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Vector-valued almost convergence and classical properties in normed spaces

A AIZPURU*, R ARMARIO*, F J GARCÍA-PACHECO and F J PÉREZ-FERNÁNDEZ

Department of Mathematics, University of Cadiz, Puerto Real 11510, Spain
E-mail: garcia.pacheco@uca.es; javier.perez@uca.es

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Abstract. In this paper we study the almost convergence and the almost summability in normed spaces. Among other things, spaces of sequences defined by the almost convergence and the almost summability are proved to be complete if the basis normed space is so. Finally, some classical properties such as completeness, reflexivity, Schur property, Grothendieck property, and the property of containing a copy of c_0 are characterized in terms of the almost convergence.

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1. Introduction

The notion of *almost convergence* appeared for the first time in the literature of the theory of series in normed spaces in 1948 (see [14]) and was introduced by Lorentz. Many results, extensions and generalizations of this concept have been provided ever since. We refer the reader to [10, 11, 16] for a wide perspective on the concept of almost convergence and some generalizations and relations with matrix methods and invariant means. For different applications of almost convergence, we refer the reader to [1, 2], where a series of results are provided involving almost convergence, almost summability and (weakly) unconditionally Cauchy series.

In the field of the theory of series in normed spaces it has been of great interest – and always will – (cf. for instance [1, 17]) the extensions, generalizations, and applications of the Hahn–Schur theorem and Orlicz–Pettis theorem. An excellent exposition of this topic can be viewed in [18]. In particular, in [4] the authors establish a generalization of the Orlicz–Pettis theorem by means of sub-algebras of $\mathcal{P}(\mathbb{N})$. In [4, 6] the authors provide Hahn–Schur theorem-like results on the uniform convergence of weakly unconditionally Cauchy series (wuC) in real Banach spaces. These results generalize previous ones from Swartz [19] and Bu and Wu [12] on the uniform convergence of unconditionally convergent series.

*Since deceased.

In [5] the authors obtain a generalization of the Orlicz–Pettis theorem by means of a regular matrix weak summability method. On the other hand, in [3] it is shown that a generalization of the above results on the uniform convergence of wuC series by means of regular matrix summability methods (we refer the reeader to [9] for a wide perspective on general matrix summability methods).

Probably, one of the most stunning characteristics of the almost convergence is that the almost summability is not representable by any matrix summability method (cf. [9]). Thus, it is legitimate to ask questions like whether or not it is possible to obtain generalizations of the Orlicz–Pettis theorem and uniform almost convergence from any particular situation of point-wise almost convergence. In this sense, positive results were provided in [1, 2].

Following [18], most of these generalizations can be interpreted in terms of multiplier convergent series. On the other hand, whenever a convergence method is introduced, it is common to study spaces of sequences defined by that convergence method. However, in the case of the almost convergence, such spaces have not yet been introduced. Only in [1] we consider spaces of bounded scalar sequences associated to a given series and defined by the almost convergence. In this manuscript we define and study spaces of vector sequences defined by the almost convergence and the almost summability, and following the lines proposed in the above references we characterize, in terms of the almost convergence, classical properties such us completeness, reflexivity, the Schur property, the Grothendieck property, and the property of containing a copy of c_0.

Throughout this paper X will stand for a real normed space unless we say otherwise. The (closed) unit ball, the open unit ball, and the unit sphere will be denoted as usual by B_X, U_X, and S_X, respectively. In a similar way, the (closed) ball of center x and radius r, the open ball of center x and radius r, and the sphere of center x and radius r will be denoted as usual by $\mathsf{B}_X(x, r)$, $\mathsf{U}_X(x, r)$ and $\mathsf{S}_X(x, r)$, respectively. The usual space of bounded sequences in X will be denoted by $\ell_\infty(X)$, which when endowed with the sup norm becomes a Banach space if X is so. The usual subspaces of $\ell_\infty(X)$ are $c_{00}(X)$ (the space of eventually null sequences in X), $c_0(X)$ (the space of null sequences in X), and $c(X)$ (the space of convergent sequences in X). It is well known that $c_{00}(X) \subset c_0(X) \subset c(X) \subset \ell_\infty(X)$. Finally, the space of absolutely summable sequences in X will be denoted as usual by $\ell_1(X)$.

In his 1932 book *Théorie des opérations linéaires* (see p. 34 of [7]), Banach extended in a natural way the limit function defined on the space of all convergent sequences to the space of all bounded sequences.

DEFINITION 1.1 [7]

A function $\varphi : \ell_\infty \to \mathbb{R}$ is called a Banach limit exactly when:

(1) $\varphi\left((\alpha x_n + \beta y_n)_{n\in\mathbb{N}}\right) = \alpha\varphi\left((x_n)_{n\in\mathbb{N}}\right) + \beta\varphi\left((y_n)_{n\in\mathbb{N}}\right)$ for every $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell_\infty$ and every $\alpha, \beta \in \mathbb{R}$,
(2) $\varphi\left((x_n)_{n\in\mathbb{N}}\right) \geq 0$ if $(x_n)_{n\in\mathbb{N}} \in \ell_\infty$ and $x_n \geq 0$ for all $n \in \mathbb{N}$,
(3) $\varphi\left((x_n)_{n\in\mathbb{N}}\right) = \varphi\left((x_{n+1})_{n\in\mathbb{N}}\right)$ if $(x_n)_{n\in\mathbb{N}} \in \ell_\infty$, and
(4) $\varphi(\mathbf{1}) = 1$, where $\mathbf{1}$ denotes the constant sequence of general term equal to 1.

Banach noticed that a linear function $\varphi : \ell_\infty \to \mathbb{R}$ is a Banach limit if and only if φ is invariant under the shift operator on ℓ_∞ and $\liminf_{n\to\infty} x_n \leq \varphi\left((x_n)_{n\in\mathbb{N}}\right) \leq \limsup_{n\to\infty} x_n$ for all $(x_n)_{n\in\mathbb{N}} \in \ell_\infty$. As a consequence, every Banach limit $\varphi : \ell_\infty \to \mathbb{R}$

verifies that $\varphi|_c = \lim$, that is, if $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, then $\varphi\left((x_n)_{n\in\mathbb{N}}\right) = \lim_{n\to\infty} x_n$. So, Banach limits are legitimate extensions of the limit function on c. In [14], Lorentz made use of the concept of Banach limit to introduce the notion of 'almost convergence'.

DEFINITION 1.2 [14]

A bounded sequence $(x_n)_{n\in\mathbb{N}} \in \ell_\infty$ is called almost convergent exactly when there exists a number $y \in \mathbb{R}$ (called the almost limit of $(x_n)_{n\in\mathbb{N}}$) such that $\varphi\left((x_n)_{n\in\mathbb{N}}\right) = y$ for all Banach limits $\varphi : \ell_\infty \to \mathbb{R}$. We will denote the limit by $\mathrm{AC}\lim_{n\to\infty} x_n = y$.

In [14], Lorentz provided an intrinsic characterization of almost convergent sequences: Given a bounded sequence $(x_n)_{n\in\mathbb{N}} \in \ell_\infty$ and a real number y, we have that $\mathrm{AC}\lim_{n\to\infty} x_n = y$ if and only if $\lim_{p\to\infty} \frac{1}{p+1} \sum_{k=0}^{p} x_{n+k} = y$ uniformly in $n \in \mathbb{N}$. Using this characterization, Boos [9] extended the concept of almost convergence to vector-valued bounded sequences:

DEFINITION 1.3 [9]

Let X be a real normed space. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ in X:

(1) $(x_n)_{n\in\mathbb{N}}$ is called almost convergent exactly when there exists $y \in X$ (called the almost limit of $(x_n)_{n\in\mathbb{N}}$) such that $\lim_{p\to\infty} \frac{1}{p+1} \sum_{k=0}^{p} x_{n+k} = y$ uniformly in $n \in \mathbb{N}$. We will denote the limit by $\mathrm{AC}\lim_{n\to\infty} x_n = y$.
(2) $(x_n)_{n\in\mathbb{N}}$ is called weakly almost convergent exactly when there exists an element $y \in X$ (called the weak almost limit of $(x_n)_{n\in\mathbb{N}}$) such that $\mathrm{AC}\lim_{n\to\infty} f(x_n) = f(y)$ for all $f \in X^*$. We will denote the limit by $\omega\mathrm{AC}\lim_{n\to\infty} x_n = y$.

Every weakly almost convergent sequence is bounded (see Theorem 1.2.18(a) of [9]) and every almost convergent sequence is weakly almost convergent. The concept of almost convergence also makes sense in the context of series in real normed spaces.

DEFINITION 1.4 [9]

Let X be a real normed space. Consider a series $\sum_{i=1}^{\infty} x_i$ in X:

(1) $\sum_{i=1}^{\infty} x_i$ is called almost convergent exactly when there exists $y \in X$ (called the almost sum of $\sum_{i=1}^{\infty} x_i$) such that $\mathrm{AC}\lim_{n\to\infty} \sum_{i=1}^{n} x_i = y$. We will denote the limit by $\mathrm{AC}\sum_{i=1}^{\infty} x_i = y$.
(2) $\sum_{i=1}^{\infty} x_i$ is called weakly almost convergent exactly when there exists an element $y \in X$ (called the weak almost sum of $\sum_{i=1}^{\infty} x_i$) such that $\omega\mathrm{AC}\lim_{n\to\infty} \sum_{i=1}^{n} x_i = y$. We will denote the limit by $\omega\mathrm{AC}\sum_{i=1}^{\infty} x_i = y$.

The reader may find it easy to check that, given a series $\sum_{i=1}^{\infty} x_i$ in a real normed space X and an element $y \in X$,

(1) $\mathrm{AC}\sum\limits_{i=1}^{\infty} x_i = y$, if and only if

$$\lim_{p\to\infty} \left(\sum_{k=1}^{n} x_k + \frac{1}{p+1} \sum_{k=1}^{p} (p-k+1)\ x_{n+k} \right) = y$$

uniformly in $n \in \mathbb{N}$.

(2) $\omega\mathrm{AC} \sum_{i=1}^{\infty} x_i = y$, if and only if

$$\lim_{p\to\infty}\left(\sum_{k=1}^{n} f(x_k) + \frac{1}{p+1}\sum_{k=1}^{p}(p-k+1)\ f(x_{n+k})\right) = f(y)$$

uniformly in $n \in \mathbb{N}$, for every $f \in X^*$.

2. Spaces of sequences defined by the almost convergence

We will let $ac(X)$ denote the set of all almost convergent sequence in X, that is,

$$ac(X) = \{(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} : \mathrm{AC}\lim_{i\to\infty} x_i \text{ exists}\}.$$

Likewise we can also consider the set of all weakly almost convergence sequences in X, that is,

$$\omega ac(X) = \{(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} : \omega\mathrm{AC}\lim_{i\to\infty} x_i \text{ exists}\}.$$

Notice that $ac(X) \subseteq \omega ac(X) \subset \ell_\infty(X)$. A natural problem to wonder is on the completeness of the previous two spaces. We take care of this in the following result.

Theorem 2.1. *Let X be a real Banach space. Then $ac(X)$ and $\omega ac(X)$ are closed subspaces of $\ell_\infty(X)$ endowed with the sup norm.*

Proof. We will only prove the closedness of $\omega ac(X)$. The closedness of $ac(X)$ can be shown in a similar way. Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $\omega ac(X)$ and consider $x^0 \in \ell_\infty(X)$ such that $\lim_{n\to\infty}\|x^n - x^0\|_\infty = 0$. We will first show that $x^0 \in \omega ac(X)$. For each natural n, there exists $x_n \in X$ satisfying $\omega\mathrm{AC}\lim_{i\to\infty} x_i^n = x_n$. We will first show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Take any $\varepsilon > 0$. An $n_0 \in \mathbb{N}$ can be found such that for each $p, q \geqslant n_0$ we have that $\|x^p - x^q\|_\infty \leqslant \varepsilon/3$. Fix $p, q \geqslant n_0$ and consider a functional $f \in \mathsf{S}_{X^*}$ such that $\|x_p - x_q\| = |f(x_p) - f(x_q)|$. There exists $i \in \mathbb{N}$ such that

$$\left| f(x_p) - \frac{1}{i+1}(f(x_j^p) + \cdots + f(x_{j+i}^p))\right| \leqslant \frac{\varepsilon}{3}$$

and

$$\left| f(x_q) - \frac{1}{i+1}(f(x_j^q) + \cdots + f(x_{j+i}^q))\right| \leqslant \frac{\varepsilon}{3}$$

for every $j \in \mathbb{N}$. It follows that

$$\begin{aligned}\|x_p - x_q\| &\leqslant \left| f(x_p) - \frac{1}{i+1}(f(x_j^p) + \cdots + f(x_{j+i}^p))\right| \\ &+ \left|\frac{1}{i+1}(f(x_j^p - x_j^q) + \cdots + f(x_{j+i}^p - x_{j+i}^q))\right| \\ &+ \left| f(x_q) - \frac{1}{i+1}(f(x_j^q) + \cdots + f(x_{j+i}^q))\right| \\ &\leqslant \varepsilon.\end{aligned}$$

Since X is complete, there exists $x_0 \in X$ such that $\lim_{n\to\infty} x_n = x_0$. Finally, we will show that $\omega\mathrm{AC}\lim_{i\to\infty} x_i^0 = x_0$. If $\varepsilon > 0$ is given and $f \in X^* \setminus \{0\}$, then we can fix $p \in \mathbb{N}$

satisfying $\|x_0 - x_p\| \leqslant \frac{\varepsilon}{3\|f\|}$ and $\|x^0 - x^p\|_\infty \leqslant \frac{\varepsilon}{3\|f\|}$. Since $\omega\mathrm{AC}\lim_{i\to\infty} x_i^p = x_p$, there exists $i_0 \in N$ such that

$$\left| f(x_p) - \frac{1}{i+1}(f(x_j^p) + \cdots + f(x_{j+i}^p)) \right| \leqslant \frac{\varepsilon}{3}$$

for every $i \geqslant i_0$ and every $j \in \mathbb{N}$. Thus

$$\begin{aligned}
&\left| f(x_0) - \frac{1}{i+1}(f(x_j^0) + \cdots + f(x_{j+i}^0)) \right| \\
&\quad \leqslant \left| f(x_0) - f(x_p) \right| \\
&\quad + \left| f(x_p) - \frac{1}{i+1}(f(x_j^p) + \cdots + f(x_{j+i}^p)) \right| \\
&\quad + \left| \frac{1}{i+1}(f(x_j^p - x_j^0) + \cdots + f(x_{j+i}^p - x_{j+i}^0)) \right| \\
&\quad \leqslant \varepsilon,
\end{aligned}$$

for every $i \geqslant i_0$ and every $j \in \mathbb{N}$. In order words, $\omega\mathrm{AC}\lim_{i\to\infty} x_i^0 = x_0$. □

As an immediate corollary we obtain the following:

COROLLARY 2.1

Let X be a real normed space. The following conditions are equivalent:

(1) *$ac\,(X)$ is complete.*
(2) *$\omega ac\,(X)$ is complete.*
(3) *X is complete.*

Proof. It suffices to show that X is a closed subspace of $\ell_\infty(X)$ (even if X is not complete). This way if anyone of $ac(X)$ or $\omega ac(X)$ is complete, then X will be so. □

The next remark shows that in c_0 the weak almost convergence can be characterized through the coordinate-wise almost convergence.

Remark 2.1.

(1) In the first place, observe that given a real normed space X, a dense vector subspace N in X^*, and a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, we have that $\omega\mathrm{AC}\lim_{n\to\infty} x_n = 0$ if for each $g \in N$ we have that $\mathrm{AC}\lim_{n\to\infty} g\,(x_n) = 0$. Indeed, fix an arbitrary $f \in X^*$ and consider $\varepsilon > 0$. There exists $A > 0$ such that $\|x_n\| < A$ for each $n \in \mathbb{N}$. By the density of N in X^* there exists $g \in N$ satisfying that $\|f - g\| < \frac{\varepsilon}{2A}$. Then

$$\begin{aligned}
\frac{1}{i+1}\left| f\left(\sum_{k=0}^{i} x_{j+k}\right)\right| &\leqslant \frac{1}{i+1}\left[\left|(f-g)\left(\sum_{k=0}^{i} x_{j+k}\right)\right| + \left|g\left(\sum_{k=0}^{i} x_{j+k}\right)\right|\right] \\
&\leqslant \frac{1}{i+1}\frac{\varepsilon}{2A}\,(i+1)\,A + \frac{1}{i+1}\left|g\left(\sum_{k=0}^{i} x_{j+k}\right)\right| \\
&= \frac{\varepsilon}{2} + \frac{1}{i+1}\left|g\left(\sum_{k=0}^{i} x_{j+k}\right)\right|
\end{aligned}$$

for every $i, j \in \mathbb{N}$. Since $\mathrm{AC}\lim_{n\to\infty} g(x_n) = 0$, we conclude that $\mathrm{AC}\lim_{n\to\infty} f(x_n) = 0$. The arbitrariness of f tells us that $\omega\mathrm{AC}\lim_{n\to\infty} x_n = 0$.

(2) From the above part and the density of c_{00} in ℓ_1 we deduce that a sequence $(x^n)_{n\in\mathbb{N}} \subset c_0$ is weakly almost convergent to $x^0 \in c_0$ if and only if $\mathrm{AC}\lim_{n\to\infty} x_i^n = x_i^0$ for every $i \in \mathbb{N}$.

(3) As an example of the previous part, the sequence $(x^n)_{n\in\mathbb{N}} \subset c_0$ is given by

$$x_i^n = \begin{cases} 0, & \text{if } i > n, \\ 1, & \text{if } n = j + (2k+1)i,\ j \in \{0, 1, \ldots, i-1\}, k \in \mathbb{N}, \\ -1, & \text{if } n = j + (2k+2)i,\ j \in \{0, 1, \ldots, i-1\}, k \in \mathbb{N}, \end{cases}$$

and satisfies that $\omega\mathrm{AC}\lim_{n\to\infty} x^n = 0$. However, it is not almost convergent.

In a natural way we can consider the almost convergence of sequences in the dual of a real normed space endowed with the weak star topology, which takes us to the concept of *weak-star almost convergence*.

DEFINITION 2.1

Let X be a real normed space. Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in X^*. We will say that $f_0 \in X^*$ is the weak-star almost limit of $(f_i)_{i\in\mathbb{N}}$ exactly when for each $x \in X$ we have that $\mathrm{AC}\lim_{i\to\infty} f_i(x) = f_0(x)$. We will denote the limit by $\omega^*\mathrm{AC}\lim_{i\to\infty} f_i = f_0$. Based upon the previous definition we have

$$\omega^* ac(X^*) := \left\{ (f_i)_{i\in\mathbb{N}} \in (X^*)^{\mathbb{N}} : \omega^*\mathrm{AC} \lim_{i\to\infty} f_i \text{ exists} \right\}.$$

Theorem 2.2. *Let X be a real normed space. The set $\omega^* ac(X^*) \cap \ell_\infty(X^*)$ is a closed subspace of $\ell_\infty(X^*)$.*

Proof. Let $(f^n)_{n\in\mathbb{N}}$ be a sequence in $\omega^* ac(X^*) \cap \ell_\infty(X^*)$ so that $\lim_{n\to\infty} \|f^n - f^0\|_\infty = 0$ for some $f^0 \in \ell_\infty(X^*)$. Our purpose is to prove that $f^0 \in \omega^* ac(X^*)$. For each natural n there exists $f_n \in X^*$ such that $\omega^*\mathrm{AC}\lim_{i\to\infty}(f_i^n) = f_n$. We will show next that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. If $\varepsilon > 0$ is given, there exists $n_0 \in \mathbb{N}$ such that $\|f^p - f^q\|_\infty \leqslant \varepsilon/6$ for any $p, q \geqslant n_0$. Fix $p, q \geqslant n_0$. We can find a vector $x \in \mathsf{S}_X$ satisfying

$$\|f_p - f_q\| - \frac{\varepsilon}{2} < |(f_p - f_q)(x)| \leqslant \|f_p - f_q\|.$$

Consider a natural i such that for every $j \in \mathbb{N}$ we have

$$\left| f_p(x) - \frac{1}{i+1}(f_j^p(x) + \cdots + f_{j+i}^p(x)) \right| \leqslant \frac{\varepsilon}{6}$$

and

$$\left| f_q(x) - \frac{1}{i+1}(f_j^q(x) + \cdots + f_{j+i}^q(x)) \right| \leqslant \frac{\varepsilon}{6}.$$

It follows that

$$\begin{aligned}\|f_p - f_q\| - \frac{\varepsilon}{2} &\leqslant \left| f_p(x) - \frac{1}{i+1}(f_j^p(x) + \cdots + f_{j+i}^p(x)) \right| \\ &+ \left| \frac{1}{i+1}((f_j^p - f_j^q)(x) + \cdots + (f_{j+i}^p - f_{j+i}^q)(x)) \right| \\ &+ \left| \frac{1}{i+1}(f_j^q(x) + \cdots + f_{j+i}^q(x)) - f_q(x) \right| \\ &\leqslant \frac{\varepsilon}{2},\end{aligned}$$

that is, $\|f_p - f_q\| \leqslant \varepsilon$ for each $p, q \geqslant n_0$. Then there exists $f_0 \in X^*$ such that $\lim_{n\to\infty} \|f_n - f_0\| = 0$. We will now show that $\omega^*\mathrm{AC}\lim_{i\to\infty} f_i^0 = f_0$. Consider $x \in X \setminus \{0\}$ and $\varepsilon > 0$. We can fix $p \in \mathbb{N}$ such that $\|f^p - f^0\|_\infty \leqslant \frac{\varepsilon}{3\|x\|}$ and $\|f_p(x) - f_0(x)\| \leqslant \frac{\varepsilon}{3}$. Since $\omega^*\mathrm{AC}\lim_{i\to\infty} f_i^p = f_0$, there exists $i_0 \in \mathbb{N}$ such that for each $i \geqslant i_0$ it is satisfied that $|f_p(x) - \frac{1}{i+1}(f_j^p(x) + \cdots + f_{j+i}^p(x))| \leqslant \frac{\varepsilon}{3}$ for every $j \in \mathbb{N}$. Therefore, if $i \geqslant i_0$, then

$$\begin{aligned}&\left| f_0(x) - \frac{1}{i+1}(f_j^0(x) + \cdots + f_{j+i}^0(x)) \right| \\ &\quad\leqslant \left| f_0(x) - f_p(x) \right| \\ &\quad+ \left| f_p(x) - \frac{1}{i+1}(f_j^p(x) + \cdots + f_{j+i}^p(x)) \right| \\ &\quad+ \left| \frac{1}{i+1}((f_j^p - f_j^0)(x) + \cdots + (f_{j+i}^p - f_{j+i}^0)(x)) \right| \\ &\quad\leqslant \varepsilon\end{aligned}$$

for every $j \in \mathbb{N}$. Thus, $\omega^*\mathrm{AC}\lim_{i\to\infty} f_i = f_0$. □

COROLLARY 2.2

Let X be a barrelled real normed space. The set $\omega^*ac(X^*)$ is a closed subspace of $\ell_\infty(X^*)$.

Proof. It suffices to show that if X is barrelled, then every weak-star almost convergent sequence in X^* is bounded and hence $\omega^*ac(X^*) \subseteq \ell_\infty(X^*)$. □

Remark 2.2. In relation to the previous corollary, an example of a weak-star convergent sequence which is not bounded is the following: Consider $X := c_{00}$ and $(ne_n)_{n\in\mathbb{N}} \subset \ell_1 = X^*$. It is easy to see that $(ne_n)_{n\in\mathbb{N}}$ is weak-star convergent in X^* to 0 but it is not bounded.

As in Remark 2.1, the weak-star almost convergence in ℓ_1 can be characterized through the coordinate-wise almost convergence.

Remark 2.3.

(1) In the first place, consider X to be a real normed space and M a dense vector subspace of X. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in X^*. If for each $y \in M$ we have that

AC $\lim_{n\to\infty} f_n(y) = 0$, then it follows that ω^*AC $\lim_{n\to\infty} f_n = 0$. We omit the proof of this fact since it is very similar to the corresponding one in Remark 2.1.

(2) Due to the previous point and the density of c_{00} in c_0, we have that a bounded sequence $(f^n)_{n\in\mathbb{N}} \subset \ell_1$ is ω^* almost convergent to $f^0 \in \ell_1$ if and only if AC $\lim_{n\to\infty} f_i^n = f_i^0$ for each $i \in \mathbb{N}$.

(3) As an example, the sequence $(f^n)_{n\in\mathbb{N}} \subset \ell_1$ is given by

$$f_i^n = \begin{cases} \dfrac{1}{2^{i-n+1}}, & \text{if } i \geqslant n, \\ (-1)^{i+n+1}\dfrac{1}{2^i}, & \text{if } i < n, \end{cases}$$

verifies that ω^*AC $\lim_{n\to\infty} f^n = 0$ (observe that $\|f^n\| < 2$ for each $n \in \mathbb{N}$). However, it is not almost convergent. In the last section of this paper we will show that $\omega ac(\ell_1) = ac(\ell_1)$, so $(f^n)_{n\in\mathbb{N}}$ is not weakly almost convergent either.

3. Spaces of sequences defined by the almost summability

Before stating and proving the main results in this section, we will remark several aspects related to sequences whose associated series have bounded partial sums, which will be crucial for the development of this section.

Remark 3.1.

(1) We will let $bps(X)$ stand for the vector space of sequences in X whose associated series have bounded partial sums, in other words:

$$bps(X) = \left\{ (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : \left(\sum_{n=1}^{k} x_n \right)_{k\in\mathbb{N}} \in \ell_\infty(X) \right\}.$$

Observe that $bps(X) \subset \ell_\infty(X)$. However, $bps(X)$ endowed with the sup norm is not complete.

(2) In [1], it was proved that

$$bps(X) = \{(z_{n+1} - z_n)_{n\in\mathbb{N}} : (z_n)_{n\in\mathbb{N}} \in \ell_\infty(X)\} \subset ac_0(X),$$

where $ac_0(X)$ stands for the set of all null almost convergent sequences in X.

(3) Finally, $bps(X)$ can actually be endowed with the following norm:

$$\|(x_i)_{i\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} x_i \right\| = \left\| \left(\sum_{i=1}^{n} x_i \right)_{n\in\mathbb{N}} \right\|_\infty, \tag{1}$$

which will make it complete if X is so.

Now that we are in possession of all the necessary tools, it is time to define spaces of almost summable sequences, which are

$$sac(X) = \left\{ (x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} : \text{AC} \sum_{i=1}^{\infty} x_i \text{ exists} \right\}$$

and

$$\omega sac\,(X) = \left\{ (x_i)_{i=1}^{\infty} \in X^{\mathbb{N}} : \omega \text{AC} \sum_{i=1}^{\infty} x_i \text{ exists} \right\}.$$

Observe the following chain of inclusions:

$$sac\,(X) \subset \omega sac\,(X) \subset bps\,(X).$$

As in the previous section, it is very natural to wonder about the completeness of the previous spaces.

Theorem 3.1. *Let X be a real Banach space. The spaces $sac\,(X)$ and $\omega sac\,(X)$ are closed in $bps\,(X)$ endowed with the norm given in (1).*

Proof. Consider a sequence $(x^n)_{n\in\mathbb{N}} \subset sac\,(X)$ and $x^0 \in bps\,(X)$ such that $\lim_{n\to\infty} \|x^n - x^0\| = 0$. For each $n \in \mathbb{N}$ fixed, we define the sequence $(y_i^n)_{i\in\mathbb{N}}$ in X given by $y_i^n = \sum_{j=1}^{i} x_j^n$ for every $i \in \mathbb{N}$. We also define the sequence $(y_i^0)_{i\in\mathbb{N}}$ in X given by $y_i^0 = \sum_{j=1}^{i} x_j^0$ for every $i \in \mathbb{N}$. We have that $(y^n)_{n\in\mathbb{N}} \subset ac\,(X)$, $(y_i^0)_{i\in\mathbb{N}} \in \ell_\infty(X)$ and $\lim_{n\to\infty} \|y^n - y^0\|_\infty = 0$. Therefore, $y^0 \in ac\,(X)$ in virtue of Theorem 2.1 and hence $x^0 \in sac\,(X)$. Similarly, $\omega sac\,(X)$ is closed in $bps\,(X)$ endowed with the norm given in (1). □

COROLLARY 3.1

Let X be a real normed space. The following conditions are equivalent:

(1) *$sac\,(X)$ is complete.*
(2) *$\omega sac\,(X)$ is complete.*
(3) *X is complete.*

Proof. It is only sufficient to notice the following:

- Any real normed space X is linearly isometric to

$$\left\{ \left(\frac{1}{2^n} x \right)_{n\in\mathbb{N}} \in bps\,(X) : x \in X \right\}.$$

- If X is any real normed space, then

$$\left\{ \left(\frac{1}{2^n} x \right)_{n\in\mathbb{N}} \in bps\,(X) : x \in X \right\}$$

  is a closed subspace of $bps\,(X)$ endowed with the norm given in (1). □

There is also the weak-star version of the almost summability.

DEFINITION 3.1

Let X be a real normed space. Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in X^*. We will say that $f_0 \in X^*$ is the weak-star almost sum of $(f_i)_{i\in\mathbb{N}}$ exactly when for each $x \in X$ we have that $\text{AC}\lim_{i\to\infty} \sum_{n=1}^{i} f_n(x) = f_0(x)$. We will denote the limit by $\omega^*\text{AC} \sum_{i=1}^{\infty} f_i = f_0$.

We can define the space

$$\omega^* sac\left(X^*\right) := \left\{(f_i)_{i\in\mathbb{N}} \in \left(X^*\right)^{\mathbb{N}} : \omega^*\mathrm{AC}\sum_{i=1}^{\infty} f_i \text{ exists}\right\}.$$

If X is barrelled, then it is easy to prove that $\omega^* sac\,(X^*)$ is a closed subspace of $bps\,(X^*)$. In case X is not, then $\omega^* sac\,(X^*) \cap bps\,(X^*)$ is closed in $bps\,(X^*)$.

Remark 3.2. Following [2], for any given series $\sum_{i=1}^{\infty} x_i$ in X we consider the spaces

$$\mathcal{S}_{\mathrm{AC}}\left(\sum_{i=1}^{\infty} x_i\right) := \{(a_i)_{i\in\mathbb{N}} \in \ell_\infty : (a_i x_i)_{i\in\mathbb{N}} \in sac\,(X)\}$$

and

$$\mathcal{S}_{\omega\mathrm{AC}}\left(\sum_{i=1}^{\infty} x_i\right) := \{(a_i)_{i\in\mathbb{N}} \in \ell_\infty : (a_i x_i)_{i\in\mathbb{N}} \in \omega sac\,(X)\},$$

both endowed with the sup norm. They will be called the spaces of almost convergence and weak almost convergence associated to the series $\sum_{i=1}^{\infty} x_i$. In [2], it is proved that if X is a real Banach space and $\sum_{i=1}^{\infty} x_i$ a series in X, then $\mathcal{S}_{\mathrm{AC}}\left(\sum_{i=1}^{\infty} x_i\right)$ and $\mathcal{S}_{\omega\mathrm{AC}}\left(\sum_{i=1}^{\infty} x_i\right)$ are complete if and only if $\sum_{i=1}^{\infty} x_i$ is a weakly unconditionally Cauchy series.

4. Some classical properties from the almost convergence viewpoint

This section is aimed at characterizing some classical properties in terms of the almost convergence and the almost summability. We will start off with a characterization of completeness.

Theorem 4.1. *Let X be a real normed space and consider $(x_n)_{n\in\mathbb{N}}$ to be a Cauchy sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is convergent if and only if it is almost convergent.*

Proof. Suppose that $(x_n)_{n\in\mathbb{N}}$ is an almost convergent Cauchy sequence in X. There exists $x_0 \in X$ such that $\mathrm{AC}\lim_{i\to\infty} x_i = x_0$. Without lack of generality, we suppose that $x_0 = 0$. If $\varepsilon > 0$ is given, then there exists $i_0 \in \mathbb{N}$ satisfying

$$\frac{1}{i_0+1}\left\|\sum_{k=0}^{i_0} x_{j+k}\right\| \leqslant \frac{\varepsilon}{2}$$

for every $j \in \mathbb{N}$. On the other hand, there exists $j_0 \in \mathbb{N}$ such that if $p, q \geqslant j_0$, then $\left\|x_p - x_q\right\| \leqslant \varepsilon/2$. Therefore

$$\left\|x_j + \frac{1}{i_0+1}\sum_{p=j+1}^{j+i_0}(x_p - x_j)\right\| = \frac{1}{i_0+1}\left\|\sum_{k=0}^{i_0} x_{j+k}\right\| \leqslant \frac{\varepsilon}{2}$$

for every $j \geqslant j_0$. If we denote $v = \sum_{p=j+1}^{j+i_0}(x_p - x_j)$, then we have that

$$\|x_j\| - \left\|\frac{v}{i_0+1}\right\| \leqslant \left\|x_j + \frac{v}{i_0+1}\right\| \leqslant \frac{\varepsilon}{2}.$$

However, $\|\frac{v}{i_0+1}\| \leqslant \frac{\varepsilon}{2}$, so we conclude that $\|x_j\| \leqslant \|\frac{v}{i_0+1}\| + \frac{\varepsilon}{2} \leqslant \varepsilon$ for every $j \geqslant j_0$. □

By definition a normed space is complete if and only if every Cauchy sequence is convergent. The following is a similar result for the almost convergence.

COROLLARY 4.1

Let X be a real normed space. Then X is complete if and only if every Cauchy sequence in X is almost convergent.

It is well-known that a normed space is reflexive if and only if every bounded sequence has a weakly convergent subsequence (see [13]). We present now a similar result for the weak almost convergence.

Theorem 4.2. *Let X be a real normed space. Then X is reflexive if and only if every bounded sequence in X has a weakly almost convergent subsequence.*

Proof. Assume that every bounded sequence in X has a weakly almost convergent subsequence. We will distinguish two parts:

(1) *X must be complete*. Indeed, let Y be the completion of X and take any $y \in Y$. There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ which converges to y. Now, $(x_n)_{n\in\mathbb{N}}$ is bounded, so by hypothesis there exists a weakly almost almost convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ to some $x \in X$. Observe then that $y = x \in X$.
(2) *Every functional on X is norm-attaining*. Indeed, let $f \in X^*$. For each natural n we can find $x_n \in \mathsf{S}_X$ such that $\|f\| - \frac{1}{n} \leqslant f(x_n) \leqslant \|f\|$. Since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ such that $\omega\mathrm{AC}\lim_{j\to\infty} x_{n_j} = x_0$ for some $x_0 \in X$. Notice that $x_0 \in \mathrm{B}_X$. On the other hand,

$$\mathrm{AC}\lim_{j\to\infty}\left(\|f\| - \frac{1}{n_j}\right) \leqslant \mathrm{AC}\lim_{j\to\infty} f(x_{n_j}) \leqslant \mathrm{AC}\lim_{j\to\infty} \|f\|,$$

that is, $\|f\| \leqslant \mathrm{AC}\lim_{j\to\infty} f\left(x_{n_j}\right) \leqslant \|f\|$. Since $\mathrm{AC}\lim_{j\to\infty} f(x_{n_j}) = f(x_0)$, we conclude that $f(x_0) = \|f\|$.

In accordance to the James' characterization of reflexivity (see [13]), we deduce that X is reflexive. □

The next classical property to be characterized is the Schur property (recall that a real normed space is said to have the Schur property when every weakly convergent sequence is norm-convergent). Before stating and proving that characterization, we will need some supporting results and definitions.

PROPOSITION 4.1

Let X be a finite dimensional real normed space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that all of its subsequences are almost convergent to the same limit. Then $(x_n)_{n\in\mathbb{N}}$ is convergent.

Proof. In the first place, observe that $(x_n)_{n\in\mathbb{N}}$ must be bounded since it is, by hypothesis, almost convergent. Let $x \in X$ such that every subsequence of $(x_n)_{n\in\mathbb{N}}$ is almost convergent to x. Suppose that $(x_n)_{n\in\mathbb{N}}$ is not convergent to x. There exist $r > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \notin \mathsf{B}_X(x,r)$ for all $k \in \mathbb{N}$. Since $(x_{n_k})_{n\in\mathbb{N}}$ is bounded, it must possess a convergent subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ to some $y \in X \setminus \mathsf{U}_X(x,r)$. Nevertheless, by hypothesis

$$y = \lim_{j\to\infty} x_{n_{k_j}} = \mathrm{AC}\lim_{j\to\infty} x_{n_{k_j}} = x,$$

which is a contradiction. □

Proposition 4.1 has a useful corollary on which we will rely later on.

COROLLARY 4.2

Let X be a real normed space X. Consider two sequences $(x_i)_{i\in\mathbb{N}} \subset X$, $(f_n)_{n\in\mathbb{N}} \subset X^*$, and two elements $x_0 \in X$, $f_0 \in X^*$. Then:

(1) $\omega\lim_{i\to\infty} x_i = x_0$ *if and only if all of its subsequences are weakly almost convergent to x_0.*
(2) $\omega^*\lim_{i\to\infty} f_i = f_0$ *if and only if all of its subsequences are weakly-star almost convergent to f_0.*

On the other hand, Proposition 4.1 does not hold in infinite dimensions as expected.

Example 4.1. Consider in c_0 or ℓ_p $(p > 1)$ the sequence $(e_i)_{i\in\mathbb{N}}$ of canonical vectors. For every infinite subset $M \subset \mathbb{N}$ we have that $\mathrm{AC}\lim_{i\in M} e_i = 0$, but $\|e_i\| = 1$ for every $i \in \mathbb{N}$.

Example 4.1 motivates the following definition.

DEFINITION 4.1

Let X be a real normed space. We say that X verifies Property (Q) if every sequence in X whose subsequences are almost convergent to the same limit is also convergent.

According to Proposition 4.1, every finite dimensional real normed space enjoys Property (Q). Since Property (Q) is clearly hereditary, every real normed space having a copy of c_0 or ℓ_p $(p > 1)$ fails to have Property (Q).

DEFINITION 4.2

Let X be a real normed space. We say that X has Property (P) if every weakly almost convergent sequence in X is almost convergent, that is, if $\omega ac(X) = ac(X)$.

We are on the right position now to provide a characterization of the Schur property in the context of almost convergence.

Theorem 4.3. *Let X be a real normed space. Then X has the Schur property if and only if X enjoys Property (P) and Property (Q).*

Proof. In the first place, suppose that X enjoys both Property (P) and Property (Q). Let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in X. From Property (P) we deduce that all of its subsequences are almost convergent to the same limit. By Property (Q), $(x_n)_{n\in\mathbb{N}}$ is convergent. Conversely, suppose that X has the Schur property. In accordance with

Corollary 4.2, X has Property (Q). We will now show that X enjoys Property (P). Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in X which is weakly almost convergent. Without lack of generality we may assume that $\omega\text{AC}\lim_{i\to\infty} a_i = 0$. For each $f \in X^*$ it is verified that $\lim_{i\to\infty} f\left(x_i^n\right) = 0$ uniformly in $n \in \mathbb{N}$, where we have established that $x_i^n := \frac{1}{i+1}\sum_{k=0}^{i} a_{n+k}$ for every $n, i \in \mathbb{N}$ in order to simplify. Since X has the Schur property, $\lim_{i\to\infty} x_i^n = 0$ for every $n \in \mathbb{N}$, so we will conclude the proof by showing that $\lim_{i\to\infty} x_i^n = 0$ uniformly in $n \in \mathbb{N}$. Suppose not. Then there exists $\varepsilon > 0$ for which the following holds:

(1) Take $n_1 = 1$. There exists $i_1 \in \mathbb{N}$ such that if $i \geqslant i_1$, then $\|x_i^{n_1}\| < \varepsilon$.
(2) There is $n_2 > n_1$ such that $y_1 := x_i^{n_2}$ satisfies that $\|y_1\| \geqslant \varepsilon$ for some $i > i_1$. Besides, there exists $i_2 > i_1$ such that if $i \geqslant i_2$, then $\|x_i^{n_2}\| < \varepsilon$.
(3) There is $n_3 > n_2$ such that $y_2 := x_i^{n_3}$ satisfies that $\|y_2\| \geqslant \varepsilon$ for some $i > i_2$. Besides, there exists $i_3 > i_2$ such that if $i \geqslant i_3$, then $\|x_i^{n_3}\| < \varepsilon$.
(4) And so on.

In this manner, two sequences $(i_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and $(y_k)_{k\in\mathbb{N}} \subset X$ are found enjoying the following:

(1) $\|y_k\| \geqslant \varepsilon$ for each $k \in \mathbb{N}$.
(2) $(i_k)_{k\in\mathbb{N}}$ is strictly increasing.
(3) For every $k \in \mathbb{N}$ there exists $i \in [i_k, i_{k+1}]$ such that $y_k = x_i^{n_{k+1}}$.

We will show next that $\omega\lim_{k\to\infty} y_k = 0$. Take $f \in X^*$ and $\eta > 0$. By hypothesis, there exists $i' \in \mathbb{N}$ so that if $i \geqslant i'$, then $|f(x_i^n)| < \eta$ for every $n \in \mathbb{N}$. On the other hand, there will exist $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$, then $i_k > i'$. So $|f(y_k)| < \eta$ for $k \geqslant k_0$. Since X has the Schur property, we have that $\lim_{k\to\infty} y_k = 0$, which contradicts that $\|y_k\| \geqslant \varepsilon$ for each $k \in \mathbb{N}$. □

Our next result is a characterization of the Grothendieck property involving weakly-star almost convergence. We remind the reader that a normed space X has the Grothendieck property if every weakly-star convergent sequence in X^* is weakly convergent.

Theorem 4.4. *Let X be a real normed space. Then X has the Grothendieck property if and only if $\omega^*ac(X^*) = \omega ac(X^*)$.*

Proof. Firstly, suppose that $\omega^*ac(X^*) = \omega ac(X^*)$. Let $(f_n)_{n\in\mathbb{N}} \subset X^*$ be such that $\omega^*\lim_{n\to\infty} f_n = f_0$. For every subsequence $(f_{n_j})_{j\in\mathbb{N}}$ we have that $\omega^*\lim_{j\to\infty} f_{n_j} = f_0$, so $\omega^*\text{AC}\lim_{j\to\infty} f_{n_j} = f_0$, which means that $\omega\text{AC}\lim_{j\to\infty} f_{n_j} = f_0$. We conclude by Corollary 4.2 that $\omega\lim_{n\to\infty} f_n = f_0$. As a consequence, X has the Grothendieck property. Conversely, assume that X has the Grothendieck property. Consider a sequence $(f_i)_{i\in\mathbb{N}} \subset X^*$ such that $\omega^*\text{AC}\lim_{i\to\infty} f_i = 0$. For each $x \in X$ it is verified that $\lim_{i\to\infty} F_i^n(x) = 0$ uniformly in $n \in \mathbb{N}$, where we have established that $F_i^n := \frac{1}{i+1}\sum_{k=0}^{i} f_{n+k}$ for every $n, i \in \mathbb{N}$ in order to simplify. We will conclude this part of the proof by showing that $\lim_{i\to\infty} g(F_i^n) = 0$ uniformly in $n \in \mathbb{N}$ for each $g \in X^{**}$. Suppose not. Then there exist $g \in X^{**}$ and $\varepsilon > 0$ satisfying the following:

(1) Take $n_1 = 1$. There exists $i_1 \in \mathbb{N}$ so that if $i \geqslant i_1$, then $|g(F_i^1)| < \varepsilon$.
(2) There is $n_2 > n_1$ such that $y_1 := F_i^{n_2}$ satisfies that $|g(y_1)| \geqslant \varepsilon$ for some $i > i_1$. Besides, there exists $i_2 > i_1$ such that if $i \geqslant i_2$, then $|g(F_i^{n_2})| < \varepsilon$.

(3) There is $n_3 > n_2$ such that $y_2 := F_i^{n_3}$ satisfies that $|g(y_2)| \geqslant \varepsilon$ for some $i > i_2$. Besides, there exists $i_3 > i_2$ such that if $i \geqslant i_3$, then $|g(F_i^{n_3})| < \varepsilon$.
(4) And so on.

In this manner, two sequences $(i_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and $(y_k)_{k\in\mathbb{N}} \subset X$ are found satisfying the following:

(1) $|g(y_k)| \geqslant \varepsilon$ for each $k \in \mathbb{N}$.
(2) $(i_k)_{k\in\mathbb{N}}$ is strictly increasing.
(3) For each $k \in \mathbb{N}$ there exists $i \in [i_k, i_{k+1}]$ such that $y_k = F_i^{n_{k+1}}$.

We will prove now that $\omega^* \lim_{k\to\infty} y_k = 0$. Take $x \in X$ and $\eta > 0$. By hypothesis, there exists $i' \in \mathbb{N}$ such that if $i \geqslant i'$, then $|F_i^n(x)| < \eta$ for every $n \in \mathbb{N}$. On the other hand, there will exist $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$, then $i_k > i'$. So $|y_k(x)| < \eta$ for $k \geqslant k_0$. Since X has the Grothendieck property, we have that $\omega \lim_{k\to\infty} y_k = 0$, which contradicts that $|g(y_k)| \geqslant \varepsilon$ for each $k \in \mathbb{N}$. □

Finally, we present a characterization of Banach spaces containing a copy of c_0. Before presenting it, we will make use of the following lemma, which is nothing else but a small variant of Proposition 4.1 and Corollary 4.2.

Lemma 4.1. *Let X be a finite dimensional real normed space. Let $x \in X$ and consider $(x_n)_{n\in\mathbb{N}}$ to be a bounded sequence in X such that every subsequence has a further subsequence almost convergent to x. Then $(x_n)_{n\in\mathbb{N}}$ is convergent to x. As a consequence, $\omega \lim_{n\to\infty} x_n = x$ if and only if every subsequence has a further subsequence weakly almost convergent to x.*

Proof. Suppose that $(x_n)_{n\in\mathbb{N}}$ is not convergent to x. There exist $r > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \notin \mathsf{B}_X(x, r)$ for all $k \in \mathbb{N}$. Since $(x_{n_k})_{n\in\mathbb{N}}$ is bounded, it has a further subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ convergent to some $y \in X \setminus \mathsf{U}_X(x, r)$. By hypothesis, $(x_{n_{k_j}})_{j\in\mathbb{N}}$ has a further subsequence $(x_{n_{k_{j_p}}})_{p\in\mathbb{N}}$ almost convergent to x, which is impossible since

$$y = \lim_{p\to\infty} x_{n_{k_{j_p}}} = \mathrm{AC}\lim_{p\to\infty} x_{n_{k_{j_p}}} = x.$$

□

Theorem 4.5. *A real Banach space X has a copy of c_0 if and only if there exists a sequence $(x_i)_{i\in\mathbb{N}} \in \ell_\infty(X) \setminus c_0(X)$ satisfying that for every infinite set $M \subset \mathbb{N}$ there exists $P \subset M$ infinite such that $\sum_{i\in P} x_i$ has bounded partial sums.*

Proof. If X has a copy of c_0, then $(x_i)_{i\in\mathbb{N}}$ can be taken as the canonical basis of c_0. Conversely, assume the existence of a sequence $(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} \setminus c_0(X)$ satisfying that for every infinite set $M \subset \mathbb{N}$ there exists $P \subset M$ infinite such that $\sum_{i\in P} x_i$ has bounded partial sums. By Lemma 4.1 and by the fact that $bps(X) \subset ac_0(X)$, we conclude that $\omega \lim_{i\to\infty} x_i = 0$. Since $(x_i)_{i\in\mathbb{N}}$ is not convergent to 0, there exists $A \subset \mathbb{N}$ infinite and $\delta > 0$ such that $\|x_i\| > \delta$ for every $i \in A$. According to a result proved by Bessaga–Pelczynski [8], there exists $B \subset A$ such that $(x_i)_{i\in B}$ is a basic sequence.

A dichotomy result by Odell [15] tells us that only one of the following conditions is satisfied:

(1) There exists $C \subset B$ infinite such that $(x_i)_{i \in C}$ is equivalent to the basis of canonical vectors in c_0.
(2) There exists $C \subset B$ infinite such that for every sequence $(\alpha_i)_{i \in C}$ of real numbers which is not convergent to zero, the series $\sum_{i \in C} \alpha_i x_i$ does not have bounded partial sums.

Thus, we deduce that X has a copy of c_0. □

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