $k \geq 2$, $1 \leq d \leq 9$ and $\ell \geq 2$.

In this paper, we prove the following theorem.

Theorem 1. *The only solutions of the diophantine equation* (2) *are*

$$(n, k, d, \ell) \in \{(5, 2, 1, 2), (5, 4, 2, 2)\}.$$

Our strategy for proving Theorem 1 is to first use known information about the sequence $F^{(k)}$ in order to obtain some properties for the sequence $L^{(k)}$. Next, we use lower bounds for linear forms in logarithms of algebraic numbers to bound n and ℓ polynomially in terms of k. When k is small, the theory of continued fractions suffices to lower such bounds and complete the calculations. When k is large, we use the fact that the dominant root of the characteristic polynomial of the sequence $G^{(k)}$ is exponentially close to 2, so we can replace this root by 2 in our calculations with linear forms in logarithms and end up with an absolute bound for k; hence, an absolute bound for all k, ℓ and n, which we then reduce using again standard facts concerning continued fractions.

In this paper, we follow the presentation described in [2].

2. Preliminary inequalities

Before proceeding further, we shall recall some facts and properties of these sequences which will be used later. First, it is known that the characteristic polynomial of the sequence $G^{(k)}$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle; the other roots are strictly inside the unit circle (see, for example, [11], [12] and [14]). Throughout this paper, $\alpha := \alpha(k)$ denotes that single root, which is located between $2(1-2^{-k})$ and 2 (see [14]). To simplify the notation, in general, we omit the dependence on k of α.

We now consider for an integer $s \geq 2$, the function

$$f_s(x) = \frac{x-1}{2+(s+1)(x-2)} \quad \text{for } x > 2(1-2^{-s}).$$

With this notation, the following 'Binet-like' formula for $F^{(k)}$ appears in [5]:

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{n-1}, \tag{3}$$

where $\alpha = \alpha_1, \ldots, \alpha_k$ are the roots of $\Psi_k(x)$. It was also proved in [5] that the contribution of the roots which are inside the unit circle to the formula (3) is very small, namely that the approximation

$$|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}| < \frac{1}{2} \quad \text{holds for all } n \geq 2-k. \tag{4}$$

When $k = 2$, one can easily prove by induction that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{and} \quad \alpha^{n-1} \leq L_n \leq 2\alpha^n \text{ for all } n \geq 1. \tag{5}$$

In [2], we proved that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \quad \text{for all } n \geq 1 \text{ and } k \geq 2, \tag{6}$$

which shows that the first assertion of (5) holds for the k-Fibonacci sequence as well. All the above information for $F^{(k)}$ motivates to search for analogous results for the sequence $L^{(k)}$.

Since the sequences $G^{(k)}$ and $F^{(k)}$ have the same recurrence relation, it is natural to ask if there is any relation between them. In this sense, we have the following result.

Lemma 1 *The following relation always holds*:

$$G_n^{(k)} = aF_{n+1}^{(k)} + (b-a)F_n^{(k)}.$$

In particular,

$$L_n^{(k)} = 2F_{n+1}^{(k)} - F_n^{(k)}.$$

Proof. We consider the formal power series representation of the generating function for $G^{(k)}$,

$$G(x) = G_0^{(k)} x + G_1^{(k)} x^2 + \cdots + G_i^{(k)} x^i + \cdots = \sum_{i=0}^{\infty} G_i^{(k)} x^i.$$

Then, it is not too difficult to see that

$$G(x) = \frac{a + (b-a)x}{1 - x - \cdots - x^k}.$$

Hence, the lemma immediately follows from the fact that the generating function for $F^{(k)}$ is

$$F(x) = \frac{x}{1 - x - \cdots - x^k}.$$

□

In the following lemma, we establish some properties of the sequence $L^{(k)}$ which will be used in the proof of Theorem 1.

Lemma 2 (*Properties of* $L^{(k)}$). *Let* $k \geq 2$ *be an integer. Then*

(a) $\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n$ *for all* $n \geq 1$;
(b) $L^{(k)}$ *satisfies the following 'Binet-like' formula*:

$$L_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 1) f_k(\alpha_i) \alpha_i^{n-1},$$

where $\alpha = \alpha_1, \ldots, \alpha_k$ *are the roots of* $\Psi_k(x)$;
(c) $|L_n^{(k)} - (2\alpha - 1) f_k(\alpha) \alpha^{n-1}| < 3/2$ *holds for all* $n \geq 2 - k$;
(d) *If* $2 \leq n \leq k$, *then* $L_n^{(k)} = 3 \cdot 2^{n-2}$.

Proof. The proof follows easily from the information we already know for the sequence $F^{(k)}$ and the relation $L_n^{(k)} = 2F_{n+1}^{(k)} - F_n^{(k)}$, which comes from Lemma 1. For example, to prove (a), we use (6) to get

$$\alpha^{n-1} = 2\alpha^{n-1} - \alpha^{n-1} \leq L_n^{(k)} = 2F_{n+1}^{(k)} - F_n^{(k)} \leq 2\alpha^n - \alpha^{n-2} < 2\alpha^n.$$

The parts (b), (c) and (d) are direct consequences of (3), (4) and (1), respectively. □

To conclude this section of preliminary inequalities, assume throughout that equation (2) holds. Since $10^{\ell-1} < L_n^{(k)} < 10^{\ell}$, we have $\ell - 1 < \log L_n^{(k)} / \log 10 < \ell$, so

$$\ell = \left\lfloor \frac{\log L_n^{(k)}}{\log 10} \right\rfloor + 1.$$

Moreover, from Lemma 2(a), we obtain

$$(n-1)\left(\frac{\log \alpha}{\log 10}\right) < \ell < n\left(\frac{\log \alpha}{\log 10}\right) + \frac{\log 2}{\log 10} + 1.$$

In addition to this, by using the fact that $3/2 < \alpha < 2$, it is a straightforward exercise to check that for all $n \geq 5$ and $k \geq 2$

$$\frac{7n}{50} < \ell < \frac{3n}{5}, \tag{7}$$

which is an estimate on ℓ in terms of n. We shall make use of it later.

3. An inequality for *n* in terms of *k*

From now on we assume that $k \geq 3$ since the case $k = 2$ was already treated by Luca in [8]. Observe that for $k \geq 6$, it follows from Lemma 2(d) that the first $k - 3$ terms which have at least 2 digits in the k-Lucas sequence are $L_4^{(k)} = 12, L_5^{(k)} = 24, \ldots, L_k^{(k)} = 3 \cdot 2^{k-2}$. These numbers are not repdigits. Indeed, since $(10^\ell - 1)/9$ is odd for all $\ell \geq 2$, it follows that the exponent of 2 in $d(10^\ell - 1)/9$ is the same as the exponent of 2 in d, in particular, it does not exceed 3. This shows that the numbers of the form $3 \cdot 2^{k-2}$ with $k \geq 6$ are not repdigits. Hence, $n > k$ when $k \geq 6$, and the same is true for $k = 3,\ 4$ and 5 also.

Using now (2) and Lemma 2(c), we get that

$$\left| \frac{d10^\ell}{9} - (2\alpha - 1) f_k(\alpha)\alpha^{n-1} \right| < \frac{3}{2} + \frac{d}{9} \leq \frac{5}{2}. \tag{8}$$

Dividing both sides of the above inequality by the second term of the left-hand side, which is positive because $\alpha > 1$ and $2^k > k+1$, so $2 > (k+1)(2-(2-2^{-k+1})) > (k+1)(2-\alpha)$, and we obtain

$$\left| 10^\ell \cdot \alpha^{-(n-1)} \cdot \frac{d}{9}(2\alpha - 1)^{-1}(f_k(\alpha))^{-1} - 1 \right| < \frac{5}{\alpha^{n-1}}, \tag{9}$$

where we used the facts $2 + (k+1)(\alpha - 2) < 2$, $1/(\alpha - 1) < 2$ and $1/(2\alpha - 1) < 1/2$, which are easily seen.

In order to prove Theorem 1, we shall use twice the following result of Matveev (see [10] or Theorem 9.4 of [4]).

Theorem 2. *Assume that $\gamma_1, \ldots, \gamma_t$ are positive numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

is not zero. Then

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t), \tag{10}$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\}$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \quad \textit{for all} \quad i = 1, \ldots, t.$$

In the above, for an algebraic number η we write $h(\eta)$ for its logarithmic height, given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\eta^{(i)}|, 1\})\right),$$

with d being the degree of η over $\mathbb{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X]$$

being the minimal primitive polynomial over the integers having positive leading coefficient a_0 and η as a root.

In a first application of Matveev's result Theorem 2, we take $t := 3$ and

$$\gamma_1 := 10, \qquad \gamma_2 := \alpha, \qquad \gamma_3 := \frac{d}{9}(2\alpha - 1)^{-1}(f_k(\alpha))^{-1}.$$

We also take $b_1 := \ell$, $b_2 := -(n-1)$ and $b_3 := 1$. Hence,

$$\Lambda := \gamma_1^{b_1} \cdot \gamma_2^{b_2} \cdot \gamma_3^{b_3} - 1. \tag{11}$$

The absolute value of Λ appears in the left-hand side of inequality (9). To see that $\Lambda \neq 0$, observe that imposing $\Lambda = 0$ we get

$$\frac{d}{9} 10^{\ell} = \frac{(2\alpha - 1)(\alpha - 1)}{2 + (k+1)(\alpha - 2)} \alpha^{n-1}.$$

Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Psi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i \geq 2$, we have

$$\frac{d}{9} 10^{\ell} = \left| \frac{(2\alpha_i - 1)(\alpha_i - 1)}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1} \right|. \tag{12}$$

But the last equality above is not possible for $i \geq 2$ because

$$|2 + (k+1)(\alpha_i - 2)| \geq (k+1)|\alpha_i - 2| - 2 \geq k - 1 \geq 2 \quad \text{and} \quad |\alpha_i - 1| < 2,$$

since $|\alpha_i| < 1$. Hence, we get that the right-hand side of (12) is at most 3, whereas its left-hand side is $\geq 100/9$, which is a contradiction. Thus, $\Lambda \neq 0$.

The algebraic number field containing $\gamma_1, \gamma_2, \gamma_3$ is $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := k$. Since $h(\gamma_1) = \log 10 = 2.302585\ldots$, we can take $A_1 := 2.31k > kh(\gamma_1)$. Further, since $h(\gamma_2) = (\log \alpha)/k < (\log 2)/k = (0.693147\ldots)/k$, we can take $A_2 := 0.7$.

We now need to estimate $h(\gamma_3)$. First of all, observe that

$$h(\gamma_3) = \log\left(\frac{d}{9}\right) + h(2\alpha - 1) + h(f_k(\alpha)) = \log 9 + h(2\alpha - 1) + h(f_k(\alpha)), \tag{13}$$

where we have used the well-known facts that $h(xy) \leq h(x) + h(y)$ and $h(x) = h(x^{-1})$. In p. 73 of [1], we found an estimate for $h(f_k(\alpha))$. More precisely, we proved that

$$h(f_k(\alpha)) < \log(k+1) + \log 4 < 3 \log k \quad \text{for all } k \geq 3.$$

On the other hand, since

$$(x+1)^k - 2(x+1)^{k-1} - 2^2(x+1)^{k-2} - \cdots - 2^{k-1}(x+1) - 2^k$$

is the minimal primitive polynomial of $2\alpha - 1$ over the integers, we get that $h(2\alpha - 1) < \log 3$. Hence, it follows from (13) that

$$h(\gamma_3) < \log 27 + 3 \log k \leq 6 \log k \quad \text{for all } k \geq 3.$$

So, we can take $A_3 := 6k \log k$. By recalling (7), we deduce $\ell < n$, so we can take $B := n - 1$. Applying inequality (10) to get a lower bound for $|\Lambda|$ and comparing this with inequality (9), we get

$$\exp\left(-C_1(k) \times (1 + \log(n-1))\, (2.31k)\, (0.7)\, (6k \log k)\right) < \frac{5}{\alpha^{n-1}},$$

where $C_1(k) := 1.4 \times 30^6 \times 3^{4.5} \times k^2 \times (1 + \log k) < 1.5 \times 10^{11}\, k^2\, (1 + \log k)$.

Taking logarithms in the above inequality, we have that

$$(n-1) \log \alpha - \log 5 < 5.83 \times 10^{12}\, k^4\, \log^2 k\, \log(n-1),$$

where we used the facts $1 + \log k < 2 \log k$ for all $k \geq 3$, $1 + \log(n-1) < 2 \log(n-1)$ and for all $n \geq 4$. Furthermore, taking into account that $1/\log \alpha < 2$, which follows by using the fact that $2(1 - 2^{-k}) < \alpha$ and $k \geq 3$, we get

$$n - 1 < 1.2 \times 10^{13}\, k^4\, \log^2 k\, \log(n-1),$$

which yields

$$\frac{n-1}{\log(n-1)} < 1.2 \times 10^{13}\, k^4\, \log^2 k. \tag{14}$$

Since the function $x \mapsto x/\log x$ is increasing for all $x > e$, it is easy to check that the inequality

$$\frac{x}{\log x} < A$$

implies $x < 2A \log A$, whenever $A \geq 3$. Thus, taking $A := 1.2 \times 10^{13}\, k^4\, \log^2 k$, inequality (14) yields

$$\begin{aligned} n - 1 &< 2\,(1.2 \times 10^{13} k^4 \log^2 k) \log(1.2 \times 10^{13}\, k^4\, \log^2 k) \\ &< (2.4 \times 10^{13} k^4 \log^2 k)(31 + 4 \log k + 2 \log\log k) \\ &< 7.92 \times 10^{14} k^4 \log^3 k. \end{aligned}$$

In the last chain of inequalities, we have used that $31 + 4 \log k + 2 \log\log k < 33 \log k$ holds for all $k \geq 3$. Now, inserting the above upper bound for n in the upper bound for ℓ

from inequality (7), we get that $\ell < 4.758 \times 10^{14} k^4 \log^3 k$. Let us record this calculation for future use.

Lemma 3 *If* (n, k, d, ℓ) *is a solution in positive integers of equation* (2) *with* $k \geq 3$ *and* $n \geq 5$, *then* $n > k$ *and both inequalities*

$$n < 8 \times 10^{14} k^4 \log^3 k \quad \textit{and} \quad \ell < 5 \times 10^{14} k^4 \log^3 k$$

hold.

4. The case of small *k*

We next treat the cases when $k \in [3, 250]$. After finding an upper bound on n the next step is to reduce it. To do this, we use several times the following lemma from [2], which is a variation of a result due to Dujella and Pethő [6].

Lemma 4 *Let* M *be a positive integer, let* p/q *be a convergent of the continued fraction of the irrational* γ *such that* $q > 6M$, *and let* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Let* $\epsilon := ||\mu q|| - M||\gamma q||$, *where* $||\cdot||$ *denotes the distance from the nearest integer. If* $\epsilon > 0$, *then there is no solution to the inequality*

$$0 < u\gamma - v + \mu < AB^{-w},$$

in positive integers u, v *and* w *with*

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

In order to apply Lemma 4, we let

$$z := \ell \log 10 - (n-1) \log \alpha + \log \mu_d, \tag{15}$$

where $\mu_d := \gamma_3$. Then $e^z - 1 = \Lambda$, where Λ is given by (11). Therefore, (9) can be rewritten as

$$|e^z - 1| < \frac{5}{\alpha^{n-1}}. \tag{16}$$

Note that $z \neq 0$ since $\Lambda \neq 0$. Thus, we distinguish the following cases. If $z > 0$, then $e^z - 1 > 0$, so from (16) we obtain

$$0 < z < \frac{5}{\alpha^{n-1}},$$

where we used the fact that $x \leq e^x - 1$ for all $x \in \mathbb{R}$. Replacing z in the above inequality by its formula (15) and dividing both sides of the resulting inequality by $\log \alpha$, we get

$$0 < \ell \left(\frac{\log 10}{\log \alpha} \right) - n + \left(1 + \frac{\log \mu_d}{\log \alpha} \right) < 5 \cdot \alpha^{-(n-1)}, \tag{17}$$

where we have used again the fact that $1/\log \alpha < 2$. With

$$\hat{\gamma}_k := \frac{\log 10}{\log \alpha}, \quad \hat{\mu}_d := 1 + \frac{\log \mu_d}{\log \alpha}, \quad A := 10 \quad \text{and} \quad B := \alpha,$$

the above inequality (17) yields

$$0 < \ell\hat{\gamma}_k - n + \hat{\mu}_d < AB^{-(n-1)}. \tag{18}$$

It is clear that $\hat{\gamma}_k$ is an irrational number because $\alpha > 1$ is a unit in $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K}$. So α and 10 are multiplicatively independent.

For each $k \in [3, 250]$, we find a good approximation of α and a convergent p_k/q_k of the continued fraction of $\hat{\gamma}_k$ such that $q_k > 6M_k$, where $M_k := \lfloor 5 \times 10^{14} k^4 \log^3 k \rfloor$, which is an upper bound on ℓ from Lemma 3. After doing this, we use Lemma 4 on (18) in order to reduce our bound on n. Indeed, a computer search with *Mathematica* revealed that if $k \in [3, 250]$, then the maximum value of $\log(Aq_k/\epsilon_k)/\log B$, where $\epsilon_k = ||\hat{\mu}_d q_k|| - M_k||\hat{\gamma}_k q_k||$ is < 110, which, according to Lemma 4, is an upper bound on $n - 1$. Hence, we deduce that the possible solutions (n, k, d, ℓ) of equation (2) for which k is in the range $[3, 250]$ and $z > 0$ all have $n \in [2, 110]$.

Next we treat the case $z < 0$. It is a straightforward exercise to check that $5/\alpha^{n-1} < 1/2$ for all $k \geq 3$ and all $n \geq 6$. Then, from (16), we have that $|e^z - 1| < 1/2$ and therefore $e^{|z|} < 2$.

Since $z < 0$, we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{10}{\alpha^{n-1}}.$$

In a similar way as in the case when $z > 0$, we obtain

$$0 < (n-1)\hat{\gamma}_k - \ell + \hat{\mu}_d < AB^{-(n-1)}, \tag{19}$$

where now

$$\hat{\gamma}_k := \frac{\log \alpha}{\log 10}, \quad \hat{\mu}_d := -\frac{\log \mu_d}{\log 10}, \quad A := 5 \quad \text{and} \quad B := \alpha.$$

Here, we take $M_k := \lfloor 8 \times 10^{14} k^4 \log^3 k \rfloor$, which is an upper bound on $n - 1$ by Lemma 3, and, as we have explained before, we apply Lemma 4 to inequality (19) for each $k \in [3, 250]$. In this case, with the help of *Mathematica*, we find that the maximum value of $\log(Aq_k/\epsilon_k)/\log B$ is also < 110. Thus, the possible solutions (n, k, d, ℓ) of equation (2) with k in the range $[3, 250]$ and $z < 0$ all have $n \in [2, 110]$.

Finally, we use *Mathematica* to display the values $L_n^{(k)} \pmod{10^{10}}$ for $1 \leq n \leq 110$, $3 \leq k \leq 250$ with $n > k$, and check that only one solution of equation (2) in this range is $(n, k, d, \ell) = (5, 4, 2, 2)$, namely $L_5^{(4)} = 22$. This completes the analysis in the case $k \in [3, 250]$.

5. An absolute upper bound on k

From now on, we assume that $k > 250$. For such k we have

$$n < 8 \times 10^{14} k^4 \log^3 k < 2^{k/2}.$$

Let $\lambda > 0$ be such that $\alpha + \lambda = 2$. Since α is located between $2(1 - 2^{-k})$ and 2, we get that $\lambda < 2 - 2(1 - 2^{-k}) = 1/2^{k-1}$, i.e., $\lambda \in (0, 1/2^{k-1})$. Moreover, $2\alpha - 1 = 3 - 2\lambda$, and so

$$3 - \frac{2}{2^{k-1}} < 2\alpha - 1 < 3.$$

On the other hand, in p. 77 of [1] we gave the following estimates

$$\alpha^{n-1} > 2^{n-1} - \frac{2^n}{2^{k/2}} \quad \text{and} \quad |f_k(\alpha) - f_k(2)| < \frac{2k}{2^k}. \tag{20}$$

Hence,

$$\begin{aligned}(2\alpha - 1)\alpha^{n-1} &> \left(3 - \frac{2}{2^{k-1}}\right)\left(2^{n-1} - \frac{2^n}{2^{k/2}}\right)\\ &= 3 \cdot 2^{n-1} - 2^n \left(\frac{3}{2^{k/2}} + \frac{1}{2^{k-1}} - \frac{2}{2^{3k/2-1}}\right) > 3 \cdot 2^{n-1} - \frac{2^{n+2}}{2^{k/2}},\end{aligned}$$

where we used the fact that

$$\frac{3}{2^{k/2}} + \frac{1}{2^{k-1}} - \frac{2}{2^{3k/2-1}} < \frac{4}{2^{k/2}} \quad \text{holds for all } k > 250.$$

It then follows that the following inequalities hold:

$$3 \cdot 2^{n-1} - \frac{2^{n+2}}{2^{k/2}} < (2\alpha - 1)\alpha^{n-1} < 3 \cdot 2^{n-1}$$

or

$$|(2\alpha - 1)\alpha^{n-1} - 3 \cdot 2^{n-1}| < \frac{2^{n+2}}{2^{k/2}}. \tag{21}$$

If we write

$$(2\alpha - 1)\alpha^{n-1} = 3 \cdot 2^{n-1} + \delta \quad \text{and} \quad f_k(\alpha) = f_k(2) + \eta,$$

then the second estimate of (20) and (21) yield

$$|\eta| < \frac{2k}{2^k} \quad \text{and} \quad |\delta| < \frac{2^{n+2}}{2^{k/2}}.$$

Besides, since $f_k(2) = 1/2$, we have

$$(2\alpha - 1) f_k(\alpha)\alpha^{n-1} = 3 \cdot 2^{n-2} + 3 \cdot 2^{n-1}\eta + \frac{\delta}{2} + \eta\,\delta.$$

So, from (8) and the above equality, we get

$$\begin{aligned}\left|3 \cdot 2^{n-2} - \frac{d10^\ell}{9}\right| &= \left|\left((2\alpha - 1) f_k(\alpha)\alpha^{n-1} - \frac{d10^\ell}{9}\right) - 3 \cdot 2^{n-1}\eta - \frac{\delta}{2} - \eta\,\delta\right|\\ &< \frac{5}{2} + \frac{3k \cdot 2^n}{2^k} + \frac{2^{n+1}}{2^{k/2}} + \frac{2^{n+3}k}{2^{3k/2}}.\end{aligned}$$

Factoring out $3 \cdot 2^{n-2}$ in the right-hand side of the above inequality and taking into account that $5/(3 \cdot 2^{n-1}) < 1/2^{k/2}$ (because $n > k$ by Lemma 3), $4k/2^k < 1/2^{k/2}$, $8/(3 \cdot 2^{k/2}) < 3/2^{k/2}$ and $32k/(3 \cdot 2^{3k/2}) < 1/2^{k/2}$ are all valid for $k > 250$, we get that

$$\left|3 \cdot 2^{n-2} - \frac{d10^\ell}{9}\right| < 18 \cdot \frac{2^{n-2}}{2^{k/2}}.$$

Consequently,

$$\left|1 - \frac{d}{27} \cdot 10^{\ell} \cdot 2^{-(n-2)}\right| < \frac{6}{2^{k/2}}. \tag{22}$$

We now set

$$\Lambda_1 := \frac{d}{27} \cdot 10^{\ell} \cdot 2^{-(n-2)} - 1. \tag{23}$$

We next justify that the amount on the left-hand side of (23) is not zero. Indeed, if this were so, we would then get that $d \cdot 10^{\ell} = 27 \cdot 2^{n-2}$ being $\ell \geq 2$ and $n \geq 5$. But this implies that 5 divides $27 \cdot 2^{n-2}$, which is false.

We lower bound the left-hand side of inequality (22) using again Matveev's result, Theorem 2. We take $t := 3$, $\gamma_1 := d/27$, $\gamma_2 := 10$ and $\gamma_3 := 2$. We also take the exponents $b_1 := 1$, $b_2 := \ell$ and $b_3 := -(n-2)$. In this application of Matveev's result, we take $D := 1$, $A_1 := \log 27$, $A_2 := \log 10$ and $A_3 := \log 2$. Also, we can take $B := n$. We thus get

$$\exp\left(-C_2\,(1 + \log n)\,(\log 27)\,(\log 10)\,(\log 2)\right) < \frac{6}{2^{k/2}},$$

where $C_2 := 1.4 \times 30^6 \times 3^{4.5} < 1.5 \times 10^{11}$.

Taking logarithms in the above inequality, we have

$$\frac{k}{2}\log 2 - \log 6 < 7.9 \times 10^{11}\,(1 + \log n).$$

This leads to

$$\begin{aligned} k &< \frac{7.9 \times 10^{11}}{\log 2} \cdot 2\,(1 + \log n) + \frac{2\log 6}{\log 2} \\ &< 3.42 \times 10^{12}\,\log n + 5.17 \\ &< 3.5 \times 10^{12}\,\log n. \end{aligned}$$

In the above, we used the inequalities $2\,(1 + \log n) < 3\log n$ (valid for all $n \geq 8$) and $3.42 \times 10^{12}\,\log n + 5.17 < 3.5 \times 10^{12}\,\log n$ (valid for all $n \geq 2$). But, recall that by Lemma 3 we have $n < 8 \times 10^{14}\,k^4\,\log^3 k$. Thus,

$$\begin{aligned} k &< 3.5 \times 10^{12}\,\log(8 \times 10^{14}\,k^4\,\log^3 k) \\ &< 3.5 \times 10^{12}\,(34.4 + 4\log k + 3\log\log k) \\ &< 4.2 \times 10^{13}\log k, \end{aligned}$$

where we used the fact that the inequality $34.4 + 4\log k + 3\log\log k < 12\log k$ holds for all $k > 250$. *Mathematica* gives $k < 1.47 \times 10^{15}$. By Lemma 3 and the right most inequality of (7), we obtain that $n < 1.6 \times 10^{80}$ and $\ell < 9.55 \times 10^{79}$, respectively. We record our conclusion as follows.

Lemma 5 *If* (n, k, d, ℓ) *is a solution in positive integers of equation* (2) *with* $k > 250$, *then all inequalities*

$$n < 1.6 \times 10^{80}, \quad k < 1.5 \times 10^{15} \quad \text{and} \quad \ell < 9.6 \times 10^{79}$$

hold.

6. Reducing the bound on k

We now want to reduce our bound on k by using again Lemma 4. In order to do this, let $z := \ell \log 10 - (n-2)\log 2 + \log(d/27)$. Thus, $e^z - 1 = \Lambda_1$, where Λ_1 is given by (23). So, from estimate (22), we deduce that

$$|e^z - 1| < \frac{6}{2^{k/2}}. \tag{24}$$

In what follows, we distinguish again two cases. First, if $\Lambda_1 < 0$, then $z < 0$; besides, $|e^z - 1| < 1/2$ implies that $e^{|z|} < 2$. Hence, from (24), we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{12}{2^{k/2}}.$$

Replacing z by its expression in the above inequality, we get

$$0 < (n-2)\gamma - \ell + \hat{\mu}_d < AB^{-k}, \tag{25}$$

where

$$\gamma := \frac{\log 2}{\log 10}, \quad \hat{\mu}_d := -\frac{\log(d/27)}{\log 10}, \quad A := 6 \quad \text{and} \quad B := 2^{1/2}.$$

Clearly, γ is an irrational number. Let p_n/q_n be the n-th convergent of the continued fraction of γ. In order to reduce the bound on k, we take $M := 1.6 \times 10^{80}$, which is an upper bound on n from Lemma 5. Now we want to find a convergent of γ whose denominator is greater than $6M = 9.6 \times 10^{80}$.

A quick inspection using *Mathematica* reveals that our desired convergent is p_{165}/q_{165}. Moreover, we get

$$M \|q_{165}\gamma\| = 0.05959 < \cdots < 0.06.$$

The minimal value of $\|q_{165}\hat{\mu}_d\|$ computed for $d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is > 0.389. Thus, we can take $\epsilon := \|q_{165}\hat{\mu}_d\| - M\|q_{165}\gamma\| > 0.389 - 0.06 = 0.383$.

It then follows from Lemma 4 that there is no solution of the inequality in (25) (and therefore for equation (2)) with

$$k \geq \left\lfloor \frac{\log(Aq_{165}/\epsilon)}{\log B} \right\rfloor + 1 = 548 \quad \text{and} \quad d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Thus, $k \leq 547$ and Lemma 3 gives $n < 1.8 \times 10^{28}$.

With this new upper bound for n we repeated the process, i.e., we applied Lemma 4 again with $M := 1.8 \times 10^{28}$. Now, our desired convergent is p_{64}/q_{64}. We also get

$$M \|q_{64}\gamma\| = 0.001290 < \cdots < 0.0013.$$

We computed the values of $||q_{64}\hat{\mu}_d||$ for $d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and we found that the minimal value of $||q_{64}\hat{\mu}_d||$ is > 0.1438. Thus, we can now take

$$\epsilon := ||q_{64}\hat{\mu}_d|| - M\,||q_{64}\gamma|| > 0.1438 - 0.0013 = 0.1425.$$

It follows from Lemma 4 that there is no solution of the inequality in (25) for

$$k \geq \left\lfloor \frac{\log(Aq_{64}/\epsilon)}{\log B} \right\rfloor + 1 = 209 \quad \text{and} \quad d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Therefore, $k \leq 208$, which is a case already treated.

In the same way, if $\Lambda_1 > 0$, we then have $z > 0$. It follows from (24) that

$$0 < z \leq e^z - 1 < \frac{6}{2^{k/2}}.$$

Thus,

$$0 < \ell\gamma - n + \hat{\mu}_d < AB^{-k}$$

with

$$\gamma := \frac{\log 10}{\log 2}, \quad \hat{\mu}_d := 2 + \frac{\log(d/27)}{\log 2}, \quad A := 9 \quad \text{and} \quad B := 2^{1/2}.$$

We now take $M := 9.6 \times 10^{79}$ which is an upper bound on ℓ by Lemma 5. Here, by making use of the convergent p_{165}/q_{165} of γ and arguing exactly as in the case when $\Lambda_1 < 0$, we obtain, in view of Lemma 4, that $k \leq 545$, and then from Lemma 3 we get $\ell < 1.2 \times 10^{28}$.

As before we apply Lemma 4 with $M := 1.2 \times 10^{28}$ once more. Now, our desired convergent is p_{63}/q_{63}. After doing this, Lemma 4 finally gives $k \leq 206$, which is a case already treated. This completes the analysis in the case $k > 250$.

Hence, we have shown that there are no solutions (n, k, d, ℓ) to equation (2) with $k > 250$. Thus, Theorem 1 is proved.

Acknowledgements

One of the authors (JJB) was partially supported by CONACyT from Mexico and Universidad del Cauca, Colciencias from Colombia, and the author (FL) was supported in part by a Marcos Moshinsky Fellowship, Project PAPIIT IN104512, UNAM, Mexico and Project MEC 80120032, CONICYT, Chile. FL also thanks the Department of Mathematics of the University of Valparaiso, Chile for their hospitality during the period when this paper was being written.

References

[1] Bravo J J and Luca F, Powers of two in generalized Fibonacci sequences, *Rev. Colombiana Mat.* **46** (2012) 67–79

[2] Bravo J J and Luca F, On a conjecture about repdigits in k-generalized Fibonacci sequences, *Publ. Math. Debrecen* **82(3–4)** (2013) 623–639

[3] Brent R P, On the periods of generalized Fibonacci recurrences, *Math. Comp.* **63(207)** (1994) 389–401

[4] Bugeaud Y, Mignotte M and Siksek S, Classical and modular approaches to exponential diophantine equations, I, Fibonacci and Lucas perfect powers, *Ann. Math. (2)* **163(3)** (2006) 969–1018

[5] Dresden G P, A simplified Binet formula for k-generalized Fibonacci numbers, Preprint 2011, arXiv:0905.0304v2

[6] Dujella A and Pethő A, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser. (2)* **49(195)** (1998) 291–306

[7] Kilic E, The Binet formula, sums and representations of generalized Fibonacci p-numbers, *European J. Combin.* **29** (2008) 701–711

[8] Luca F, Fibonacci and Lucas numbers with only one distinct digit, *Port. Math.* **57(2)** (2000) 243–254

[9] Marques D, On k-generalized Fibonacci numbers with only one distinct digit, to appear in *Util. Math.*

[10] Matveev E M, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (**6**) (2000) 125–180; translation in *Izv. Math*. **64** (**6**) (2000) 1217–1269

[11] Miles E P Jr, Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* **67** (1960) 745–752

[12] Miller M D, Mathematical notes: on generalized Fibonacci numbers, *Amer. Math. Monthly* **78** (1971) 1108–1109

[13] Muskat J B, Generalized Fibonacci and Lucas sequences and root finding methods, *Math. Comp.* **61(203)** (1993) 365–372

[14] Wolfram D A, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* **36(2)** (1998) 129–145