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A statistic on n-color compositions and related sequences

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Abstract. A composition of a positive integer in which a part of size n may be assigned one of n colors is called an *n-color composition*. Let a_m denote the number of n-color compositions of the integer m. It is known that $a_m = F_{2m}$ for all $m \geq 1$, where F_m denotes the Fibonacci number defined by $F_m = F_{m-1} + F_{m-2}$ if $m \geq 2$, with $F_0 = 0$ and $F_1 = 1$. A statistic is studied on the set of n-color compositions of m, thus providing a polynomial generalization of the sequence F_{2m}. The statistic may be described, equivalently, in terms of statistics on linear tilings and lattice paths. The restriction to the set of n-color compositions having a prescribed number of parts is considered and an explicit formula for the distribution is derived. We also provide q-generalizations of relations between a_m and the number of self-inverse n-compositions of $2m + 1$ or $2m$. Finally, we consider a more general recurrence than that satisfied by the numbers a_m and note some particular cases.

Keywords. Compositions; n-color compositions; q-generalization.

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1. Introduction

A partition of an integer m is a decreasing sequence of positive integers whose sum is m. Compositions, as originally defined by MacMahon [11], are ordered partitions. For example, there are 5 partitions of the number 4, namely, 4, 31, 2^2, 21^2, 1^4, whereas there are 8 compositions, which are given by 4, 31, 2^2, 21^2, 13, 121, 1^{22}, 1^4.

Agarwal and Andrews [3] defined an n-color partition as one in which a part of size n can come in n different colors. We use a subscript to index the color and write $n_1, n_2, \ldots, n_n$ to denote the possible parts of size n. Agarwal [1] later defined an n-color composition as an n-color ordered partition, in analogy to MacMahon's ordinary compositions. Thus, for example, there are 8 n-color compositions of 3, and they are given by

$$\begin{aligned}&3_1, 3_2, 3_3,\\ &1_1 2_1, 1_1 2_2, 2_1 1_1, 2_2 1_1,\\ &1_1 1_1 1_1.\end{aligned}$$

Further properties of n-color compositions may be found in [2]. See also [9, 10, 12].

Given $m \geq 1$, let $\mathcal{A}_m$ denote the set of n-color compositions of m, and let $a_m = |\mathcal{A}_m|$, with $a_0 = 1$. In [1], Agarwal showed that $a_m = F_{2m}$ for all $m \geq 1$, where F_m denotes the

Fibonacci number defined by the recurrence $F_m = F_{m-1} + F_{m-2}$ if $m \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$ (see, for example, A000045 or A001906 in [15]). Conditioning on the size i of the first part within a member of $\mathcal{A}_m$ yields the recurrence

$$a_m = \sum_{i=1}^{m} i a_{m-i}, \qquad m \geq 1, \tag{1}$$

with $a_0 = 1$. Let $i_q := 1 + q + \cdots + q^{i-1}$ if $i \geq 1$, with $0_q := 0$. Here, we consider the q-generalization of a_m obtained by replacing i with i_q in (1). We will denote it by $a_m(q)$. Note that $a_m(q)$ is then a polynomial generalization of the sequence F_{2m} for $m \geq 1$. We show that $a_m(q)$ arises as a distribution polynomial for a statistic on $\mathcal{A}_m$, which we denote by σ and may be described, equivalently, in terms of linear tilings or lattice paths. The restriction of the σ statistic to the subset of $\mathcal{A}_m$ whose members have a prescribed number of parts is considered, and an explicit formula for the distribution polynomial is derived. A combinatorial proof of a binomial identity is obtained as a result.

Next, we provide q-generalizations of relations (see [12]) between the number of self-inverse n-compositions of $2m+1$ or $2m$ and the number of n-compositions of m. We make use of a combinatorial argument to show this, which in the case $q = 1$ differs from the argument given in [12] for that case. In the third section, we consider the more general recurrence

$$u_m = \sum_{i=1}^{m} r_i u_{m-i}, \qquad m \geq 1, \tag{2}$$

where r_i is a sequence of non-negative integers, and look at some particular cases. Note that u_m reduces to a_m and to $a_m(q)$, respectively, when $r_i = i$ and $r_i = i_q$ for all i. We term the resulting combinatorial objects as *r_n-compositions of m*.

2. A statistic on *n*-color compositions

We first define a statistic σ on $\mathcal{A}_m$ as follows. Given $\alpha = ([a_1]_{b_1}, [a_2]_{b_2}, \ldots) \in \mathcal{A}_m$, let

$$\sigma(\alpha) = \sum_i (b_i - 1).$$

For example, if $m = 18$ and $\alpha = 2_1 + 5_3 + 1_1 + 4_2 + 6_6 \in \mathcal{A}_{18}$, then $\sigma(\alpha) = 0 + 2 + 0 + 1 + 5 = 8$. Let

$$a_m(q) := \sum_{\alpha \in \mathcal{A}_m} q^{\sigma(\alpha)}, \qquad m \geq 1,$$

with $a_0(q) := 1$.

Theorem 2.1.

(i) *We have*

$$\sum_{m \geq 0} a_m(q) x^m = 1 + \frac{x}{1 - (2+q)x + qx^2}. \tag{3}$$

(ii) *The sequence $a_m(q)$ satisfies the recurrence $a_m(q) = (2+q)a_{m-1}(q) - qa_{m-2}(q)$ if $m \geq 3$, with initial values $a_1(q) = 1$ and $a_2(q) = 2 + q$.*

Proof. Conditioning on the size of the first part within a member of $\mathcal{A}_m$ gives the recurrence

$$a_m(q) = \sum_{i=1}^{m} i_q a_{m-i}(q), \qquad m \geq 1, \tag{4}$$

with $a_0(q) = 1$. Let $f(x) = \sum_{m\geq 0} a_m(q)x^m$. Then equation (4) may be rewritten in terms of generating functions as $f(x) - 1 = g(x)f(x)$, where

$$g(x) = \sum_{i\geq 1} i_q x^i = \frac{x}{(1-x)(1-qx)}.$$

Solving for $f(x)$ gives (3). The second statement follows easily from the first. □

Letting $q = 0$ in (3) above gives $\sum_{m\geq 0} a_m(0)x^m = \frac{1-x}{1-2x}$, which implies that there are 2^{m-1} members α of $\mathcal{A}_m$ for which $\sigma(\alpha) = 0$ if $m \geq 1$. This is also seen upon noting that such α are synonymous with the usual compositions of m numbering 2^{m-1}.

COROLLARY 2.2

If $m \geq 1$, *then* $a_m(-1) = F_m$.

Proof. This follows from substituting $q = -1$ in (3) and recalling $\sum_{m\geq 0} F_m x^m = \frac{x}{1-x-x^2}$. Alternatively, a combinatorial proof may be given as follows by defining an involution of $\mathcal{A}_m$ which changes the parity of σ values. Given $\alpha = ([a_1]_{b_1}, [a_2]_{b_2}, \ldots) \in \mathcal{A}_m$, let i_o denote the smallest index $i \geq 1$, if it exists, such that either (i) a_i is even, or (ii) a_i is odd and $b_i > 1$. If a_{i_o} is even (resp., odd), then replace b_{i_o} with $b_{i_o} + 1$ if b_{i_o} is odd (resp., even) or replace b_{i_o} with $b_{i_o} - 1$ if it is even (resp., odd). Let α' denote the resulting member of $\mathcal{A}'_m$. Then α and α' have opposite σ-parity (their σ values differing by one), and the mapping $\alpha \mapsto \alpha'$ is an involution, where defined. The mapping is not defined for those members α of $\mathcal{A}_m$ such that a_i is odd and $b_i = 0$ for all i. Note that the cardinality of the set of all such α is given by F_m, since they are synonymous with the compositions of m having odd parts (see for e.g., p. 46 of [16]). □

COROLLARY 2.3

If $m \geq 1$, *then the sum of the* σ *values of all the members of* $\mathcal{A}_m$ *is given by*

$$\frac{(m-1)F_{2m} + mF_{2m-2}}{5}.$$

Proof. It is well known that $\frac{x}{1-3x+x^2} = \sum_{m\geq 0} F_{2m}x^m$. Thus,

$$\frac{3-2x}{(1-3x+x^2)^2} = \sum_{m\geq 0}(m-1)F_{2m}x^{m-2}$$

and

$$\frac{2x-3x^2}{(1-3x+x^2)^2} = \frac{3x^2-2x^3}{(1-3x+x^2)^2} + \frac{2x}{1-3x+x^2} = \sum_{m\geq 0}(m+1)F_{2m}x^m.$$

Hence, by (3) we have

$$\frac{\mathrm{d}}{\mathrm{d}q}\sum_{m\geq 0} a_m(q)x^m\Bigg|_{q=1} = \frac{x^2(1-x)}{(1-3x+x^2)^2}$$
$$= \frac{x^2(3-2x)}{5(1-3x+x^2)^2} + \frac{x(2x-3x^2)}{5(1-3x+x^2)^2}$$
$$= \sum_{m\geq 0}\frac{(m-1)F_{2m}}{5}x^m + \sum_{m\geq 0}\frac{(m+1)F_{2m}}{5}x^{m+1},$$

which completes the proof. □

A modified, weighted version of the combinatorial proof occurring in Theorem 3.1 of [12] may be given to explain statement (ii) in Theorem 2.1. One may also show this directly via a bijection with linear tilings as follows. We first describe the σ statistic on $\mathcal{A}_m$ in terms of a statistic on tilings. Let $\mathcal{F}_m$ denote the set of coverings of the numbers $1, 2, \ldots, m$, arranged in a row, by indistinguishable dominos and indistinguishable squares, where pieces do not overlap, a domino is a piece covering two numbers, and a square is a piece covering a single number. The members of $\mathcal{F}_m$ are also called (linear) *tilings* or *domino arrangements* (see Chapter 1 of [4]). Note that $|\mathcal{F}_m| = F_{m+1}$ for all m. Furthermore, let s and d stand for *square* and *domino*, respectively. It is easily seen that members of $\mathcal{F}_m$ correspond uniquely to words in the alphabet $\{d, s\}$ containing k d's and $m-2k$ s's for some k, $0 \leq k \leq \frac{m}{2}$. In what follows, we will identify tilings π by such words $\pi_1\pi_2\cdots$.

By a *run* of a letter, we will mean a maximal sequence of consecutive occurrences of the letter. Thus, if $\pi \in \mathcal{F}_m$ contains exactly j squares, it may be expressed as $\pi = d^{i_1}sd^{i_2}s\cdots d^{i_j}sd^{i_{j+1}}$, where i_s denotes the length of the s-th run (possibly empty) of dominos, $1 \leq s \leq j+1$. Given $\pi \in \mathcal{F}_m$ as described, let

$$\rho(\pi) = \sum_{s=1}^{(j+2)/2} i_{2s-1},$$

i.e., $\rho(\pi)$ gives the sum of the lengths of the odd-indexed runs of π. We have the following result.

PROPOSITION 2.4

If $m \geq 1$, *then*

$$a_m(q) = \sum_{\pi\in\mathcal{F}_{2m-1}} q^{\rho(\pi)}. \tag{5}$$

Proof. We define a bijection between $\mathcal{A}_m$ and $\mathcal{F}_{2m-1}$ showing the equivalence of the σ and ρ statistics on the respective structures. To do so, suppose $\alpha = ([a_1]_{b_1}, [a_2]_{b_2}, \ldots, [a_r]_{b_r}) \in \mathcal{A}_m$. Convert the first part $[a_1]_{b_1}$ to the word $d^{b_1-1}sd^{a_1-b_1}$ of length $2a_1-1$. If $2 \leq i \leq r$, then convert the i-th part $[a_i]_{b_i}$ to the word $sd^{b_i-1}sd^{a_i-b_i}$ of length $2a_i$. Concatenate the resulting words and consider the member of $\mathcal{F}_{2m-1}$ so obtained, denoting it by $g(\alpha)$. Then it may be verified that g is a bijection from $\mathcal{A}_m$ to $\mathcal{F}_{2m-1}$ such that $\sigma(\alpha) = \rho(g(\alpha))$ for all $\alpha \in \mathcal{A}_m$. □

Using (5), it is possible to provide a bijective proof of the recurrence in Theorem 2.1.

Combinatorial proof of Theorem 2.1(*ii*). We will show that if $m \geq 3$, then the right-hand side of the recurrence in Theorem 2.1(ii) gives the total ρ-weight of all the members of $\mathcal{F}_{2m-1}$. Note first that the weight of all members of $\mathcal{F}_{2m-1}$ ending in $-ss$ or in $-d$ is $2a_{m-1}(q)$ for adding d to the end of some $\sigma \in \mathcal{F}_{2m-3}$ does not change the ρ value since σ containing an odd number of s's implies this d must belong to a run of even-index. So it remains to show that the weight of all members of $\mathcal{F}_{2m-1}$ ending in $-ds$ is given by $q(a_{m-1}(q) - a_{m-2}(q))$. To do so, first observe that the weight of all members of $\mathcal{F}_{2m-3}$ ending in a domino is $a_{m-2}(q)$, and thus the weight of those members ending in a square is given by $a_{m-1}(q) - a_{m-2}(q)$, by subtraction. Since there are an odd number of squares in any member of $\mathcal{F}_{2m-3}$, inserting a domino just prior to a terminal square increases the length of the final odd-indexed run of dominos by one and thus increases the value of ρ by one, whence the factor of q appearing, which completes the proof. □

The statistic σ may also be described, equivalently, in terms of lattice paths as follows. Let R denote the diagonal step $(1, 1)$, S the step $(1, 0)$, and T_a the step $(0, a)$ for any $a \geq 1$. Let $\mathcal{H}_m$ denote the set of lattice paths starting from the origin and taking first b $(0 \leq b \leq m - 1)$ R steps followed by S or T_a steps such that the path ends on the line $x + y = m + b - 1$. It was shown in [13] that $|\mathcal{H}_m| = \mathcal{A}_m$ for all $m \geq 1$, by both algebraic and combinatorial arguments. See the description of the bijection ψ from $\mathcal{H}_m$ to $\mathcal{A}_m$ given in §3 of [13].

We briefly recall some of the terminology used in this description. A *beam* is a maximal sequence of T_a steps wherein $a = 1$ for all but possibly the first step and a *plain* is a maximal sequence of S steps. The restriction of ψ to paths $\lambda \in \mathcal{H}_m$ not starting with R (i.e., $b = 0$) is denoted by ϕ. From ϕ, we see that each beam of λ contributes towards the value of $\sigma(\phi(\lambda))$ the number of vertical steps contained within it except for beams directly preceded by plains, which contribute one less than the number of vertical steps contained therein towards this value (see (3.3) and (3.4) of [13]). Note further that the extension from ϕ to ψ entails only adding some parts of size one at the beginning, and hence there is no change in the value of $\sigma(\psi(\lambda))$ when R steps are added to the beginning of $\lambda \in \mathcal{H}_i$ for some $i < m$. Since each run of vertical steps within a member of $\mathcal{H}_m$ consists of one or more beams, it follows that the σ statistic on $\mathcal{A}_m$ is equivalent to the statistic μ on $\mathcal{H}_m$ defined by

$$\mu(\lambda) = \begin{cases} \text{\# of vertical steps} - \text{\# of runs of vertical steps}, & \text{if } \lambda = R^b S \lambda'; \\ \\ 1 + \text{\# of vertical steps} - \text{\# of runs of vertical steps}, & \text{if } \lambda = R^b T_a \lambda'. \end{cases}$$

Thus, we have the following result.

PROPOSITION 2.5

If $m \geq 1$, then the σ statistic on $\mathcal{A}_m$ is equivalent to the μ statistic on $\mathcal{H}_m$. In particular, $\sigma(\alpha) = \mu(\psi^{-1}(\alpha))$ for all $\alpha \in \mathcal{A}_m$, where ψ is defined in [13].

Let $\mathcal{A}_{m,r}$ denote the subset of $\mathcal{A}_m$ whose members contain exactly r parts. We consider the restriction of the σ statistic to $\mathcal{A}_{m,r}$. Let

$$a_{m,r}(q) = \sum_{\sigma \in \mathcal{A}_{m,r}} q^{\sigma(\alpha)}, \qquad m \geq r \geq 1.$$

We have the following explicit formula for the polynomials $a_{m,r}(q)$.

Theorem 2.6. *If* $m \geq r \geq 1$, *then*

$$a_{m,r}(q) = \sum_{a=0}^{m-r} \binom{r-1+a}{r-1}\binom{m-1-a}{r-1} q^a. \tag{6}$$

Proof. First observe that $a_{m,r}(q)$ satisfies the following recurrence relation:

$$a_{m,r}(q) = a_{m-1,r}(q) + \sum_{i=1}^{m-r+1} q^{i-1} a_{m-i,r-1}(q), \qquad m \geq r \geq 2, \tag{7}$$

with $a_{m,1}(q) = m_q$ for all $m \geq 1$ and $a_{m,r}(q) = 0$ if $m < r$. To show this, first note that the weight of those members of $\mathcal{A}_{m,r}$ ending in a part of the form i_j, where $1 \leq j < i$, is $a_{m-1,r}(q)$ (simply increase the last part of a member of $\mathcal{A}_{m-1,r}$ by one, leaving its subscript unchanged). Members of $\mathcal{A}_{m,r}$ ending in a part i_i, where $1 \leq i \leq m-r+1$, have weight $q^{i-1}a_{m-i,r-1}(q)$, and summing over i gives (7).

If $m \geq 1$, then let $a_m(q,u) = \sum_{r=1}^{m} a_{m,r}(q)u^r$. Multiplying (7) by u^r, and summing over $r = 2, 3, \ldots, m$, gives

$$\begin{aligned} a_m(q,u) &= q^{m-1}u + ua_{m-1}(q,u) + a_{m-1}(q,u) \\ &\quad + \sum_{r=2}^{m} u^r \sum_{i=2}^{m-r+1} q^{i-1} a_{m-i,r-1}(q) = q^{m-1}u + a_{m-1}(q,u) \\ &\quad + u \sum_{i=1}^{m-1} q^{m-1-i} a_i(q,u), \qquad m \geq 2, \end{aligned}$$

which is also seen to hold for $m = 1$, upon letting $a_0(q,u) = 0$.

Define $A(x;q,u) = \sum_{m\geq 1} a_m(q,u)x^m$. Multiplying the last recurrence by x^m, and summing over $m \geq 1$, yields

$$\begin{aligned} A(x;q,u) &= \frac{ux}{1-qx} + xA(x;q,u) + u\sum_{m\geq 1} x^m \sum_{i=1}^{m-1} q^{m-1-i} a_i(q,u) \\ &= \frac{ux}{1-qx} + xA(x;q,u) + \frac{ux}{1-qx}A(x;q,u), \end{aligned}$$

which implies

$$A(x;q,u) = \frac{ux}{(1-qx)(1-x)-ux} = \sum_{j\geq 1} \frac{(ux)^j}{(1-x)^j(1-qx)^j}.$$

Thus

$$A(x;q,u) = \sum_{j\geq 1}\sum_{b\geq 0}\sum_{c\geq 0}\binom{j-1+b}{j-1}\binom{j-1+c}{j-1}u^j q^c x^{j+b+c},$$

and extracting the coefficient of $x^m u^r$ gives

$$a_{m,r}(q) = \sum_{a=0}^{m-r}\binom{r-1+a}{r-1}\binom{m-1-a}{r-1}q^a,$$

which completes the proof. □

Taking $q = 1$ in (6), and using Identity 5.26 of [8], gives the formula for $|\mathcal{A}_{m,r}|$ found in [1].

COROLLARY 2.7

If $m \geq r \geq 1$, *then* $|\mathcal{A}_{m,r}| = \binom{m+r-1}{2r-1}$.

We obtain the following simple expression for the total value of the σ statistic taken over all the members of $\mathcal{A}_{m,r}$.

COROLLARY 2.8

If $m \geq r \geq 1$, *then the sum of the* σ *values of all the members of* $\mathcal{A}_{m,r}$ *is given by* $r\binom{m+r-1}{2r}$.

Proof. By (6), the total σ value is given by

$$\begin{aligned}
&\sum_{a=1}^{m-r} a \binom{r-1+a}{r-1}\binom{m-1-a}{r-1}\\
&\quad = r\sum_{a=1}^{m-r}\frac{r+a}{r}\binom{r-1+a}{r-1}\binom{m-1-a}{r-1} - r\sum_{a=1}^{m-r}\binom{r-1+a}{r-1}\binom{m-1-a}{r-1}\\
&\quad = r\sum_{a=0}^{m-r}\binom{r+a}{r}\binom{m-1-a}{r-1} - r\sum_{a=0}^{m-r}\binom{r-1+a}{r-1}\binom{m-1-a}{r-1}\\
&\quad = r\left[\binom{m+r}{2r} - \binom{m+r-1}{2r-1}\right] = r\binom{m+r-1}{2r},
\end{aligned}$$

where we have used Identity 5.26 of [8]. □

We also obtain the following binomial identity by means of a combinatorial proof.

COROLLARY 2.9

If $m \geq r \geq 1$, *then*

$$\sum_{a=0}^{m-r}(-1)^a\binom{r-1+a}{r-1}\binom{m-1-a}{r-1} = \begin{cases} \binom{\frac{m+r}{2}-1}{r-1}, & \text{if } m \text{ and } r \text{ have the same parity;} \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Proof. Restricting the involution used in the second proof of Corollary 2.2 above to $\mathcal{A}_{m,r}$ shows that $a_{m,r}(-1)$ is zero if m and r differ in parity or equals the cardinality of the set of compositions of m having r parts, each of which is odd, if m and r have the same parity. Note that this cardinality is given by $\binom{\frac{m-r}{2}+r-1}{r-1} = \binom{\frac{m+r}{2}-1}{r-1}$. Comparing this with (6) when $q = -1$ yields (8). □

An n-color composition is said to be *self-inverse* (or *palindromic*) if it reads the same from right to left as it does from left to right (see [9, 12]). Let $\mathcal{B}_m$ denote the subset of $\mathcal{A}_m$ comprising the self-inverse members. Let $b_m(q)$ denote the polynomial that results when one restricts the σ statistic to $\mathcal{B}_m$, i.e.,

$$b_m(q) = \sum_{\lambda \in \mathcal{B}_m} q^{\sigma(\lambda)}.$$

Theorem 2.10. *If* $m \geq 1$, *then*

$$b_{2m}(q) = (2+q)a_m(q^2) \tag{9}$$

and

$$b_{2m+1}(q) = a_{m+1}(q^2) + qa_m(q^2). \tag{10}$$

Proof. To prove (9), first note that the weight of all members of $\mathcal{B}_{2m}$ containing an even number of parts is $a_m(q^2)$. Now suppose $\lambda \in \mathcal{B}_{2m}$ contains an odd number of parts and is thus of the form $\alpha t \alpha$, where α is a composition and t is an even number. If $t = (2a)_{2b+1}$, where $a \geq 1$ and $0 \leq b \leq a-1$, then one may break this part into the parts a_{b+1}, a_{b+1} and append one to each copy of α, which implies that the total weight of all such λ in this case is also $a_m(q^2)$. If $t = (2a)_{2b}$, where $a \geq 1$ and $1 \leq b \leq a$, then the total weight is $qa_m(q^2)$, upon breaking this part into two parts of a_b. Note that the part $(2a)_{2b}$ contributes $2b-1$ towards the $\sigma(\lambda)$ value, but the two parts a_b contribute only $2(b-1)$ towards the combined σ value of the pair $(\alpha a_b, \alpha a_b)$, whence the factor of q appears. Adding the three cases gives (9).

We now show (10). Let

$$F_m(q) := \sum_{\pi \in \mathcal{F}_{m-1}} q^{\rho(\pi)}, \qquad m \geq 2,$$

with $F_1(q) = 1$. By Proposition 2.4, $a_m(q) = F_{2m}(q)$, so we show, equivalently,

$$b_{2m+1}(q) = F_{2m+2}(q^2) + qF_{2m}(q^2), \qquad m \geq 1. \tag{11}$$

To show (11), first let T be the subset of $\mathcal{B}_{2m+1}$ consisting of those $\lambda = \alpha r \alpha$ in which the middle part r is either (i) $(2i+1)_1$, where $i \geq 0$, or (ii) $(2i+1)_t$, where $i \geq 1$ and t is odd, $3 \leq t \leq 2i+1$. We define a mapping u between T and $\mathcal{F}_{2m+1}$ as follows. If r is of the form (i), then let $u(\lambda) = sd^i sg(\alpha)$, where g is the mapping used in the proof of Proposition 2.4 above, whereas if r is of the form (ii), then let $u(\lambda) = d^{\frac{t-1}{2}} sd^{i-\frac{t-1}{2}} sg(\alpha)$. It may be verified that u is a bijection such that $\sigma(\lambda) = 2\rho(u(\lambda))$ for all λ, and thus the total weight of all members of T is given by $F_{2m+2}(q^2)$.

To complete the proof of (11), we define a bijection v between $\mathcal{B}_{2m+1} - T$ and $\mathcal{F}_{2m-1}$ as follows. Suppose $\lambda = \alpha r \alpha$, where $r = (2i+1)_t$ for some $i \geq 1$ and t even, $2 \leq t \leq 2i$. Let $v(\lambda) = d^{\frac{t}{2}-1} sd^{i-\frac{t}{2}} sg(\alpha)$. Then v is a bijection and $\sigma(\lambda) = 2\rho(v(\lambda)) + 1$ for all λ. This implies that the weight of all members of $\mathcal{B}_{2m+1} - T$ is $qF_{2m}(q^2)$, which completes the proof. □

Remark 2.11. Let $L_m = F_{m+1} + F_{m-1}$ denote the m-th Lucas number (see A000032 in [15]). Setting $q = 1$ in (9) and (10) implies

$$|\mathcal{B}_{2m}| = 3|\mathcal{A}_m| = 3 \cdot F_{2m}$$

and

$$|\mathcal{B}_{2m+1}| = L_{2m+1},$$

which was shown in Theorem 6.2 of [12] in a different manner using generating functions.

We have the following additional formulas for $b_m(q)$ in terms of $a_m(q)$.

PROPOSITION 2.12

If $m \geq 1$, *then*

$$b_{2m}(q) = a_m(q^2) + \sum_{\ell=1}^{m} (2\ell)_q a_{m-\ell}(q^2) \tag{12}$$

and

$$b_{2m+1}(q) = \sum_{\ell=0}^{m} (2\ell+1)_q a_{m-\ell}(q^2). \tag{13}$$

Proof. To show (12), first note that the weight of the members of $\mathcal{B}_{2m}$ containing an even number of parts is $a_m(q^2)$. If a member of $\mathcal{B}_{2m}$ contains an odd number of parts, then the middle part is of size 2ℓ for some ℓ, $1 \leq \ell \leq m$, with the remaining parts comprising two copies of some member of $\mathcal{A}_{m-\ell}$. Thus, the sum gives the weight of all members of $\mathcal{B}_{2m}$ having an odd number of parts, which implies (12). Noting that a member of $\mathcal{B}_{2m+1}$ must contain an odd number of parts gives (13), by similar reasoning. □

The Chebyshev polynomials of the second kind $U_m(t)$ are given by the recurrence $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ if $m \geq 2$, with $U_0(t) = 1$ and $U_1(t) = 2t$ (see, e.g., [14]).

Dividing the recurrence for $a_m(q)$ in part (ii) of Theorem 2.1 above by $q^{\frac{m-1}{2}}$ shows that

$$a_m(q) = q^{\frac{m-1}{2}} U_{m-1}\left(\frac{2+q}{2\sqrt{q}}\right), \qquad m \geq 1. \tag{14}$$

We conclude this section with the following pair of Chebyshev polynomial identities.

COROLLARY 2.13

If $m \geq 1$, *then*

$$(2m+2)_q + \sum_{\ell=1}^{m} q^{m-\ell}(2\ell)_q U_{m-\ell}\left(\frac{2+q^2}{2q}\right) = q^m(1+q)U_m\left(\frac{2+q^2}{2q}\right) \tag{15}$$

and

$$\begin{aligned}(2m+1)_q &+ \sum_{\ell=1}^{m} q^{m-\ell}(2\ell-1)_q U_{m-\ell}\left(\frac{2+q^2}{2q}\right)\\ &= q^m\left(U_m\left(\frac{2+q^2}{2q}\right) + U_{m-1}\left(\frac{2+q^2}{2q}\right)\right).\end{aligned} \tag{16}$$

Proof. By (12) and (9), we have

$$(2m)_q + a_m(q^2) + \sum_{\ell=1}^{m-1}(2\ell)_q a_{m-\ell}(q^2) = b_{2m}(q) = (2+q)a_m(q^2).$$

Cancelling $a_m(q^2)$ from both sides, using (14), and replacing m with $m+1$ gives (15). By (14), (13) and (10), we have

$$\begin{aligned}(2m+1)_q &+ \sum_{\ell=1}^{m} q^{m-\ell}(2\ell-1)_q U_{m-\ell}\left(\frac{2+q^2}{2q}\right)\\ &= (2m+1)_q + \sum_{\ell=0}^{m-1}(2\ell+1)_q a_{m-\ell}(q^2)\\ &= b_{2m+1}(q) = a_{m+1}(q^2) + q a_m(q^2)\\ &= q^m\left(U_m\left(\frac{2+q^2}{2q}\right) + U_{m-1}\left(\frac{2+q^2}{2q}\right)\right),\end{aligned}$$

which gives (16). □

3. Some related sequences

In this section, we consider, more generally, sequences $\{u_m\}_{m\geq 0}$ determined by $u_0 = 1$ and the recurrence

$$u_m = \sum_{i=1}^{m} r_i u_{m-i}, \qquad m \geq 1, \tag{17}$$

where $\{r_i\}_{i\geq 1}$ is an arbitrary sequence of non-negative integers. Note that when $r_i = i$ for all i, then u_m is the sequence a_m which enumerates the n-color compositions of m, and when $r_i = i_q = 1+q+\cdots+q^{i-1}$, then one obtains the q-generalization of a_m considered in the previous section.

One could consider other sequences r_i in (17) above and thereby obtain additional analogues of a_m. Let $f(x) := \sum_{m\geq 0} u_m x^m$ and $g(x) := \sum_{i\geq 1} r_i x^i$. Then multiplying (17) by x^m, and summing over $m \geq 1$, implies

$$f(x) - 1 = \sum_{m\geq 1} \left(\sum_{i=1}^{m} r_i u_{m-i} \right) x^m = f(x)g(x),$$

so that

$$f(x) = \frac{1}{1-g(x)}. \tag{18}$$

Note that, combinatorially, the sequence u_m enumerates the compositions of m in which a part of size n is marked in one of r_n ways, $1 \leq n \leq m$. We will term such compositions as *r_n-compositions of m*. See also the related discussion of inverse transformations in [5]. Note that the construction described here is essentially equivalent to the "sequence" construction SEQ for combinatorial objects described in Section I.2 of [7].

Let G_i denote the generalized Fibonacci sequence defined by the initial values $G_0 = 0$ and $G_1 = 1$ and the recurrence

$$G_i = aG_{i-1} + bG_{i-2}, \qquad i \geq 2. \tag{19}$$

Recall that when a and b are positive integers, the number G_i counts the square-and-domino tilings of length $i-1$ where squares are painted in one of a colors and dominos in one of b colors (see Chapter 3 of [4]). Note that when $a = b = 1$, G_i is the i-th Fibonacci number F_i, and when $a = 2$ and $b = 1$, then G_i is the i-th Pell number p_i (see A000129 of [15]).

We first consider the case of u_m when $r_i = G_i$ in (17) above.

PROPOSITION 3.1

If $m \geq 1$, then the number of G_n-compositions of m, where $G_i = G_i(a, b)$ is defined by (19) above, is given by $G_m(a + 1, b)$.

Proof. We supply two proofs. It is well-known and easily shown that

$$\sum_{m\geq 1} G_m(a, b)x^m = \frac{x}{1-ax-bx^2}. \tag{20}$$

Thus, by (18), we have

$$\sum_{m\geq 0} u_m x^m = \frac{1}{1-\frac{x}{1-ax-bx^2}} = \frac{1-ax-bx^2}{1-(a+1)x-bx^2} = 1 + \frac{x}{1-(a+1)x-bx^2},$$

which implies the result, by (20).

For a combinatorial proof, first note that $G_m(a+1, b)$ counts square-and-domino tilings of length $m-1$ in which dominos are painted with one of b colors, and either a square is painted with one of a colors or is left unpainted. Let us denote this set of tilings by $\mathcal{G}_{m-1}$. We will define a bijection between $\mathcal{G}_{m-1}$ and the set of G_n-compositions of m. To do so, suppose $\lambda \in \mathcal{G}_{m-1}$. We append an unpainted square u to the front of λ and then decompose $u\lambda$ as $ut_1ut_2\cdots ut_j$ for some $j \geq 1$, where each subtiling t_i of λ contains no unpainted squares. Then associate with λ the G_n-composition of m given by $(|t_1|+1, |t_2|+1, \ldots, |t_j|+1)$ in which the i-th part $|t_i|+1$ is marked with the subtiling t_i of λ for all i. This association is seen to be a bijection. □

COROLLARY 3.2

If $m \geq 1$, then the number of F_n-compositions of m is the Pell number p_m.

Now we consider the case of u_m when $r_i = C_i$ in (17), where $C_i = \frac{1}{i+1}\binom{2i}{i}$ denotes the i-th Catalan number. Recall that C_i counts the lattice paths from $(0,0)$ to $(2i, 0)$ using up $(1,1)$ steps and down $(1,-1)$ steps that never go below the x-axis, which are often called *Dyck paths* (see A000108 of [15]). Thus, a C_n-composition of m is one in which a part of size n is assigned a Dyck path of semi-length n. The C_n-compositions of m are equinumerous with other more familiar combinatorial structures.

PROPOSITION 3.3

The following three sets have the same cardinality for all $m \geq 1$:

(i) *The C_n-compositions of m;*
(ii) *The set of compositions of m having m parts, where parts of size 0 are allowed;*
(iii) *The set of functions f from $[m]$ to $[m]$ such that $x < y$ implies $f(x) \leq f(y)$.*

Proof. Given a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of m having non-negative parts, we form the sequence $1^{\lambda_1}2^{\lambda_2}\cdots m^{\lambda_m}$, which is seen to be a non-decreasing function from $[m]$ to $[m]$. This yields a bijection between the sets in (ii) and (iii).

Let $\mathcal{L}$ be the set of lattice paths from $(0,0)$ to $(2m-1, 1)$ using up steps $(1,1)$ and down steps $(1,-1)$, which we will denote by u and d, respectively. Then a composition of the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ above may be construed as a member λ' of $\mathcal{L}$ upon writing $\lambda' = u^{\lambda_1}du^{\lambda_2}\cdots du^{\lambda_m}$.

To complete the proof, we then define a bijection between the set $\mathcal{K}$ of C_n-compositions of m and the set $\mathcal{L}$. First note that a member of $\mathcal{K}$ may be viewed as a Dyck path α of semi-length m in which some of the points of return to the x-axis are marked, with the last return always marked as well as the initial position at the origin. To see this, suppose there are $j+1$ marks for some $j \geq 1$, and let α_i, $1 \leq i \leq j$, denote the subpath of α between the i-th and $(i+1)$-st marked points, inclusive. Let $|\beta|$ denote the semi-length of a Dyck path β. Then $(|\alpha_1|, |\alpha_2|, \ldots, |\alpha_j|)$ is a composition of m having j parts, where the i-th part is associated with the Dyck path α_i. Thus, α is synonymous with a member of $\mathcal{K}$.

To define a bijection between members of $\mathcal{K}$ (construed as marked Dyck paths) and of $\mathcal{L}$, first decompose $\alpha \in \mathcal{K}$ as $\alpha = \beta_1\beta_2\cdots\beta_t$ for some t, where each β_i is a Dyck path having no internal points of return to the x-axis. If the terminal point of the subpath β_i, $1 \leq i \leq t$, is not marked, then reflect β_i in the x-axis, whereas if it is marked, then leave β_i unchanged. Let β_i' denote the resulting path in either case. Then $\alpha' = \beta_1'\beta_2' \cdots \beta_t'$ is a lattice path from $(0, 0)$ to $(2m, 0)$ using u and d steps in which the final step is u, which we delete. The mapping $\alpha \mapsto \alpha'$ yields the desired bijection between the sets $\mathcal{K}$ and $\mathcal{L}$. □

From (18), we see that the generating function $f(x)$ for the number of C_n-compositions of m is given by

$$f(x) = \frac{1}{2 - c(x)},$$

where

$$c(x) = \sum_{m \geq 0} \frac{1}{m+1}\binom{2m}{m}x^m = \frac{1 - \sqrt{1-4x}}{2x}.$$

From the generating function or from Proposition 3.3, we see that the C_n-compositions of m number $\binom{2m-1}{m}$ for all $m \geq 0$. The sequence $\binom{2m-1}{m}$, $m \geq 0$, occurs as entry A088218 in [15], where structures (ii) and (iii) in the last proposition are noted. See also A001700 of [15] for additional related structures.

Other choices for r_i in (17) above will lead to further sequences. For example, if $r_i = B_i$, the i-th Bell number, then u_m is given by sequence A129247 in [15], a type of inverted Bell number. See also the recent paper [6] for additional choices of r_i. Perhaps by introducing statistics on the structures enumerated by u_m for various r_i, one may obtain q-analogues of some of the sequences discussed in this section or of others.

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