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General L_p-harmonic Blaschke bodies

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Abstract. Lutwak introduced the harmonic Blaschke combination and the harmonic Blaschke body of a star body. Further, Feng and Wang introduced the concept of the L_p-harmonic Blaschke body of a star body. In this paper, we define the notion of general L_p-harmonic Blaschke bodies and establish some of its properties. In particular, we obtain the extreme values concerning the volume and the L_p-dual geominimal surface area of this new notion.

Keywords. General L_p-harmonic Blaschke bodies; L_p-harmonic Blaschke bodies; extreme values.

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1. Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. $\mathcal{K}^n_s$ denotes the set of origin-symmetric convex bodies in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$.

If K is a compact star shaped (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ is defined by (see [4, 16])

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}. \tag{1.1}$$

If ρ_K is positive and continuous, then K will be called a star body (about the origin). For the set of star bodies containing the origin in their interiors, the set of star bodies whose centroid lie at the origin and the set of origin-symmetric star bodies in $\mathbb{R}^n$, we write $\mathcal{S}^n_o$, $\mathcal{S}^n_e$ and $\mathcal{S}^n_s$, respectively. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

From (1.1), we obtain that for a constant $c > 0$ and any $u \in S^{n-1}$,

$$\rho(cK, u) = c\rho(K, u). \tag{1.2}$$

If $\phi \in GL(n)$, then for all $u \in S^{n-1}$ (see [4]),

$$\rho(\phi K, u) = \rho(K, \phi^{-1}u), \tag{1.3}$$

where $GL(n)$ denotes the group of general (nonsingular) linear transformations and ϕ^{-1} denotes the reverse of ϕ.

The notion of harmonic Blaschke combination was given by Lutwak (see [10]). For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero), the harmonic Blaschke combination, $\lambda \circ K \hat{+} \mu \circ L \in \mathcal{S}_o^n$, of K and L is defined by

$$\frac{\rho(\lambda \circ K \hat{+} \mu \circ L, \cdot)^{n+1}}{V(\lambda \circ K \hat{+} \mu \circ L)} = \lambda \frac{\rho(K, \cdot)^{n+1}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+1}}{V(L)}, \tag{1.4}$$

where the operation '$\hat{+}$' is called the harmonic Blaschke addition and $\lambda \circ K$ denotes the harmonic Blaschke scalar multiplication.

Based on the definition of harmonic Blaschke combination, Lutwak [10] gave the concept of harmonic Blaschke body as follows: For $K \in \mathcal{S}_o^n$, the harmonic Blaschke body, $\hat{\nabla} K$ of K is defined by

$$\hat{\nabla} K = \frac{1}{2} \circ K \hat{+} \frac{1}{2} \circ (-K).$$

Further, Feng and Wang introduced the notion of L_p-harmonic Blaschke combination (see [3]). For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $\lambda, \mu \geq 0$ (both not zero), the L_p-harmonic Blaschke combination, $\lambda \circ K \hat{+}_p \mu \circ L \in \mathcal{S}_o^n$, of K and L is defined by

$$\frac{\rho(\lambda \circ K \hat{+}_p \mu \circ L, \cdot)^{n+p}}{V(\lambda \circ K \hat{+}_p \mu \circ L)} = \lambda \frac{\rho(K, \cdot)^{n+p}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+p}}{V(L)}, \tag{1.5}$$

where the operation '$\hat{+}_p$' is called the L_p-harmonic Blaschke addition. From (1.2) and (1.5), we obtain that $\lambda \circ K = \lambda^{\frac{1}{p}} K$.

Let $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (1.5), then the L_p-harmonic Blaschke body, $\hat{\nabla}_p K$ of $K \in \mathcal{S}_o^n$ is given by (see [3])

$$\hat{\nabla}_p K = \frac{1}{2} \circ K \hat{+}_p \frac{1}{2} \circ (-K). \tag{1.6}$$

Associated with the definitions of (1.5) and (1.6), Feng and Wang [3] proved the following results.

Theorem 1.A. *If* $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $\lambda, \mu \geq 0$ (*both not zero*), *then*

$$V(\lambda \circ K \hat{+}_p \mu \circ L)^{\frac{p}{n}} \geq \lambda V(K)^{\frac{p}{n}} + \mu V(L)^{\frac{p}{n}}, \tag{1.7}$$

with equality if and only if K and L are dilates.

Theorem 1.B. *If* $K \in \mathcal{S}_o^n$, $p \geq 1$, *then*

$$V(\hat{\nabla}_p K) \geq V(K), \tag{1.8}$$

with equality if and only if K is an origin-symmetric.

In this article, we extend the notion of L_p-harmonic Blaschke body, that is, the general L_p-harmonic Blaschke bodies is given. Further, we study some of its properties and establish the extreme values of the volume and the L_p-dual geominimal surface area, respectively.

Now, we define the general L_p-harmonic Blaschke bodies as follows: For $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, the general L_p-harmonic Blaschke body of K is defined by

$$\frac{\rho(\hat{\nabla}_p^\tau K, \cdot)^{n+p}}{V(\hat{\nabla}_p^\tau K)} = f_1(\tau)\frac{\rho(K, \cdot)^{n+p}}{V(K)} + f_2(\tau)\frac{\rho(-K, \cdot)^{n+p}}{V(-K)}, \tag{1.9}$$

where

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \quad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.10}$$

From (1.10), we easily have that

$$f_1(\tau) + f_2(\tau) = 1, \tag{1.11}$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau). \tag{1.12}$$

Associated with the definition of L_p-harmonic Blaschke combination, it easily follows that

$$\hat{\nabla}_p^\tau K = f_1(\tau) \circ K \hat{+}_p f_2(\tau) \circ (-K). \tag{1.13}$$

Besides, by (1.6), (1.13) and (1.10), we get that if $\tau = 0$, then $\hat{\nabla}_p^0 K = \hat{\nabla}_p K$; if $\tau = \pm 1$, then $\hat{\nabla}_p^{+1} K = K$, $\hat{\nabla}_p^{-1} K = -K$.

We note that operators of this type and related maps compatible with linear transformations appear naturally in the theory of valuations in connection with isoperimetric and analytic inequalities (see [1, 2, 5–9, 11–13, 17–19]).

The main results can be stated as follows: First, we give the extreme values of the volume of general L_p-harmonic Blaschke bodies.

Theorem 1.1. *If $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$, then*

$$V(\hat{\nabla}_p K) \geq V(\hat{\nabla}_p^\tau K) \geq V(K). \tag{1.14}$$

If K is not origin-symmetric, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$.

Moreover, based on L_p-dual geominimal surface area (see eq. (2.6)), we give some of its extreme values for general L_p-harmonic Blaschke bodies.

Theorem 1.2. *If $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$, then*

$$\tilde{G}_{-p}(\hat{\nabla}_p K) \geq \tilde{G}_{-p}(\hat{\nabla}_p^\tau K) \geq \tilde{G}_{-p}(K). \tag{1.15}$$

If $\tau \neq \pm 1$, equality holds in the right inequality if and only if K is an origin-symmetric body, and if $\tau \neq 0$, with equality in the left inequality if and only if K is also an origin-symmetric body.

For $M \in \mathcal{S}_o^n$, let cent M denote the centroid of M, and M^* denotes the polar of M (see eq. (2.2)). Thus combining with Theorem 1.1 and the extended Blaschke-Santaló inequality (see eq. (2.3)), we finally get the following result.

Theorem 1.3. *If $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$, then*

$$V(K)V(\hat{\nabla}_p^{\tau,c} K) \leq \omega_n^2, \tag{1.16}$$

if $\tau = \pm 1$, then with equality if and only if K is an ellipsoid. Here $\hat{\nabla}_p^{\tau,c} K = (\hat{\nabla}_p^{\tau} K)^c$, and for any $Q \in \mathcal{S}_o^n$, $Q^c = (Q - \text{cent } Q)^*$, and $Q - \text{cent } Q \in \mathcal{S}_e^n$.

The proofs of Theorems 1.1–1.3 will be given in §4 of this paper. In §3, we also get some properties of general L_p-harmonic Blaschke bodies which may be required in the proofs of main results.

2. Preliminaries

2.1 *Support function and polar set*

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see [4, 16])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

If $E \subseteq \mathbb{R}^n$ is a non-empty set, the polar set, E^* of E is defined by (see [4])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \quad \text{for all } y \in E\}. \tag{2.2}$$

For $K \in \mathcal{S}_e^n$, an extension of the Blaschke–Santaló inequality takes the following form (see [14]):

Theorem 2.A. *If $K \in \mathcal{S}_e^n$, then*

$$V(K)V(K^*) \leq \omega_n^2, \tag{2.3}$$

with equality if and only if K is an ellipsoid.

2.2 *L_p-dual mixed volume*

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (both not zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in \mathcal{S}_o^n$ of K and L is defined by (see [15])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda\rho(K, \cdot)^{-p} + \mu\rho(L, \cdot)^{-p}, \tag{2.4}$$

where the operation '$+_{-p}$' is called the L_p-harmonic radial addition and $\lambda \star K$ denotes the L_p-harmonic radial scalar multiplication. From (1.2) and (2.4), we get that the L_p-harmonic radial and the usual scalar multiplication are related by $\lambda \star K = \lambda^{-\frac{1}{p}} K$.

Associated with the L_p-harmonic radial combination of star bodies, Lutwak [15] introduced the notion of L_p-dual mixed volume as follows: For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\tilde{V}_{-p}(K, L)$ of K and L is defined by

$$\frac{n}{-p}\tilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

The definition above and l'Hôpital's rule give the following integral representation of L_p-dual mixed volume (see [15]):

$$\tilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u)\rho_L^{-p}(u)\mathrm{d}u, \tag{2.5}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From formula (2.5), we get

$$\tilde{V}_{-p}(K, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n}(u)\mathrm{d}u.$$

2.3 *L_p-dual geominimal surface area*

The notion of L_p-dual geominimal surface area was given by Wang and Qi [20]. For $K \in \mathcal{S}_o^n$ and $p \geq 1$, the L_p-dual geominimal surface area, $\tilde{G}_{-p}(K)$ of K is defined by

$$\omega_n^{-\frac{p}{n}} \tilde{G}_{-p}(K) = \inf\{n\tilde{V}_{-p}(K, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_s^n\}. \tag{2.6}$$

3. Properties of general L_p-harmonic Blaschke bodies

In this section, we establish some properties of general L_p-harmonic Blaschke bodies. Here the proof of Theorem 3.1 requires the following lemma.

Lemma 3.1 [11]. *If $K, L \in \mathcal{S}_o^n$ and $p \geq 1$, then for any $Q \in \mathcal{S}_o^n$,*

$$\frac{\tilde{V}_{-p}(K, Q)}{V(K)} = \frac{\tilde{V}_{-p}(L, Q)}{V(L)}$$

if and only if $K = L$.

Theorem 3.1. *If $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$ and $\phi \in GL(n)$, then*

$$\hat{\nabla}_p^\tau \phi K = \phi \hat{\nabla}_p^\tau K.$$

Proof. From (1.3) and (1.9), it follows that for all $u \in S^{n-1}$,

$$\begin{aligned}
\frac{\rho(\hat{\nabla}_p^\tau \phi K, u)^{n+p}}{V(\hat{\nabla}_p^\tau \phi K)} &= f_1(\tau)\frac{\rho(\phi K, u)^{n+p}}{V(\phi K)} + f_2(\tau)\frac{\rho(-\phi K, u)^{n+p}}{V(-\phi K)} \\
&= \frac{1}{\|\det \phi\|}\left[f_1(\tau)\frac{\rho(K, \phi^{-1}u)^{n+p}}{V(K)}\right. \\
&\quad \left. + f_2(\tau)\frac{\rho(-K, \phi^{-1}u)^{n+p}}{V(-K)}\right] \\
&= \frac{1}{\|\det \phi\|}\frac{\rho(\hat{\nabla}_p^\tau K, \phi^{-1}u)^{n+p}}{V(\hat{\nabla}_p^\tau K)} = \frac{\rho(\phi \hat{\nabla}_p^\tau K, u)^{n+p}}{V(\phi \hat{\nabla}_p^\tau K)}.
\end{aligned} \tag{3.1}$$

For any $Q \in \mathcal{S}_o^n$, equality (3.1) can be written by

$$\frac{\rho(\hat{\nabla}_p^\tau \phi K, u)^{n+p} \rho(Q, u)^{-p}}{V(\hat{\nabla}_p^\tau \phi K)} = \frac{\rho(\phi \hat{\nabla}_p^\tau K, u)^{n+p} \rho(Q, u)^{-p}}{V(\phi \hat{\nabla}_p^\tau K)}.$$

Use formula (2.5) to get that for any $Q \in \mathcal{S}_o^n$,

$$\frac{\tilde{V}_{-p}(\hat{\nabla}_p^\tau \phi K, Q)}{V(\hat{\nabla}_p^\tau \phi K)} = \frac{\tilde{V}_{-p}(\phi \hat{\nabla}_p^\tau K, Q)}{V(\phi \hat{\nabla}_p^\tau K)}. \tag{3.2}$$

Thus from Lemma 3.1, this yields

$$\hat{\nabla}_p^\tau \phi K = \phi \hat{\nabla}_p^\tau K.$$

□

Theorem 3.2. *If* $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$, *then*

$$\hat{\nabla}_p^{-\tau} K = \hat{\nabla}_p^\tau(-K) = -\hat{\nabla}_p^\tau K.$$

Proof. From (1.12) and (1.13), we obtain that for $p \geq 1$ and $\tau \in [-1, 1]$,

$$\begin{aligned} \hat{\nabla}_p^{-\tau} K &= \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p} \circ K \hat{+}_p \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p} \circ (-K) \\ &= \hat{\nabla}_p^\tau(-K). \end{aligned}$$

Further, we prove the right equality. It easily follows that for any $u \in S^{n-1}$,

$$\begin{aligned} \frac{\rho(-\hat{\nabla}_p^\tau K, u)^{n+p}}{V(-\hat{\nabla}_p^\tau K)} &= \frac{\rho(\hat{\nabla}_p^\tau K, -u)^{n+p}}{V(\hat{\nabla}_p^\tau K)} \\ &= f_1(\tau)\frac{\rho(K, -u)^{n+p}}{V(K)} + f_2(\tau)\frac{\rho(-K, -u)^{n+p}}{V(-K)} \\ &= f_1(\tau)\frac{\rho(-K, u)^{n+p}}{V(-K)} + f_2(\tau)\frac{\rho(-(-K), u)^{n+p}}{V(-(-K))} \\ &= \frac{\rho(\hat{\nabla}_p^\tau(-K), u)^{n+p}}{V(\hat{\nabla}_p^\tau(-K))}. \end{aligned}$$

Similarly, using the proof method of (3.2), it follows that for any $Q \in \mathcal{S}_o^n$,

$$\frac{\tilde{V}_{-p}(-\hat{\nabla}_p^\tau K, Q)}{V(-\hat{\nabla}_p^\tau K)} = \frac{\tilde{V}_{-p}(\hat{\nabla}_p^\tau(-K), Q)}{V(\hat{\nabla}_p^\tau(-K))}.$$

Hence, from Lemma 3.1, we get

$$\hat{\nabla}_p^\tau(-K) = -\hat{\nabla}_p^\tau K.$$

□

Theorem 3.3. *If $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$ and $\tau \neq 0$, then $\hat{\nabla}_p^{\tau} K = \hat{\nabla}_p^{-\tau} K$ if and only if K is an origin-symmetric star body.*

Proof. From (1.9) and (1.12), we get that for all $u \in S^{n-1}$,

$$\frac{\rho(\hat{\nabla}_p^{\tau} K, u)^{n+p}}{V(\hat{\nabla}_p^{\tau} K)} = f_1(\tau)\frac{\rho(K, u)^{n+p}}{V(K)} + f_2(\tau)\frac{\rho(-K, u)^{n+p}}{V(-K)}, \tag{3.3}$$

$$\frac{\rho(\hat{\nabla}_p^{-\tau} K, u)^{n+p}}{V(\hat{\nabla}_p^{-\tau} K)} = f_2(\tau)\frac{\rho(K, u)^{n+p}}{V(K)} + f_1(\tau)\frac{\rho(-K, u)^{n+p}}{V(-K)}. \tag{3.4}$$

Hence, K is an origin-symmetric star body, i.e. $K = -K$. Combining with (3.3) and (3.4), we get

$$\frac{\rho(\hat{\nabla}_p^{\tau} K, u)^{n+p}}{V(\hat{\nabla}_p^{\tau} K)} = \frac{\rho(\hat{\nabla}_p^{-\tau} K, u)^{n+p}}{V(\hat{\nabla}_p^{-\tau} K)}.$$

Thus, we use Lemma 3.1 to get

$$\hat{\nabla}_p^{\tau} K = \hat{\nabla}_p^{-\tau} K.$$

Conversely, if $\hat{\nabla}_p^{\tau} K = \hat{\nabla}_p^{-\tau} K$, then together with (3.3) and (3.4) we get

$$[f_1(\tau) - f_2(\tau)]\frac{\rho(K, u)^{n+p}}{V(K)} = [f_1(\tau) - f_2(\tau)]\frac{\rho(-K, u)^{n+p}}{V(-K)}. \tag{3.5}$$

Since $f_1(\tau) - f_2(\tau) \neq 0$ when $\tau \neq 0$, thus it follows that $\rho(K, u) = \rho(-K, u)$ for all $u \in S^{n-1}$, i.e., K is an origin-symmetric star body. □

From Theorem 3.3, the following result immediately follows.

COROLLARY 3.1

For $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$, if K is not an origin-symmetric star body, then $\hat{\nabla}_p^{\tau} K = \hat{\nabla}_p^{-\tau} K$ if and only if $\tau = 0$.

Theorem 3.4. *If $K \in \mathcal{S}_s^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$\hat{\nabla}_p^{\tau} K = K.$$

Proof. Since $K \in \mathcal{S}_s^n$, i.e. $K = -K$, by (1.9) and (1.11), we get for any $u \in S^{n-1}$,

$$\begin{aligned}\frac{\rho(\hat{\nabla}_p^{\tau} K, u)^{n+p}}{V(\hat{\nabla}_p^{\tau} K)} &= f_1(\tau)\frac{\rho(K, u)^{n+p}}{V(K)} + f_2(\tau)\frac{\rho(-K, u)^{n+p}}{V(-K)}.\\ &= \frac{\rho(K, u)^{n+p}}{V(K)}.\end{aligned}$$

Using Lemma 3.1, we get

$$\hat{\nabla}_p^\tau K = K. \qquad \square$$

4. Proofs of main results

In this section, we complete the proofs of Theorems 1.1–1.3.

Proof of Theorem 1.1. From (1.11), we use (1.7) to get for any $\tau \in [-1, 1]$,

$$\begin{aligned} V(\hat{\nabla}_p^\tau K)^{\frac{p}{n}} &= V(f_1(\tau) \circ K \hat{+}_p f_2(\tau) \circ (-K))^{\frac{p}{n}} \\ &\geq f_1(\tau) V(K)^{\frac{p}{n}} + f_2(\tau) V(-K)^{\frac{p}{n}} \\ &= V(K)^{\frac{p}{n}}, \end{aligned}$$

i.e.,

$$V(\hat{\nabla}_p^\tau K) \geq V(K). \tag{4.1}$$

Clearly, equality holds in (4.1) if $\tau = \pm 1$. Besides, if $\tau \neq \pm 1$, then by the condition of equality in (1.7), we see that equality holds in (4.1) if and only if K and $-K$ are dilates, this yields $K = -K$, i.e., K is an origin-symmetric star body. This means that if K is not an origin-symmetric body, then equality holds in (4.1) if and only if $\tau = \pm 1$.

Now, we prove the left inequality of (1.14). Since L_p-harmonic Blaschke sum is distributive, it follows from (1.6) and (1.13) that for any $u \in S^{n-1}$,

$$\begin{aligned} \hat{\nabla}_p K &= \frac{1}{2} \frac{(1+\tau)^p + (1-\tau)^p}{(1+\tau)^p + (1-\tau)^p} \circ K \hat{+}_p \frac{1}{2} \frac{(1-\tau)^p + (1+\tau)^p}{(1+\tau)^p + (1-\tau)^p} \circ (-K) \\ &= \frac{1}{2} \circ \hat{\nabla}_p^\tau K \hat{+}_p \frac{1}{2} \circ \hat{\nabla}_p^{-\tau} K. \end{aligned} \tag{4.2}$$

From Theorem 3.2 and (4.2), we use (1.7) which yields

$$\begin{aligned} V(\hat{\nabla}_p K)^{\frac{p}{n}} &= V\left(\frac{1}{2} \circ \hat{\nabla}_p^\tau K \hat{+}_p \frac{1}{2} \circ \hat{\nabla}_p^{-\tau} K\right)^{\frac{p}{n}} \\ &\geq \frac{1}{2} V(\hat{\nabla}_p^\tau K)^{\frac{p}{n}} + \frac{1}{2} V(\hat{\nabla}_p^{-\tau} K)^{\frac{p}{n}} \\ &= \frac{1}{2} V(\hat{\nabla}_p^\tau K)^{\frac{p}{n}} + \frac{1}{2} V(-\hat{\nabla}_p^\tau K)^{\frac{p}{n}} \\ &= V(\hat{\nabla}_p^\tau K)^{\frac{p}{n}}. \end{aligned}$$

We get

$$V(\hat{\nabla}_p K) \geq V(\hat{\nabla}_p^\tau K). \tag{4.3}$$

According to the equality condition of (1.7), we see that equality holds in (4.3) if and only if $\hat{\nabla}_p^\tau K$ and $\hat{\nabla}_p^{-\tau} K$ are dilates. Since $\hat{\nabla}_p^{-\tau} K = -\hat{\nabla}_p^\tau K$, $\hat{\nabla}_p^\tau K = \hat{\nabla}_p^{-\tau} K$. Therefore, using Corollary 3.1, we obtain that if K is not origin-symmetric body, then equality holds in (4.3) if and only if $\tau = 0$. $\square$

Proof of Theorem 1.2. From definition (2.6), we get

$$\begin{aligned}
&\frac{\omega_n^{-\frac{p}{n}}\tilde{G}_{-p}(\hat{\nabla}_p^\tau K)}{V(\hat{\nabla}_p^\tau K)}\\
&= \inf\left\{n\frac{\tilde{V}_{-p}(\hat{\nabla}_p^\tau K, Q)}{V(\hat{\nabla}_p^\tau K)}V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_s^n\right\}\\
&= \inf\left\{n\frac{\tilde{V}_{-p}(f_1(\tau)\circ K\hat{+}_p f_2(\tau)\circ(-K), Q)}{V(f_1(\tau)\circ K\hat{+}_p f_2(\tau)\circ(-K))}V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_s^n\right\}\\
&= \inf\left\{\int_{S^{n-1}}\left[\frac{\rho(f_1(\tau)\circ K\hat{+}_p f_2(\tau)\circ(-K), u)^{n+p}}{V(f_1(\tau)\circ K\hat{+}_p f_2(\tau)\circ(-K))}\rho(Q,u)^{-p}\right]\right.\\
&\quad\left.\times du V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_s^n\right\}\\
&= \inf\left\{\int_{S^{n-1}}\left[f_1(\tau)\frac{\rho(K,u)^{n+p}}{V(K)} + f_2(\tau)\frac{\rho(-K,u)^{n+p}}{V(-K)}\right]\right.\\
&\quad\left.\times \rho(Q,u)^{-p} du V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_s^n\right\}\\
&\geq \frac{f_1(\tau)}{V(K)}\inf\left\{n\tilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_s^n\right\}\\
&\quad + \frac{f_2(\tau)}{V(-K)}\inf\left\{n\tilde{V}_{-p}(-K,Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_s^n\right\}. \qquad (4.4)
\end{aligned}$$

Since $Q \in \mathcal{K}_s^n$, we use $\rho(Q,u) = \rho(-Q,u) = \rho(Q,-u)$ to get

$$\tilde{V}_{-p}(-K, Q) = \tilde{V}_{-p}(K, Q). \qquad (4.5)$$

Combining (4.4) and (4.5), we get

$$\frac{\tilde{G}_{-p}(\hat{\nabla}_p^\tau K)}{V(\hat{\nabla}_p^\tau K)} \geq \frac{\tilde{G}_{-p}(K)}{V(K)}, \qquad (4.6)$$

i.e.,

$$\frac{\tilde{G}_{-p}(\hat{\nabla}_p^\tau K)}{\tilde{G}_{-p}(K)} \geq \frac{V(\hat{\nabla}_p^\tau K)}{V(K)}. \qquad (4.7)$$

Together with the right inequality of (1.14), this yields

$$\tilde{G}_{-p}(\hat{\nabla}_p^\tau K) \geq \tilde{G}_{-p}(K). \qquad (4.8)$$

Equality of (4.4) holds if and only if $\hat{\nabla}_p^\tau K$ are dilates with K and $-K$, respectively i.e., equality holds in (4.4) if and only if K and $-K$ are dilates. This yields $K = -K$, thus K is an origin-symmetric star body. Since (4.4) and (4.7) are equivalent, hence equality holds in (4.7) if and only if K is an origin-symmetric star body. Besides, if $\tau \neq \pm 1$, we know that equality holds in the right-hand side of (1.14) if and only if K is also an origin-symmetric star body. Therefore, if $\tau \neq \pm 1$, equality holds in (4.8) if and only if K is an origin-symmetric star body.

Further, we prove the left inequality of (1.15). From Theorem 3.2, we know that

$$\hat{\nabla}_p^{-\tau} K = -\hat{\nabla}_p^{\tau} K.$$

Thus, (4.2) can be written as

$$\hat{\nabla}_p K = \frac{1}{2} \circ \hat{\nabla}_p^{\tau} K \hat{+}_p \frac{1}{2} \circ (-\hat{\nabla}_p^{\tau} K).$$

Similar to the proof of inequality (4.6), we have

$$\frac{\tilde{G}_{-p}(\hat{\nabla}_p K)}{V(\hat{\nabla}_p K)} \geq \frac{\tilde{G}_{-p}(\hat{\nabla}_p^{\tau} K)}{V(\hat{\nabla}_p^{\tau} K)}, \tag{4.9}$$

that is

$$\frac{\tilde{G}_{-p}(\hat{\nabla}_p K)}{\tilde{G}_{-p}(\hat{\nabla}_p^{\tau} K)} \geq \frac{V(\hat{\nabla}_p K)}{V(\hat{\nabla}_p^{\tau} K)}. \tag{4.10}$$

By the left inequality of (1.14), this yields

$$\tilde{G}_{-p}(\hat{\nabla}_p K) \geq \tilde{G}_{-p}(\hat{\nabla}_p^{\tau} K). \tag{4.11}$$

Similar to the equality proof of inequality (4.7), we easily get that equality holds in (4.10) if and only if $\hat{\nabla}_p^{\tau} K = \hat{\nabla}_p^{-\tau} K$. Besides, we also know that equality holds in the left inequality of (1.14) if and only if $\hat{\nabla}_p^{\tau} K = \hat{\nabla}_p^{-\tau} K$. Hence, if $\tau \neq 0$, using Theorem 3.3 to get that equality holds in (4.11) if and only if K is an origin-symmetric star body. □

Proof of Theorem 1.3. Inequality (2.3) implies that for $M \in \mathcal{S}_o^n$, then

$$V(M)V(M^c) \leq \omega_n^2, \tag{4.12}$$

with equality if and only if M is an ellipsoid. Since $\hat{\nabla}_p^{\tau} K \in \mathcal{S}_o^n$, thus, from the right inequality of (1.14) and (4.12) we obtain that

$$V(K)V(\hat{\nabla}_p^{\tau,c} K) \leq V(\hat{\nabla}_p^{\tau} K)V(\hat{\nabla}_p^{\tau,c} K) \leq \omega_n^2. \tag{4.13}$$

If $\tau = \pm 1$, equality holds in the first inequality of (4.13), thus together with the equality condition of inequality (4.12), we show that if $\tau = \pm 1$, equality holds in (1.16) if and only if K is an ellipsoid. □

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