Proc. Indian Acad. Sci. (Math. Sci.) Vol. 124, No. 1, February 2014, pp. 81–92.


# Boundedness of composition of operators associated with different homogeneities on weighted Besov and Triebel–Lizorkin spaces

XINFENG WU

Department of Mathematics, China University of Mining and Technology, Beijing 100083, China
E-mail: wuxf@cumtb.edu.cn

MS received 22 October 2012; revised 19 December 2012

**Abstract.** In this paper, we introduce weighted Besov spaces and weighted Triebel–Lizorkin spaces associated with different homogeneities and prove that the composition of two Calderón–Zygmund operators is bounded on these spaces. This extends a recent result in Han *et al*, *Revista Mat. Iber.*

**Keywords.** Composition of operators; different homogeneities; weighted Besov spaces; weighted Triebel–Lizorkin spaces.

**Mathematics Subject Classification.** Primary: 42B20; Secondary: 42B35

## 1. Introduction and statement of main results

For $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\delta > 0$, we consider two kinds of homogeneities on $\mathbb{R}^n$:

$$\delta \circ_e (x', x_n) = (\delta x', \delta x_n),$$
$$\delta \circ_h (x', x_n) = (\delta x', \delta^2 x_n).$$

The first are the classical isotropic dilations occurring in the classical Calderón–Zygmund singular integrals, while the second are non-isotropic and related to the heat equations (also Heisenberg groups). Let $e(\xi)$ be a function on $\mathbb{R}^n$ homogeneous of degree 0 in the isotropic sense and smooth away from the origin. Similarly, let us suppose that $h(\xi)$ is a function on $\mathbb{R}^n$ homogeneous of degree 0 in the non-isotropic sense, and also smooth away from the origin. Then it is well-known that the Fourier multipliers $T_1$ defined by $\widehat{T_1(f)}(\xi) = e(\xi)\hat{f}(\xi)$ and $T_2$ given by $\widehat{T_2(f)}(\xi) = h(\xi)\hat{f}(\xi)$ are both bounded on $L^p$ for $1 < p < \infty$, and satisfy various other regularity properties such as being of weak-type $(1, 1)$ and bounded on the classical isotropic and non-isotropic Hardy spaces, respectively. Rivieré in [8] asked the question: is the composition $T_1 \circ T_2$ still of weak-type $(1, 1)$? Phong and Stein in [6] answered this question and gave a necessary and sufficient condition for which $T_1 \circ T_2$ is of weak-type $(1, 1)$. The operators they studied are in fact composition of operators with different kind of homogeneities which arise naturally in the $\bar{\partial}$-Neumann problem. Recently, Han *et al* [3, 4] have developed a theory of Hardy spaces and Carleson measure spaces associated with different homogeneities and proved that the composition of two Calderón–Zygmund operators with different homogeneities is bounded on these new spaces. More recently, the author has characterized a new class of Muckenhoupt weights associated with different homogeneities and established weighted norm inequalities for composition of operators on weighted Hardy and Carleson measure spaces (see [10, 11]).

The purpose of this paper is to study the boundedness of composition of two Calderón–Zygmund operators associated with different homogeneities on weighted Triebel–Lizorkin spaces and weighted Besov spaces. To achieve this, we shall use the discrete Calderón's reproducing formula and weighted Littlewood–Paley theory. These ideas and methods have been used before in other multiparameter setting, see [1, 5, 9, 11]. It is worthwhile to point out that the classical method using Peetre's maximal function and Plancherel–Pôlya–Nikols'kij inequality (see [7, 12]) do *not* work for our case. Indeed, a variant of multiparameter Plancherel–Pôlya–Nikols'kij inequality holds only for the strong maximal operator $\mathcal{M}_S$, and not for $\mathcal{M}_{\mathcal{C}}$. However $\mathcal{M}_S$ are, in general, not bounded on $L^p_w$ when $w \in A^{\mathcal{C}}_p$.

In order to describe more precisely the results studied in this paper, we begin with recalling some notions and notations. The Calderón–Zygmund operators associated with different homogeneities are defined as follows.

DEFINITION 1.1

A locally integrable function $\mathcal{K}_1$ on $\mathbb{R}^n\backslash\{0\}$ is said to be a Calderón–Zygmund kernel associated with isotropic homogeneity if it satisfies

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\mathcal{K}_1(x)\right| \le A|x|_e^{-n-|\alpha|}, \quad \text{for all } |\alpha| \ge 0, \tag{1}$$

and

$$\int_{r_1<|x|_e<r_2} \mathcal{K}_1(x)\mathrm{d}x = 0, \quad \text{for all } 0 < r_1, r_2 < \infty. \tag{2}$$

We say that an operator $T_1$ is a Calderón–Zygmund singular integral operator associated with the isotropic homogeneity if $T_1(f)(x) = p \cdot v \cdot (\mathcal{K}_1 * f)(x)$, where $\mathcal{K}_1$ satisfies conditions of (1)–(2).

DEFINITION 1.2

A locally integrable function $\mathcal{K}_2$ on $\mathbb{R}^n\backslash\{0\}$ is said to be a Calderón–Zygmund kernel associated with anisotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial (x')^\alpha}\frac{\partial^\beta}{\partial (x_n)^\beta}\mathcal{K}_2(x', x_n)\right| \le B|x|_h^{-n-1-|\alpha|-2\beta}, \quad \text{for all } |\alpha| \ge 0, \beta \ge 0, \tag{3}$$

and

$$\int_{r_3<|x|_h<r_4} \mathcal{K}_2(x)\mathrm{d}x = 0, \quad \text{for all } 0 < r_3, r_4 < \infty. \tag{4}$$

We say that an operator $T_2$ is a Calderón–Zygmund singular integral operator associated with the anisotropic homogeneity if $T_2(f)(x) = p \cdot v \cdot (\mathcal{K}_2 * f)(x)$, where $\mathcal{K}_2$ satisfies conditions of (3)–(4).

We recall next the Muckenhoupt class of $A_p$ weights associated with different homogeneities introduced in [11]. Let $j \wedge k = \min\{j, k\}$ and $j \vee k = \max\{j, k\}$. A rectangle in $\mathbb{R}^n$ is said to be acceptable if $R = I \times J$, where $I, J$ are cubes in $\mathbb{R}^{n-1}$ and $\mathbb{R}$, respectively with sidelength $\ell(I) = 2^{j\vee k}$ and $\ell(J) = 2^{j\vee 2k}$ for some $j, k \in \mathbb{Z}$. Let $\mathcal{R}_{\mathcal{C}}$ denote the set of all acceptable rectangles. The Hardy–Littlewood maximal operator associated with different homogeneities is defined by

$$\mathcal{M}_{\mathcal{C}}(f)(x) = \sup_{\substack{R\in\mathcal{R}_{\mathcal{C}}\\ R\ni x}} \frac{1}{|R|}\int_R |f(y)|\mathrm{d}y.$$

DEFINITION 1.3

Let $w$ be a nonnegative locally integrable function on $\mathbb{R}^n$. We say that $w$ is in $A_p^{\mathcal{C}}(\mathbb{R}^n)$ if it satisfies

$$\sup_{R\in\mathcal{R}_{\mathcal{C}}}\left(\frac{1}{|R|}\int_R w(x)\mathrm{d}x\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\mathrm{d}x\right)^{p-1} < \infty, \quad \text{if } 1 < p < \infty,$$

$$\mathcal{M}_{\mathcal{C}}(w)(x) \le Cw(x), \text{ for almost all } x \in \mathbb{R}^n, \quad \text{if } p = 1.$$

Define $A_\infty^{\mathcal{C}}(\mathbb{R}^n) = \bigcup_{1\le p<\infty} A_p^{\mathcal{C}}(\mathbb{R}^n)$. Let $q_w = \inf\{q : w \in A_q^{\mathcal{C}}(\mathbb{R}^n)\}$ denote the critical index of $w$.

It is well-known that the classical isotropic Hardy–Littlewood maximal oeprator $\mathcal{M}_e$ is bounded on $L_w^p$ if and only if $w$ belongs to the isotropic Muckenhoupt weight class $A_p^e$. Similarly, the anisotropic maximal operator $\mathcal{M}_h$ is bounded on $L_w^p$ if and only if $w$ belongs to the anisotropic Muckenhoupt weight class $A_p^h$. For the maximal operator $\mathcal{M}_{\mathcal{C}}$ and Muckenhoupt weight class $A_p^{\mathcal{C}}$, the following result holds (see [11]).

**Theorem A.** *Let $w$ be a nonnegative locally integrable function. Then the following four statements are equivalent*:

(1) $w \in A_p^{\mathcal{C}}$;
(2) $w \in A_p^e \cap A_p^h$;
(3) $\mathcal{M}_{\mathcal{C}}$ *is bounded on* $L_w^p$ (*or* $L_w^p(\ell^q)$);
(4) $\mathcal{M}_e \circ \mathcal{M}_h$ *is bounded on* $L_w^p$ (*or* $L_w^p(\ell^q)$).

*In particular, the following weighted Fefferman–Stein vector-valued inequality*

$$\int_{\mathbb{R}^n} |\mathcal{M}_{\mathcal{C}}(f)(x)|_{\ell^q}^p w(x)dx \le C\int_{\mathbb{R}^n} |f(x)|_{\ell^q}^p w(x)dx, \quad 1 < p, q < \infty \tag{5}$$

*holds if and only if $w \in A_p^{\mathcal{C}}(\mathbb{R}^n)$, provided that $f = (f_1, f_2, \dots) \in L_w^p(\ell^q)$. Moreover, in term of the Muckenhoupt class $A_p^{\mathcal{C}}$, the author developed the theory of weighted Hardy spaces $H_{\mathcal{C},w}^p$ associated with different homogeneities and proved the boundedness of $T_1 \circ T_2$ on $H_{\mathcal{C},w}^p$.*

In the present paper, we shall introduce weighted Besov spaces and weighted Triebel–Lizorkin spaces associated with different homogeneities and prove the boundedness of the composition operator $T_1 \circ T_2$ on these spaces. More precisely, for $x = (x', x_n) \in \mathbb{R}^n$, we denote $|x|_e = (|x'|^2 + |x_n|^2)^{1/2}$ and $|x|_h = (|x'|^2 + |x_n|)^{1/2}$. Let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp} \widehat{\psi^{(1)}}(\xi) \subseteq \{\xi : 1/2 < |\xi|_e \leq 2\}, \tag{6}$$

and

$$\sum_{j\in\mathbb{Z}} \widehat{\psi^{(1)}}(2^j \circ_e \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n \backslash \{0\}, \tag{7}$$

and let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp} \widehat{\psi^{(2)}}(\xi) \subseteq \{\xi : 1/2 < |\xi|_h \leq 2\}, \tag{8}$$

and

$$\sum_{k\in\mathbb{Z}} \widehat{\psi^{(2)}}(2^k \circ_h \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n \backslash \{0\}. \tag{9}$$

Let $\psi_j^{(1)}(x) = 2^{-jn}\psi^{(1)}(2^{-j} \circ_e x)$, $\psi_k^{(2)}(x) = 2^{-kn}\psi^{(2)}(2^{-k} \circ_h x)$ and $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}$. We now introduce *weighted Besov spaces and weighted Triebel–Lizorkin spaces associated with different homogeneities.* For $j, k \in \mathbb{Z}$, let $\mathcal{R}_{\mathcal{C}}^{j,k}$ denote the set of all *dyadic* acceptable rectangles $R = I \times J$ in $\mathbb{R}^n$ with sidelength $\ell(I) = 2^{j\vee k}$ and $\ell(J) = 2^{j\vee 2k}$. A corner of a rectangle $R$ is called the *minimal corner* of $R$ in $\mathbb{R}^n$ if it has the minimal value of each coordinate component.

DEFINITION 1.4

Let $s = (s_1, s_2) \in \mathbb{R}^2$ and $p,\ q \in (0, \infty)$ and let $w \in A_\infty^{\mathcal{C}}(\mathbb{R}^n)$. The weighted Triebel–Lizorkin space associated with different homogeneities $\dot{\mathcal{F}}_{p,q}^{s,w}(\mathbb{R}^n)$ is defined to be the collection of all $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|^{\psi}_{\dot{\mathcal{F}}_{p,q}^{s,w}} \equiv \left\| \left( \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j,k}} 2^{js_1q}2^{ks_2q}|\psi_{j,k} * f(x_R)|^q \chi_R \right)^{\frac{1}{q}} \right\|_{L_w^p} < \infty, \tag{10}$$

where $x_R$ denotes the minimal corner of $R$. The weighted Besov space associated with different homogeneities $\dot{\mathcal{B}}_{p,q}^{s,w}(\mathbb{R}^n)$ is defined to be the collection of all $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|^{\psi}_{\dot{\mathcal{B}}_{p,q}^{s,w}} \equiv \left( \sum_{j,k\in\mathbb{Z}} 2^{js_1q}2^{ks_2q} \left\| \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j,k}} |\psi_{j,k} * f(x_R)|\chi_R \right\|_{L_w^p}^q \right)^{\frac{1}{q}} < \infty. \tag{11}$$

The following theorem shows that the definition of the weighted Besov spaces $\dot{\mathcal{B}}_{p,q}^{s,w}$ and Triebel–Lizorkin spaces $\dot{\mathcal{F}}_{p,q}^{s,w}$ are independent of the choice of $(\psi^{(1)}, \psi^{(2)})$ and hence they are well defined.

**Theorem 1.5.** *Let $w \in A_\infty^{\mathcal{C}}(\mathbb{R}^n)$. If $\varphi_{j,k}$ satisfy the same conditions as $\psi_{j,k}$, then for $s = (s_1, s_2) \in \mathbb{R}^2$ and $p, q \in (0, \infty)$ and $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$,*

$$\|f\|^{\psi}_{\dot{\mathcal{F}}^{s,w}_{p,q}} \sim \|f\|^{\varphi}_{\dot{\mathcal{F}}^{s,w}_{p,q}}, \quad \|f\|^{\psi}_{\dot{\mathcal{B}}^{s,w}_{p,q}} \sim \|f\|^{\varphi}_{\dot{\mathcal{B}}^{s,w}_{p,q}}.$$

Our main result in this paper is the following.

**Theorem 1.6.** *Let $w \in A_\infty^{\mathcal{C}}(\mathbb{R}^n)$. Let $T_1$ and $T_2$ be Calderón–Zygmund singular integral operators with the isotropic and non-isotropic homogeneities, respectively. Then for $s = (s_1, s_2) \in \mathbb{R}^2$ and $p, q \in (0, \infty)$, both $T_1$ and $T_2$ are bounded on $\dot{\mathcal{B}}^{s,w}_{p,q}$ and $\dot{\mathcal{F}}^{s,w}_{p,q}$. Hence the composition operator $T = T_1 \circ T_2$ is also bounded on $\dot{\mathcal{B}}^{s,w}_{p,q}$ and $\dot{\mathcal{F}}^{s,w}_{p,q}$.*

*Remark* 1.7. When $q = 2$, $s = (0, 0)$ and $w \equiv 1$, then the Triebel–Lizorkin spaces $\dot{\mathcal{F}}^{s,w}_{p,2}$ coincides with the Hardy spaces $H^p_{\text{com}}$ studied in [3]. Hence, our result in Theorem 1.6 extends the result in [3].

The following paper is organized as follows. In §2, we give some lemmas. The proof of Theorem 1.5 is presented in §3. Theorem 1.6 is proved in §4.

## 2. Some lemmas

To prove Theorem 1.5, we need the following discrete Calderón's reproducing formula, whose proof can be found in [3].

**Theorem 2.1.** *Suppose that $\varphi^{(1)}$ and $\varphi^{(2)}$ are functions satisfying conditions (6)–(7) and (8)–(9), respectively. Let $\varphi_{j,k} = \varphi^{(1)}_j * \varphi^{(2)}_k$. Then*

$$f(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}^{j,k}_C} |R| \varphi_{j,k} * f(x_R) \varphi_{j,k}(x - x_R),$$

*where $x_R$ is the minimal corner of $R$ and the series converges in $L^2(\mathbb{R}^n)$, $\mathcal{S}_\infty(\mathbb{R}^n)$ and $\mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$.*

We also need the following almost orthogonality estimates (see [3]).

*Lemma* 2.2. *Suppose that $\psi^{(1)}, \varphi^{(1)}$ are functions satisfying conditions (6)–(7) and $\psi^{(2)}, \varphi^{(2)}$ are functions satisfying conditions (8)–(9). Then for any given positive integers $L$ and $M$, there exists a constant $C = C(L, M) > 0$ such that*

$$\begin{aligned} |\psi_{j,k} * \varphi_{j',k'}(x)| \ &\leq \ C 2^{-|j-j'|L} 2^{-|k-k'|L} \frac{2^{(j \vee j' \vee k \vee k')M}}{(2^{j \vee j' \vee k \vee k'} + |x'|)^{n-1+M}} \\ &\quad \times \frac{2^{(j \vee j' \vee 2k \vee 2k')M}}{(2^{j \vee j' \vee 2k \vee 2k'} + |x_n|)^{1+M}}. \end{aligned}$$

*Remark* 2.3. If the functions $\phi^{(1)}$, $\phi^{(2)}$, $\psi^{(1)}$ and $\psi^{(2)}$ only satisfy moment conditions up to order $L_0$, that is, for $i = 1, 2$,

$$\int_{\mathbb{R}^n} \psi^{(i)}(x) x^\alpha \mathrm{d}x = 0 = \int_{\mathbb{R}^n} \varphi^{(i)}(y) y^\beta \mathrm{d}y \quad \text{for all } |\alpha|, |\beta| \leq L_0,$$

then the above almost orthogonality estimate holds for all $M > 0$ and all $L \leq L_0 + 1$.

The following estimate provides a substitute of Plancherel–Pôlya–Nikols'kij inequality.

*Lemma* 2.4 [3, 11]. *Let* $R = I \times J \in \mathcal{R}_{\mathcal{C}}^{j,k}$. *Let* $x_R = (x_I, x_J)$ *and* $x_{R'} = (x_{I'}, x_{J'})$ *be the minimal corners of* $R$ *and* $R'$, *respectively. Then for any* $\frac{n-1}{M+n-1} < \delta \le 1$,

$$
\begin{aligned}
\sum_{R' \in \mathcal{R}_{\mathcal{C}}^{j',k'}} & |R'| \frac{2^{M(j \vee j' \vee k \vee k')}}{(2^{j \vee j' \vee k \vee k'} + |x_I - x_{I'}|)^{n-1+M}} \\
& \times \frac{2^{M(j \vee j' \vee 2k \vee 2k')}}{(2^{j \vee j' \vee 2k \vee 2k'} + |x_J - x_{J'}|)^{1+M}} |g(x_{R'})| \chi_R(x) \\
& \lesssim \left\{ [2^{(n-1)(j' \vee k' - j \vee k)} \vee 1][2^{j' \vee 2k' - j \vee 2k} \vee 1] \right\}^{\frac{1}{\delta} - 1} \\
& \quad \times \left\{ \mathcal{M}_{\mathcal{C}} \left[ \left( \sum_{R' \in \mathcal{R}_{\mathcal{C}}^{j',k'}} |g(x_{R'})|^q \chi_{R'} \right)^{\frac{\delta}{q}} \right](x) \right\}^{\frac{1}{\delta}},
\end{aligned}
$$

*where the implicit constant depends only on* $M$ *and* $n$.

## 3. Proof of Theorems 1.5

Let $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$ and let $w \in A_{\infty}^{\mathcal{C}}(\mathbb{R}^n)$. Let $x_R$ and $x_{R'}$ denote the minimal corner of $R$ and $R'$, respectively. Applying the discrete Calderón's reproducing formula in Theorem 2.1, the almost orthogonality estimates in Lemma 2.2 with $M > (n-1)\{(q_w/(p \wedge q) - 1) \vee 0\}$ and $L$ large enough to be determined later, and Lemma 2.4 with $g = \varphi_{j',k'} * f$, we deduce that for $(n-1)/(n-1+M) < \delta < (\{(p \wedge q)/q_w\} \wedge 1)$,

$$
\begin{aligned}
& |(\psi_{j,k} * f)(x_R)| \chi_R(x) \\
& \sim \Big| \sum_{j',k' \in \mathbb{Z}} \sum_{R' \in \mathcal{R}_{\mathcal{C}}^{j',k'}} |R'| \varphi_{j,k} * f(x_{R'}) \psi_{j,k} * \varphi_{j',k'}(x_R - x_{R'}) \Big| \chi_R(x) \\
& \lesssim \sum_{j',k' \in \mathbb{Z}} 2^{-|j-j'|L} 2^{-|k-k'|L} \sum_{R' \in \mathcal{R}_{\mathcal{C}}^{j',k'}} |R'| \frac{2^{M(j \vee j' \vee k \vee k')}}{(2^{j \vee j' \vee k \vee k'} + |x_I - x_{I'}|)^{n-1+M}} \\
& \quad \times \frac{2^{M(j \vee j' \vee 2k \vee 2k')}}{(2^{j \vee j' \vee 2k \vee 2k'} + |x_J - x_{J'}|)^{1+M}} |\varphi_{j',k'} * f(x_{R'})| \chi_R(x) \\
& \lesssim \sum_{j',k' \in \mathbb{Z}} 2^{-|j-j'|L} 2^{-|k-k'|L} \{ [2^{(n-1)(j' \vee k' - j \vee k)} \vee 1][2^{j' \vee 2k' - j \vee 2k} \vee 1] \}^{\frac{1}{\delta} - 1} \\
& \quad \times \left\{ \mathcal{M}_{\mathcal{C}} \left[ \left( \sum_{R' \in \mathcal{R}_{\mathcal{C}}^{j',k'}} |\varphi_{j',k'} * f(x_{R'})|^q \chi_{R'} \right)^{\frac{\delta}{q}} \right](x) \right\}^{\frac{1}{\delta}} \\
& \lesssim \sum_{j',k' \in \mathbb{Z}} 2^{-|j-j'|L'} 2^{-|k-k'|L'} \left\{ \mathcal{M}_{\mathcal{C}} \left[ \left( \sum_{R' \in \mathcal{R}_{\mathcal{C}}^{j',k'}} |\varphi_{j',k'} * f(x_{R'})|^q \chi_{R'} \right)^{\frac{\delta}{q}} \right](x) \right\}^{\frac{1}{\delta}},
\end{aligned}
$$

where in the last inequality we have used the inequalities

$$2^{j'\vee k'-j\vee k}\vee 1\le 2^{|j-j'|+|k-k'|},\quad 2^{j'\vee 2k'-j\vee 2k}\vee 1\le 2^{|j-j'|+2|k-k'|},$$

and $L'=L-(n+1)(1/\delta-1)$ can be chosen arbitrarily large.

Applying Hölder's inequality, we have

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_{\mathcal{C}}^{j,k}}2^{js_1q}2^{ks_2q}|\psi_{j,k}*f(x_R)|^q\chi_R(x)\\
&\lesssim\sum_{j,k\in\mathbb{Z}}\sum_{j',k'\in\mathbb{Z}}2^{-|j-j'|(L'-|s_1|q)}2^{-|k-k'|(L'-|s_2|q)}\\
&\quad\times\Big\{\mathcal{M}_{\mathcal{C}}\Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}}2^{j's_1q}2^{k's_2q}|\varphi_{j',k'}*f(x_{R'})|^q\chi_{R'}\Big)^{\frac{\delta}{q}}\Big](x)\Big\}^{\frac{q}{\delta}}\\
&\lesssim\sum_{j,k\in\mathbb{Z}}\Big\{\sum_{j',k'\in\mathbb{Z}}2^{-|j-j'|L''}2^{-|k-k'|L''}\Big\}\sum_{j',k'\in\mathbb{Z}}2^{-|j-j'|L''}2^{-|k-k'|L''}\\
&\quad\times\Big\{\mathcal{M}_{\mathcal{C}}\Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}}2^{j's_1q}2^{k's_2q}|\varphi_{j',k'}*f(x_{R'})|^q\chi_{R'}\Big)^{\frac{\delta}{q}}\Big](x)\Big\}^{\frac{q}{\delta}}\\
&\lesssim\sum_{j',k'\in\mathbb{Z}}\Big\{\mathcal{M}_{\mathcal{C}}\Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}}2^{j's_1q}2^{k's_2q}|\varphi_{j',k'}*f(x_{R'})|^q\chi_{R'}\Big)^{\frac{\delta}{q}}\Big](x)\Big\}^{\frac{q}{\delta}},
\end{aligned}\tag{12}$$

where $L'$ is chosen sufficiently large so that $L''=L'-q(|s_1|\vee|s_2|)>0$, and in the last inequality we have used $\sum_{j',k'\in\mathbb{Z}}2^{-|j-j'|L''}2^{-|k-k'|L''}\le C$ and $\sum_{j,k\in\mathbb{Z}}2^{-|j-j'|L''}2^{-|k-k'|L''}\le C$. Finally take the $\frac{1}{q}$-th power and apply the weighted Fefferman–Stein's inequality (5) on $L_w^{p/\delta}(\ell^{q/\delta})$ (note that $(q\wedge p)/\delta>q_w$ implies $w\in A_{p/\delta}^{\mathcal{C}}$) to obtain $\|f\|^{\psi}_{\dot{\mathcal{F}}^{s,w}_{p,q}}\lesssim\|f\|^{\varphi}_{\dot{\mathcal{F}}^{s,w}_{p,q}}$. The converse inequality follows by symmetry. This finishes the proof of Theorem 1.5 for the weighted Triebel–Lizorkin spaces.

Now, we turn to the proof for the weighted Besov spaces. Arguing as above, we have for $(n-1)/(n-1+M)<\delta<((p\wedge q)/q_w\wedge 1)$,

$$\begin{aligned}
&\sum_{R\in\mathcal{R}_{\mathcal{C}}^{j,k}}|\psi_{j,k}*f(x_R)|\chi_R(x)\\
&\lesssim\sum_{j',k'\in\mathbb{Z}}2^{-|j-j'|L'}2^{-|k-k'|L'}\Big\{\mathcal{M}_{\mathcal{C}}\Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}}|\varphi_{j',k'}*f(x_{R'})|\chi_{R'}\Big)^{\delta}\Big](x)\Big\}^{\frac{1}{\delta}}.
\end{aligned}$$

Note that $p/\delta > q_w$ implies $w \in A^{\mathcal{C}}_{p/\delta}(\mathbb{R}^n)$. Applying $L^{p/\delta}_w$ boundedness of $\mathcal{M}_{\mathcal{C}}$ yields

$$\begin{aligned}
&\Big\| \sum_{R\in\mathcal{R}^{j,k}_{\mathcal{C}}} |\psi_{j,k} * f(x_R)|\chi_R \Big\|_{L^p_w} \\
&\quad\lesssim \sum_{j',k'\in\mathbb{Z}} 2^{-|j-j'|L'} 2^{-|k-k'|L'} \Big\| \Big\{ \mathcal{M}_{\mathcal{C}} \Big[ \Big( \sum_{R'\in\mathcal{R}^{j',k'}_{\mathcal{C}}} |\varphi_{j',k'} * f(x_{R'})|\chi_{R'} \Big)^{\delta} \Big] \Big\}^{\frac{1}{\delta}} \Big\|_{L^p_w} \\
&\quad\lesssim \sum_{j',k'\in\mathbb{Z}} 2^{-|j-j'|L'} 2^{-|k-k'|L'} \Big\| \sum_{R'\in\mathcal{R}^{j',k'}_{\mathcal{C}}} |\varphi_{j',k'} * f(x_{R'})|\chi_{R'} \Big\|_{L^p_w}
\end{aligned}$$

for all $L' > 0$. Therefore,

$$\begin{aligned}
&\Big( \sum_{j,k\in\mathbb{Z}} 2^{js_1q} 2^{ks_2q} \Big\| \sum_{R\in\mathcal{R}^{j,k}_{\mathcal{C}}} |\psi_{j,k} * f(x_R)|\chi_R \Big\|^q_{L^p_w} \Big)^{\frac{1}{q}} \\
&\quad\lesssim \Big\{ \sum_{j,k\in\mathbb{Z}} 2^{js_1q} 2^{ks_2q} \Big( \sum_{j',k'\in\mathbb{Z}} 2^{-|j-j'|L'} 2^{-|k-k'|L'} \\
&\qquad\qquad \times \Big\| \sum_{R'\in\mathcal{R}^{j',k'}_{\mathcal{C}}} |\varphi_{j',k'} * f(x_{R'})|\chi_{R'} \Big\|_{L^p_w} \Big)^q \Big\}^{\frac{1}{q}}.
\end{aligned}$$

If $q > 1$, we apply Hölder's inequality and if $0 < q \le 1$ we use the inequality $(\sum_i a_i)^q \le \sum_i a_i^q$, then

$$\begin{aligned}
&\Big( \sum_{j,k\in\mathbb{Z}} 2^{js_1q} 2^{ks_2q} \Big\| \sum_{R\in\mathcal{R}^{j,k}_{\mathcal{C}}} |\psi_{j,k} * f(x_R)|\chi_R \Big\|^q_{L^p_w} \Big)^{\frac{1}{q}} \\
&\quad\lesssim \Big( \sum_{j,k\in\mathbb{Z}} 2^{js_1q} 2^{ks_2q} \sum_{j',k'\in\mathbb{Z}} 2^{-|j-j'|(1\wedge q)L'} 2^{-|k-k'|(1\wedge q)L'} \\
&\qquad \times \Big\| \sum_{R'\in\mathcal{R}^{j',k'}_{\mathcal{C}}} |\varphi_{j',k'} * f(x_{R'})|\chi_{R'} \Big\|^q_{L^p_w} \Big)^{\frac{1}{q}} \\
&\quad\lesssim \Big( \sum_{j',k'\in\mathbb{Z}} \Big\{ \sum_{j,k\in\mathbb{Z}} 2^{-|j-j'|L''} 2^{-|k-k'|L''} \Big\} 2^{j's_1q} 2^{k's_2q} \\
&\qquad \times \Big\| \sum_{R'\in\mathcal{R}^{j',k'}_{\mathcal{C}}} |\varphi_{j',k'} * f(x_{R'})|\chi_{R'} \Big\|^q_{L^p_w} \Big)^{\frac{1}{q}} \\
&\quad\lesssim \Big( \sum_{j',k'\in\mathbb{Z}} 2^{j's_1q} 2^{k's_2q} \Big\| \sum_{R'\in\mathcal{R}^{j',k'}_{\mathcal{C}}} |\varphi_{j',k'} * f(x_{R'})|\chi_{R'} \Big\|^q_{L^p_w} \Big)^{\frac{1}{q}},
\end{aligned} \tag{13}$$

where $L'$ is chosen sufficiently large so that $L'' = (1 \wedge q)L' - q(|s_1| \vee |s_2|) > 0$. This concludes the proof of Theorem 1.5. □

As a corollary, we can get the following density lemma, which will be used in the proof of Theorem 1.6.

*Lemma* 3.1. *Let $w \in A_\infty^{\mathcal{C}}(\mathbb{R}^n)$. Then for $s = (s_1, s_2) \in \mathbb{R}^2$ and $p, q \in (0, \infty)$, $\mathcal{S}_\infty(\mathbb{R}^n)$ is dense in $\dot{\mathcal{F}}_{p,q}^{s,w}(\mathbb{R}^n)$ and $\dot{\mathcal{B}}_{p,q}^{s,w}(\mathbb{R}^n)$.*

*Proof.* Suppose $f \in \dot{\mathcal{F}}_{p,q}^{s,w}(\mathbb{R}^n)$. Set $W_L = \{(j, k, R) : |j| \le L, |k| \le L, R \in \mathcal{R}_{\mathcal{C}}^{j,k}, R \subseteq B(0, L)\}$, where $B(0, L)$ are isotropic (Euclidean) balls in $\mathbb{R}^n$ centered at the origin with radius $L$. It is easy to see that

$$f_L(x) \equiv \sum_{(j,k,R)\in W_L} |R|\psi_{j,k}(x - x_R)\psi_{j,k} * f(x_R) \in \mathcal{S}_\infty(\mathbb{R}^n)$$

for each $L$. To show the lemma, is thus suffices to prove

$$f - f_L = \sum_{(j,k,R)\in W_L^c} |R|\psi_{j,k}(\cdot - x_R)\psi_{j,k} * f(x_R)$$

tends to zero in $\dot{\mathcal{F}}_{p,q}^{s,w}(\mathbb{R}^n)$ norm as $L$ goes to infinity. Repeating the same proof as in Theorem 1.5 yields

$$\Big\| \sum_{(j,k,R)\in W_L^c} |R|\psi_{j,k}(\cdot - x_R)\psi_{j,k} * f(x_R) \Big\|_{\dot{\mathcal{F}}_{p,q}^{s,w}}$$
$$\lesssim \Big\| \Big( \sum_{(j,k,R)\in W_L^c} 2^{js_1q}2^{ks_2q}|\psi_{j,k} * f(x_R)|^q \chi_R \Big)^{\frac{1}{q}} \Big\|_{L_w^p},$$

where the last term tends to zero as $L$ tends to infinity, whenever $f \in \dot{\mathcal{F}}_{p,q}^{s,w}(\mathbb{R}^n)$.

Suppose $f \in \dot{\mathcal{B}}_{p,q}^{s,w}(\mathbb{R}^n)$, set $U_L = \{(j, k) : |j| \le L, |k| \le L\}$ and $V_L = \{R \in \mathcal{R}_{\mathcal{C}}^{j,k} : R \subseteq B(0, L)\}$. Then

$$\Big\| \sum_{(j,k)\in U_L^c, R\in V_L^c} |R|\psi_{j,k}(\cdot - x_R)\psi_{j,k} * f(x_R) \Big\|_{\dot{\mathcal{B}}_{p,q}^{s,w}}$$
$$\lesssim \Big( \sum_{(j,k)\in U_L^c} 2^{js_1q}2^{ks_2q} \Big\| \sum_{R\in V_L^c} |\psi_{j,k} * f(x_R)|\chi_R \Big\|_{L_w^p}^q \Big)^{\frac{1}{q}},$$

where the last term tends to zero as $L$ goes to infinity, whenever $f \in \dot{\mathcal{B}}_{p,q}^{s,w}(\mathbb{R}^n)$. □

As a consequence of Lemma 3.1, $L^2 \cap \dot{\mathcal{F}}_{p,q}^{s,w}(\mathbb{R}^n)$ is dense in $\dot{\mathcal{F}}_{p,q}^{s,w}(\mathbb{R}^n)$ and $L^2 \cap \dot{\mathcal{B}}_{p,q}^{s,w}(\mathbb{R}^n)$ is dense in $\dot{\mathcal{B}}_{p,q}^{s,w}(\mathbb{R}^n)$.

## 4. Proof of Theorem 1.6

To show Theorem 1.6, we need a discrete Calderón-type identity on $L^2$ involving bump functions and almost orthogonality estimates involving the kernels $\mathcal{K}_1$ and $\mathcal{K}_2$. To do this, let $\phi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ with supp $\phi^{(1)} \subseteq B(0, 1)$,

$$\sum_{j\in\mathbb{Z}} \widehat{\phi^{(1)}}(2^j \circ_e \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n\backslash\{0\} \tag{14}$$

and

$$\int_{\mathbb{R}^n} \phi^{(1)}(x)x^{\alpha}\mathrm{d}x = 0, \quad \text{for all } |\alpha| \le L_0, \tag{15}$$

and let $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^n)$ with supp $\phi^{(2)} \subseteq B(0, 1)$,

$$\sum_{k\in\mathbb{Z}} \widehat{\phi^{(2)}}(2^k \circ_h \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n\backslash\{0\} \tag{16}$$

and

$$\int_{\mathbb{R}^n} \phi^{(2)}(x)x^{\beta}\mathrm{d}x = 0, \quad \text{for all } |\beta| \le L_0. \tag{17}$$

Let $\phi_j^{(1)}(x) = 2^{-jn}\phi^{(1)}(2^{-j} \circ_e x)$, $\phi_k^{(2)}(x) = 2^{-kn}\phi^{(2)}(2^{-k} \circ_h x)$ and $\phi_{j,k} = \phi_j^{(1)} * \phi_k^{(2)}$. The discrete Calderón-type identity is as follows.

PROPOSITION 4.1

*Let $0 < p \le 1$ and $w \in A_\infty^{\mathcal{C}}(\mathbb{R}^n)$. Let $\phi_{j,k}$ be defined as above with $L_0$ sufficiently large. Then for any $f \in L^2 \cap \dot{\mathcal{F}}^{s,w}_{p,q}$, there exists $h \in \dot{\mathcal{F}}^{s,w}_{p,q}$ such that for a sufficiently large $N \in \mathbb{N}$,*

$$f(x) = \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j-N,k-N}} |R|\phi_{j,k}(x - x_R)\phi_{j,k} * h(x_R),$$

*where $x_R$ denotes the minimal corner of $R$ and the series converges in $L^2(\mathbb{R}^n)$. Moreover, $\|f\|_{L^2} \sim \|h\|_{L^2}$, $\|f\|_{\dot{\mathcal{F}}^{s,w}_{p,q}} \sim \|h\|_{\dot{\mathcal{F}}^{s,w}_{p,q}}$. Similar result holds if weighted Triebel–Lizorkin spaces $\dot{\mathcal{F}}^{s,w}_{p,q}$ is replaced by weighted Besov spaces $\dot{\mathcal{B}}^{s,w}_{p,q}$.*

Proposition 4.1 can be proved as in the Hardy space case, see [3]. Using this proposition and repeating the same proof as in Theorem 1.5, we get

*Lemma* 4.2. *Let $w \in A_\infty^{\mathcal{C}}(\mathbb{R}^n)$. If $L_0$ and $N$ are chosen sufficiently large, then for $s = (s_1, s_2) \in \mathbb{R}^2$ and $p, q \in (0, \infty)$,*

$$\|f\|_{\dot{\mathcal{F}}^{s,w}_{p,q}} \sim \left\| \left( \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j-N,k-N}} 2^{js_1q}2^{ks_2q}|\phi_{j,k} * f(x_R)|^q \chi_R \right)^{\frac{1}{q}} \right\|_{L^p_w},$$

$$\|f\|_{\dot{\mathcal{F}}^{s,w}_{p,q}} \sim \left( \sum_{j,k\in\mathbb{Z}} 2^{js_1q}2^{ks_2q} \left\| \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j-N,k-N}} |\phi_{j,k} * f(x_R)|\chi_R \right\|^q_{L^p_w} \right)^{\frac{1}{q}}.$$

The fact that $\phi^{(1)}$ and $\phi^{(2)}$ have compact support enables us to prove the following almost orthogonality estimates (see [2, 3]).

*Lemma* 4.3. *Let $\phi_{j,k}$ be defined as above. Then for any $L \le L_0 - 1$ and any $M > 0$,*

$$|\phi_j^{(1)} * \mathcal{K}_1 * \phi_{j'}^{(1)}(x)| \le C2^{-L|j-j'|}\frac{2^{(j\vee j')M}}{(2^{j\vee j'} + |x|)^{n+M}} \tag{18}$$

*and*

$$|\phi_k^{(2)} * \mathcal{K}_2 * \phi_{k'}^{(2)}(x)| \le C2^{-L|k-k'|} \frac{2^{(k\vee k')M}}{(2^{k\vee k'} + |x'|)^{n-1+M}} \frac{2^{(2k\vee 2k')M}}{(2^{2k\vee 2k'} + |x_n|)^{1+M}}. \tag{19}$$

Now, we are ready to give

*Proof of Theorem* 1.6. We first assume that $f \in L^2 \cap \dot{\mathcal{F}}_{p,q}^{s,w}(\mathbb{R}^n)$. Let $\mathbf{K}$ be either $\mathcal{K}_1$ or $\mathcal{K}_2$ and $\mathbf{T}$ be the operator with convolution kernel $\mathbf{K}$. Set $\tilde{\phi}_{j,k} = \phi_{j,k} * \mathbf{K}$, then Lemma 4.3 says that $\psi_{j,k} * \tilde{\phi}_{j',k'}$ satisfy the almost orthogonality estimates in Lemma 2.2. Applying Proposition 4.1 and repeating the same argument as in (12), we have

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j,k}} 2^{js_1q} 2^{ks_2q} |\psi_{j,k} * \mathbf{K} * f(x_R)|^q \chi_R(x) \\
&\quad \lesssim \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j,k}} 2^{js_1q} 2^{ks_2q} |\sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j-N,k-N}} |R'| \phi_{j',k'} * h(x_{R'}) \\
&\quad\quad \times \psi_{j,k} * \tilde{\phi}_{j',k'}(x_R - x_{R'})|^q \chi_R(x) \\
&\quad \lesssim \sum_{j',k'\in\mathbb{Z}} \left\{ \mathcal{M}_{\mathcal{C}} \left[ \left( \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j'-N,k'-N}} 2^{j's_1q} 2^{k's_2q} |\phi_{j',k'} * h(x_{R'})|^q \chi_{R'} \right)^{\frac{\delta}{q}} \right](x) \right\}^{\frac{q}{\delta}}.
\end{aligned}$$

Applying weighted Fefferman–Stein inequality (5), Lemmas 4.2 and Proposition 4.1 yields

$$\begin{aligned}
\|\mathbf{T}f\|_{\dot{\mathcal{F}}_{p,q}^{s,w}} &= \left\| \left\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j,k}} 2^{js_1q} 2^{ks_2q} |\psi_{j,k} * f(x_R)|^q \chi_R \right\}^{\frac{1}{q}} \right\|_{L_w^p} \\
&\lesssim \left\| \left\{ \sum_{j',k'\in\mathbb{Z}} \left\{ \mathcal{M}_{\mathcal{C}} \left[ \left( \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j'-N,k'-N}} 2^{j's_1q} 2^{k's_2q} \right.\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left.\left. \times |\phi_{j',k'} * h(x_{R'})|^q \chi_{R'} \right)^{\frac{\delta}{q}} \right] \right\}^{\frac{q}{\delta}} \right\}^{\frac{1}{q}} \right\|_{L_w^p} \\
&\lesssim \left\| \left\{ \sum_{j',k'\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathcal{C}}^{j'-N,k'-N}} 2^{j's_1q} 2^{k's_2q} |\phi_{j',k'} * h(x_{R'})|^q \chi_{R'} \right\}^{\frac{1}{q}} \right\|_{L_w^p} \\
&\sim \|h\|_{\dot{\mathcal{F}}_{p,q}^{s,w}} \sim \|f\|_{\dot{\mathcal{F}}_{p,q}^{s,w}}.
\end{aligned}$$

By Lemma 3.1, $f \in L^2 \cap \dot{\mathcal{F}}_{p,q}^{s,w}$ is dense in $\dot{\mathcal{F}}_{p,q}^{s,w}$, thus a density argument yields the boundedness of $\mathbf{T}$ on $\dot{\mathcal{F}}_{p,q}^{s,w}$.

The proof of the boundedness of $\mathbf{T}$ on weighted Besov spaces is similar to Theorem 1.5. Indeed, applying Proposition 4.1 and Lemmas 2.4, 4.3 and weighted Fefferman–Stein's inequality (5) yields

$$\begin{aligned}\|\mathbf{T}f\|_{\dot{\mathcal{B}}^{s,w}_{p,q}} &= \Bigg(\sum_{j,k\in\mathbb{Z}} 2^{js_1q}2^{ks_2q}\Bigg\|\sum_{R\in\mathcal{R}^{j,k}_C}|\psi_{j,k}*f(x_R)|\chi_R\Bigg\|^q_{L^p_w}\Bigg)^{\frac{1}{q}}\\ &\lesssim \Bigg(\sum_{j',k'\in\mathbb{Z}} 2^{j's_1q}2^{k's_2q}\Bigg\|\sum_{R'\in\mathcal{R}^{j'-N,k'-N}_C}|\phi_{j',k'}*h(x_{R'})|\chi_{R'}\Bigg\|^q_{L^p_w}\Bigg)^{\frac{1}{q}}\\ &\sim \|h\|_{\dot{\mathcal{B}}^{s,w}_{p,q}} \sim \|f\|_{\dot{\mathcal{B}}^{s,w}_{p,q}},\end{aligned}$$

which implies the boundedness of $\mathbf{T}$ on $\dot{\mathcal{B}}^{s,w}_{p,q}(\mathbb{R}^n)$. Hence the proof of Theorem 1.6 is complete. □


## Acknowledgements

The author would like to express his deep gratitude to the referees for their valuable comments and suggestions. This research was supported by NNSF-China (Grant No. 11101423), the Fundamental Research Funds for the Central Universities of China (Grant No. 2009QS12) and supported in part by NNSF-China (Grant No. 11171345).


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