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Positive solutions for system of *2n*-th order Sturm–Liouville boundary value problems on time scales

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Abstract. Intervals of the parameters λ and μ are determined for which there exist positive solutions to the system of dynamic equations

$$(-1)^n u^{\Delta^{2n}}(t) + \lambda p(t) f(v(\sigma(t))) = 0, \quad t \in [a, b],$$
$$(-1)^n v^{\Delta^{2n}}(t) + \mu q(t) g(u(\sigma(t))) = 0, \quad t \in [a, b],$$

satisfying the Sturm–Liouville boundary conditions

$$\alpha_{i+1} u^{\Delta^{2i}}(a) - \beta_{i+1} u^{\Delta^{2i+1}}(a) = 0,\ \gamma_{i+1} u^{\Delta^{2i}}(\sigma(b)) + \delta_{i+1} u^{\Delta^{2i+1}}(\sigma(b)) = 0,$$
$$\alpha_{i+1} v^{\Delta^{2i}}(a) - \beta_{i+1} v^{\Delta^{2i+1}}(a) = 0,\ \gamma_{i+1} v^{\Delta^{2i}}(\sigma(b)) + \delta_{i+1} v^{\Delta^{2i+1}}(\sigma(b)) = 0,$$

for $0 \le i \le n-1$. To this end we apply a Guo–Krasnosel'skii fixed point theorem.

Keywords. Time scales; system of equations; boundary value problem; eigenvalue intervals; positive solution; cone.

Mathematics Subject Classifications. 39A10, 34B15, 34A40.

1. Introduction

The theory of dynamic equations on time scales (more generally, on measure chains) was introduced in 1988 by Stefan Hilger in his Ph.D. thesis (see [20,21]). The theory presents a structure where, once a result is established for a general time scale, then special cases can be obtained by taking the particular time scale. If $\mathbb{T} = \mathbb{R}$, then we have the result for differential equations. Choosing $\mathbb{T} = \mathbb{Z}$ we immediately get the result for difference equations. A great deal of work has been done since 1988, unifying and extending the theories of differential and difference equations, and many results are now available in the general setting of time scales and references therein.

On a larger scale, there has been a great deal of study focused on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [14,16,17,22,29] and as applications for which only positive solutions are meaningful [1,15,23,24]. These considerations are cast primarily for scalar problems, but much attention has been given to boundary value problems for systems of differential equations [18,19,27,28,30].

In this paper, we are concerned with determining values of λ and μ for which there exist positive solutions for the system of dynamic equations,

$$(-1)^n u^{\Delta^{2n}}(t) + \lambda p(t) f(v(\sigma(t))) = 0, \quad t \in [a, b],$$
$$(-1)^n v^{\Delta^{2n}}(t) + \mu q(t) g(u(\sigma(t))) = 0, \quad t \in [a, b], \tag{1.1}$$

satisfying the boundary conditions

$$\alpha_{i+1} u^{\Delta^{2i}}(a) - \beta_{i+1} u^{\Delta^{2i+1}}(a) = 0, \ \gamma_{i+1} u^{\Delta^{2i}}(\sigma(b)) + \delta_{i+1} u^{\Delta^{2i+1}}(\sigma(b)) = 0,$$
$$\alpha_{i+1} v^{\Delta^{2i}}(a) - \beta_{i+1} v^{\Delta^{2i+1}}(a) = 0, \ \gamma_{i+1} v^{\Delta^{2i}}(\sigma(b)) + \delta_{i+1} v^{\Delta^{2i+1}}(\sigma(b)) = 0, \tag{1.2}$$

with $\alpha_i, \beta_i, \gamma_i, \delta_i \geq 0$ such that

$$d_i = \gamma_i \beta_i + \alpha_i \delta_i + \alpha_i \gamma_i (\sigma(b) - a) > 0. \tag{1.3}$$

We will use the following assumptions:

(A1) $f, g \in C([0, \infty), [0, \infty))$;
(A2) $p, q \in C([a, \sigma(b)], [0, \infty))$, and each function does not vanish identically on any closed subinterval of $[a, \sigma(b)]$;
(A3) The following limits exist as positive real numbers:

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x}, \ g_0 := \lim_{x \to 0^+} \frac{g(x)}{x},$$
$$f_\infty := \lim_{x \to \infty} \frac{f(x)}{x} \text{ and } g_\infty := \lim_{x \to \infty} \frac{g(x)}{x}.$$

We also, assume that $\sigma(b)$ is right dense so that $\sigma^k(b) = \sigma(b)$, for $k \geq 1$. Throughout this paper, we let $\mathbb{T}$ be any time scale (nonempty closed subset of $\mathbb{R}$) and $[a, b]$ be a subset of $\mathbb{T}$ such that $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$.

The main tool in this paper is an application of the Krasnosel'skii fixed point theorem for operators leaving a Banach space cone invariant [14]. A Green function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. Some preliminaries

In this section, we state some preliminaries and we will apply the following fixed point theorem which can be found in the book by Krasnosel'skii [25] as well as in the book by Deimling [11].

Theorem 2.1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either

(i) $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

We will apply Theorem 2.1 to a completely continuous operator whose kernel is the Green's function for

$$\begin{aligned}&(-1)^n u^{\Delta^{2n}}(t) = 0, \quad t \in [a, b],\\ &\alpha_{i+1} u^{\Delta^{2i}}(a) - \beta_{i+1} u^{\Delta^{2i+1}}(a) = 0,\ \gamma_{i+1} u^{\Delta^{2i}}(\sigma(b)) + \delta_{i+1} u^{\Delta^{2i+1}}(\sigma(b)) = 0.\end{aligned} \tag{2.1}$$

For $1 \le i \le n$, let $G_i(t, s)$ be the Green's function for the boundary value problems

$$\begin{aligned}&-u^{\Delta\Delta}(t) = 0, \quad t \in [a, b],\\ &\alpha_i u(a) - \beta_i u^{\Delta}(a) = 0,\ \gamma_i u(\sigma(b)) + \delta_i u^{\Delta}(\sigma(b)) = 0.\end{aligned}$$

Then, for $1 \le i \le n$,

$$G_i(t, s) = \frac{1}{d_i}\begin{cases}\{\alpha_i(t-a) + \beta_i\}\{\gamma_i(\sigma(b) - \sigma(s)) + \delta_i\}, & t \le s,\\ \{\alpha_i(\sigma(s) - a) + \beta_i\}\{\gamma_i(\sigma(b) - t) + \delta_i\}, & \sigma(s) \le t,\end{cases} \tag{2.2}$$

where d_i is defined by (1.3). It was shown in [12] that, for $1 \le i \le n$,

$$G_i(t, s) > 0, \quad (t, s) \in (a, \sigma(b)) \times (a, \sigma(b)) \tag{2.3}$$

and

$$G_i(t, s) \le G_i(\sigma(s), s) = \frac{[\alpha_i(\sigma(s) - a) + \beta_i][\gamma_i(\sigma(b) - \sigma(s)) + \delta_i]}{d_i} \tag{2.4}$$

for $t \in [a, \sigma(b)]$, $s \in [a, \sigma(b)]$. Let $I = \left[\frac{3a+\sigma(b)}{4}, \frac{a+3\sigma(b)}{4}\right]$. Then

$$G_i(t, s) \ge k_i G_i(\sigma(s), s) = k_i \frac{[\alpha_i(\sigma(s) - a) + \beta_i][\gamma_i(\sigma(b) - \sigma(s)) + \delta_i]}{d_i} \tag{2.5}$$

for $t \in I$, $s \in [a, \sigma(b)]$, where

$$k_i = \min\left\{\frac{\alpha_i(\sigma(b) - a) + 4\beta_i}{4(\alpha_i(\sigma(b) - a) + \beta_i)}, \frac{\gamma_i(\sigma(b) - a) + 4\delta_i}{4(\gamma_i(\sigma(b) - a) + \delta_i)}\right\} < 1. \tag{2.6}$$

3. Positive solutions in a cone

Assume throughout that $[a, \sigma(b)]$ is such that

$$\begin{aligned}\xi &= \min\left\{t \in \mathbb{T} : t \ge \frac{3a + \sigma(b)}{4}\right\},\\ \omega &= \max\left\{t \in \mathbb{T} : t \le \frac{a + 3\sigma(b)}{4}\right\};\end{aligned}$$

both exist and satisfy

$$\frac{3a+\sigma(b)}{4} \leq \xi < \omega \leq \frac{a+3\sigma(b)}{4}.$$

Next, let $\tau \in [\xi, \omega]$ be defined by

$$\int_{\xi}^{\omega} G_i(\tau, s)p(s)\Delta s = \max_{t\in[\xi,\omega]} \int_{\xi}^{\omega} G_i(t, s)p(s)\Delta s.$$

Finally, we define, for $1 \leq i \leq n$,

$$l_i = \min_{s\in[a,\sigma(b)]} \frac{G_i(\sigma(\omega), s)}{G_i(\sigma(s), s)},$$

and set

$$M_i = \min\{k_i, l_i\}.$$

Next, set

$$H_1(t, s) = G_1(t, s),$$

and for $2 \leq j \leq n$, we recursively define

$$H_j(t, s) = \int_a^{\sigma(b)} H_{j-1}(t, r)G_j(r, s)\Delta r. \tag{3.1}$$

Then, $H_n(t, s)$ is the Green's function for the boundary value problem (1.1), (1.2).

If we define

$$L_j = \int_a^{\sigma(b)} G_j(\sigma(r), r)\Delta r, \qquad 1 \leq j \leq n$$

and

$$K_j = \int_{\xi}^{\omega} G_j(\sigma(r), r)\Delta r, \qquad 1 \leq j \leq n,$$

it follows from arguments in [12] and [10] that

$$0 \leq H_n(t, s) \leq \eta G_n(\sigma(s), s), \quad (t, s) \in [a, \sigma(b)] \times [a, b] \tag{3.2}$$

and

$$H_n(t, s) \geq M\eta G_n(\sigma(s), s), \quad (t, s) \in [\xi, \sigma(\omega)] \times [a, b], \tag{3.3}$$

where

$$\eta = \prod_{j=1}^{n-1} L_j \quad \text{and} \quad M = M_n \prod_{j=1}^{n-1} \frac{M_j K_j}{L_j} < 1.$$

We note that a pair $(u(t), v(t))$ is a solution of the eigenvalue problem (1.1), (1.2) if and only if

$$u(t) = \lambda \int_a^{\sigma(b)} H_n(t,s)p(s)f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s, \quad a \le t \le \sigma(b),$$

where

$$v(t) = \mu \int_a^{\sigma(b)} H_n(t,s)q(s)g(u(\sigma(s)))\Delta s, \quad a \le t \le \sigma(b).$$

Values of λ and μ for which there are positive solutions (positive with respect to a cone) of (1.1), (1.2) will be determined via applications of the Krasnosel'skii fixed point theorem.

Throughout this paper, we let $\mathcal{B} = \{x : [a, \sigma(b)] \to \mathbb{R}\}$. Then $\mathcal{B}$ is a Banach space with the norm $\|x\| = \sup_{t\in[a,\sigma(b)]} |x(t)|$. We define a set $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{x \in \mathcal{B} : x(t) \ge 0 \text{ on } [a,\sigma(b)] \text{ and } \min_{t\in[\xi,\sigma(\omega)]} x(t) \ge M\|x\|\right\}. \tag{3.4}$$

Clearly, $\mathcal{P}$ is a cone.

For our first result, define positive numbers Λ_1 and Λ_2 by

$$\Lambda_1 := \max\left\{\left[M^2\eta \int_\xi^\omega G_n(\tau,s)p(s)\Delta s f_\infty\right]^{-1}, \left[M^2\eta \int_\xi^\omega G_n(\tau,s)q(s)\Delta s g_\infty\right]^{-1}\right\},$$

$$\Lambda_2 := \min\left\{\left[\eta \int_a^{\sigma(b)} G_n(\sigma(s),s)p(s)\Delta s f_0\right]^{-1}, \left[\eta \int_a^{\sigma(b)} G_n(\sigma(s),s)q(s)\Delta s g_0\right]^{-1}\right\}.$$

Theorem 3.1. *Assume that conditions* (A1)–(A3) *are satisfied. Then, for each* λ, μ *satisfying*

$$\Lambda_1 < \lambda, \mu < \Lambda_2, \tag{3.5}$$

there exists a pair (u, v) *satisfying* (1.1), (1.2) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(a, \sigma(b))$.

Proof. Let λ, μ be as in (3.5) and let $\epsilon > 0$ be chosen such that

$$\max\left\{\left[M^2\eta \int_\xi^\omega G_n(\tau,s)p(s)\Delta s(f_\infty - \epsilon)\right]^{-1}, \left[M^2\eta \int_\xi^\omega G_n(\tau,s)q(s)\Delta s(g_\infty - \epsilon)\right]^{-1}\right\} \le \lambda, \mu$$

and

$$\lambda, \mu \le \min\left\{\left[\eta \int_a^{\sigma(b)} G_n(\sigma(s), s) p(s) \Delta s (f_0 + \epsilon)\right]^{-1}, \right.$$

$$\left.\left[\eta \int_a^{\sigma(b)} G_n(\sigma(s), s) q(s) \Delta s (g_0 + \epsilon)\right]^{-1}\right\}.$$

Define an integral operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) = \lambda \int_a^{\sigma(b)} H_n(t, s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r\right) \Delta s. \tag{3.6}$$

We seek a fixed point of T which belongs to $\mathcal{P}$ and so we first claim $T : \mathcal{P} \to \mathcal{P}$. From the nonnegativity of $H_n(t, s)$ and from assumptions (A1) and (A2), if $u \in \mathcal{P}$, then $Tu(t) \ge 0$ on $[a, \sigma(b)]$.

Next, if we choose $u \in \mathcal{P}$, then from (3.2),

$$\begin{aligned} Tu(t) &:= \lambda \int_a^{\sigma(b)} H_n(t, s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r\right) \Delta s \\ &\le \lambda \eta \int_a^{\sigma(b)} G_n(\sigma(s), s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r\right) \Delta s \end{aligned}$$

and so

$$\|Tu\| \le \lambda \eta \int_a^{\sigma(b)} G_n(\sigma(s), s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r\right) \Delta s.$$

Next, if $u \in \mathcal{P}$, we have from (3.3) and (3.6) that

$$\begin{aligned} &\min_{t \in [\xi, \sigma(\omega)]} Tu(t) \\ &= \min_{t \in [\xi, \sigma(\omega)]} \lambda \int_a^{\sigma(b)} H_n(t, s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r\right) \Delta s \\ &\ge \lambda M \eta \int_a^{\sigma(b)} G_n(\sigma(s), s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r\right) \Delta s \\ &\ge M \|Tu\|. \end{aligned}$$

Thus $Tu \in \mathcal{P}$ if $u \in \mathcal{P}$ and we conclude that $T : \mathcal{P} \to \mathcal{P}$. In addition, standard arguments show that T is completely continuous.

Now, from the definitions of f_0 and g_0, there exists an $H_1 > 0$ such that

$$f(x) \le (f_0 + \epsilon)x \quad \text{and} \quad g(x) \le (g_0 + \epsilon)x, \quad 0 < x \le H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (3.2) and choice of ϵ, for $a \le s \le \sigma(b)$, that

$$\begin{aligned}
&\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r \\
&\quad \le \mu\eta \int_a^{\sigma(b)} G_n(\sigma(r), r) q(r) g(u(\sigma(r)))\Delta r \\
&\quad \le \mu\eta \int_a^{\sigma(b)} G_n(\sigma(r), r) q(r) (g_0 + \epsilon) u(r)\Delta r \\
&\quad \le \mu\eta \int_a^{\sigma(b)} G_n(\sigma(r), r) q(r)\Delta r (g_0 + \epsilon)\|u\| \\
&\quad \le \|u\| \\
&\quad = H_1.
\end{aligned}$$

As a consequence, we next have from (3.2) and choice of ϵ, for $a \le t \le \sigma(b)$, that

$$\begin{aligned}
Tu(t) &= \lambda \int_a^{\sigma(b)} H_n(t, s) p(s) f\Big(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r\Big)\Delta s \\
&\le \lambda\eta \int_a^{\sigma(b)} G_n(\sigma(s), s) p(s) (f_0 + \epsilon)\mu \\
&\qquad \times \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r \Delta s \\
&\le \lambda\eta \int_a^{\sigma(b)} G(\sigma(s), s) p(s) (f_0 + \epsilon) H_1 \Delta s \\
&\le H_1 \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \le \|u\|$. If we set

$$\Omega_1 = \{x \in \mathcal{B} : \|x\| < H_1\},$$

then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.7}$$

Next, from the definitions of f_∞ and g_∞, there exists $\bar{H}_2 > 0$ such that

$$f(x) \ge (f_\infty - \epsilon)x \quad \text{and} \quad g(x) \ge (g_\infty - \epsilon)x, \;\; x \ge \bar{H}_2.$$

Set

$$H_2 = \max\{2H_1, \bar{H}_2/M\},$$

and let $u \in \mathcal{P}$ with $\|u\| = H_2$. Then

$$\min_{t \in [\xi, \sigma(\omega)]} u(t) \ge M\|u\| \ge \bar{H}_2.$$

Consequently, from (3.3) and choice of ϵ, for $a \le s \le \sigma(b)$, we have that

$$\begin{aligned}
&\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r \\
&\ge \mu \int_\xi^{\omega} H_n(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r \\
&\ge \mu M \eta \int_\xi^{\omega} G_n(\tau, r) q(r)(g_\infty - \epsilon) u(\sigma(r))\Delta r \\
&\ge \mu M^2 \eta \int_\xi^{\omega} G_n(\tau, r) q(r)(g_\infty - \epsilon)\Delta r \|u\| \\
&\ge \|u\| \\
&= H_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Tu(\tau) &= \lambda \int_a^{\sigma(b)} H_n(\tau, s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r\right)\Delta s \\
&\ge \lambda \int_\xi^{\omega} H_n(\tau, s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r\right)\Delta s \\
&\ge \lambda \int_a^{\sigma(b)} H_n(\tau, s) p(s)(f_\infty - \epsilon)\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r \Delta s \\
&\ge \lambda M \eta \int_a^{\sigma(b)} G_n(\tau, s) p(s)(f_\infty - \epsilon) M H_2 \Delta s \\
&\ge H_2 \\
&= \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \ge \|u\|$. So if we set

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\},$$

then

$$\|Tu\| \ge \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.8}$$

Applying Theorem 2.1 to (3.7) and (3.8), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$. As such, and with v being defined by

$$v(t) = \mu \int_a^{\sigma(b)} H_n(t, s) q(s) g(u(\sigma(s)))\Delta s,$$

the pair (u, v) is a desired solution of (1.1), (1.2) for the given λ and μ. The proof is complete. □

Prior to our next result, we introduce another hypothesis.

(A4) $g(0) = 0$, and f is an increasing function.

We now define positive numbers Λ_3 and Λ_4 by

$$\Lambda_3 := \max\left\{\left[M^2\eta\int_\xi^\omega G_n(\tau,s)p(s)\Delta s f_0\right]^{-1},\right.$$
$$\left.\left[M^2\eta\int_\xi^\omega G_n(\tau,s)q(s)\Delta s g_0\right]^{-1}\right\},$$
$$\Lambda_4 := \min\left\{\left[\eta\int_a^{\sigma(b)} G_n(\sigma(s),s)p(s)\Delta s f_\infty\right]^{-1},\right.$$
$$\left.\left[\eta\int_a^{\sigma(b)} G_n(\sigma(s),s)q(s)\Delta s g_\infty\right]^{-1}\right\}.$$

Theorem 3.2. *Assume that conditions* (A1)–(A4) *are satisfied. Then, for each* λ, μ *satisfying*

$$\Lambda_3 < \lambda, \mu < \Lambda_4, \tag{3.9}$$

there exists a pair (u, v) *satisfying* (1.1), (1.2) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(a, \sigma(b))$.

Proof. Let λ, μ be as in (3.9) and let $\epsilon > 0$ be chosen such that

$$\max\left\{\left[M^2\eta\int_\xi^\omega G_n(\tau,s)p(s)\Delta s(f_0-\epsilon)\right]^{-1},\right.$$
$$\left.\left[M^2\eta\int_\xi^\omega G_n(\tau,s)q(s)\Delta s(g_0-\epsilon)\right]^{-1}\right\} \le \lambda, \mu$$

and

$$\lambda, \mu \le \min\left\{\left[\eta\int_a^{\sigma(b)} G_n(\sigma(s),s)p(s)\Delta s(f_\infty+\epsilon)\right]^{-1},\right.$$
$$\left.\left[\eta\int_a^{\sigma(b)} G_n(\sigma(s),s)q(s)\Delta s(g_\infty+\epsilon)\right]^{-1}\right\}.$$

Let T be defined as in (3.6). Then T is a cone preserving, completely continuous operator.

By the definitions of f_0 and g_0, there exists $\bar{H}_3 > 0$ such that

$$f(x) \ge (f_0-\epsilon)x \quad \text{and} \quad g(x) \ge (g_0-\epsilon)x,\ 0 < x \le \bar{H}_3.$$

Now, $g(0) = 0$, and so there exists $0 < H_3 < \bar{H}_3$ such that

$$\mu g(x) \le \frac{\bar{H}_3}{\eta\int_a^{\sigma(b)} G_n(\sigma(s),s)q(s)\Delta s}, \quad 0 \le x \le H_3.$$

Choose $u \in \mathcal{P}$ with $\|u\| = H_3$. Then, for $a \leq s \leq \sigma(b)$, we have

$$\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r \leq \frac{\int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) \bar{H}_3 \Delta r}{\eta \int_a^{\sigma(b)} G_n(\sigma(s), s) q(s) \Delta s} \leq \bar{H}_3.$$

Hence,

$$\begin{aligned}
Tu(\tau) &= \lambda \int_a^{\sigma(b)} H_n(\tau, s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r\right) \Delta s \\
&\geq \lambda \int_\xi^{\omega} H_n(\tau, s) p(s) f\left(\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r\right) \Delta s \\
&\geq \lambda \int_\xi^{\omega} H_n(\tau, s) p(s) (f_0 - \epsilon) \mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r \Delta s \\
&\geq \lambda \int_\xi^{\omega} H_n(\tau, s) p(s) (f_0 - \epsilon) \mu M \eta \int_\xi^{\omega} G_n(\tau, r) q(r) g(u(\sigma(r))) \Delta r \Delta s \\
&\geq \lambda \int_\xi^{\omega} H_n(\tau, s) p(s) (f_0 - \epsilon) \mu M^2 \eta \int_\xi^{\omega} G_n(\tau, r) q(r) (g_0 - \epsilon) \|u\| \Delta r \Delta s \\
&\geq \lambda M^2 \eta \int_\xi^{\omega} G_n(\tau, s) p(s) (f_0 - \epsilon) \|u\| \Delta s \\
&\geq \|u\|.
\end{aligned}$$

So, $\|Tu\| \geq \|u\|$. If we put $\Omega_1 = \{x \in \mathcal{B} : \|x\| < H_3\}$, then

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.10}$$

Next, by definitions of f_∞ and g_∞, there exists $\bar{H}_4$ such that

$$f(x) \leq (f_\infty - \epsilon) x \quad \text{and} \quad g(x) \leq (g_\infty - \epsilon) x, \quad x \geq \bar{H}_4$$

There are two cases:

(i) g is bounded, and
(ii) g is unbounded.

For case (i), suppose $N > 0$ is such that $g(x) \leq N$ for all $0 < x < \infty$. Then, for $a \leq s \leq \sigma(b)$ and $u \in \mathcal{P}$,

$$\mu \int_a^{\sigma(b)} H_n(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r \leq N \mu \eta \int_a^{\sigma(b)} G_n(\sigma(r), r) q(r) \Delta r.$$

Let

$$J = \max\left\{ f(x) : 0 \leq x \leq N \mu \eta \int_a^{\sigma(b)} G_n(\sigma(r), r) q(r) \Delta r \right\}$$

and let

$$H_4 > \max\left\{ 2H_3, J \lambda \eta \int_a^{\sigma(b)} G_n(\sigma(s), s) p(s) \Delta s \right\}.$$

Then, for $u \in \mathcal{P}$ with $\|u\| = H_4$,

$$Tu(t) \le \lambda\eta \int_a^{\sigma(b)} G_n(\sigma(s), s)p(s)J\Delta s \le H_5 = \|u\|$$

so that $\|Tu\| \le \|u\|$. If

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_4\},$$

then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.11}$$

Next, consider case (ii). There exists $H_4 > \max\{2H_3, \bar{H}_4\}$ such that $g(x) \le g(H_4)$, for $0 < x \le H_4$. Similarly, there exists $H_5 > \max\{H_4, \mu\eta \int_a^{\sigma(b)} G_n(\sigma(r), r)q(r)g(H_4)\Delta r\}$ such that $f(x) \le f(H_5)$, for $0 < x \le H_5$. Choosing $u \in \mathcal{P}$ with $\|u\| = H_5$, we have by (A4) that

$$\begin{aligned}
Tu(t) &\le \lambda \int_a^{\sigma(b)} H_n(t, s)p(s)f\left(\mu\eta \int_a^{\sigma(b)} G_n(\sigma(r), r)q(r)g(H_5)\Delta r\right)\Delta s \\
&\le \lambda \int_a^{\sigma(b)} H_n(t, s)p(s)f(H_5)\Delta s \\
&\le \lambda\eta \int_a^{\sigma(b)} G_n(\sigma(s), s)p(s)\Delta s(f_\infty + \epsilon)H_5 \\
&\le H_5 \\
&= \|u\|,
\end{aligned}$$

and so $\|Tu\| \le \|u\|$. For this case, if we let

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_5\},$$

then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.12}$$

In either case, application of part (ii) of Theorem 2.1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$, which in turn yields a pair (u, v) satisfying (1.1), (1.2) for the chosen values of λ and μ. The proof is complete. □

4. Example

Let us introduce an example to illustrate the usage of Theorem 3.1. Let $n = 2$, $\mathbb{T} = \{(\frac{2}{5})^n : n \in \mathbb{N}_0\} \cup \{0\} \cup [1, 2]$, $\alpha_1 = \beta_2 = \frac{1}{2}$, $\beta_1 = \frac{1}{8}$, $\gamma_1 = \frac{3}{2}$, $\alpha_2 = \frac{1}{10}$, $\gamma_2 = 2$, $\delta_1 = \frac{2}{3}$, $\delta_2 = \frac{1}{4}$.

The Green function $G_1(t, s)$ in (2.2) is

$$G_1(t, s) = \frac{1200}{1381} \begin{cases} \left(\frac{t}{2} + \frac{9}{200}\right)\left(\frac{5}{6} - \frac{15s}{4}\right) : t \le s \\ \left(\frac{5s}{4} + \frac{9}{200}\right)\left(\frac{13}{6} - \frac{3t}{2}\right) : \frac{5s}{2} \le t \end{cases}$$

and the Green function $G_2(t,s)$ in (2.2) is

$$G_2(t,s) = \frac{1000}{1193}\begin{cases} \left(\frac{t}{10}+\frac{23}{50}\right)\left(\frac{9}{4}-5s\right) : t \le s, \\ \left(\frac{s}{10}+\frac{23}{50}\right)\left(\frac{9}{4}-2t\right) : \frac{5s}{2} \le t, \end{cases}$$

From (2.6), (3.2) and (3.3), we get

$$\begin{aligned} &m_1 = 0.4220, \quad K_1 = 0.1326, \quad L_1 = 0.2058, \\ &m_2 = 0.3471, \quad K_2 = 0.0706, \quad L_2 = 0.39207, \\ &\eta = 0.0806, \quad M = 0.3471. \end{aligned}$$

Also let, $f(v) = \frac{kve^{2v}}{c+e^v+e^{2v}}$, $g(u) = \frac{kue^{2u}}{c+e^u+e^{2u}}$, $p(t) = q(t) = \frac{1}{10}t$, $k = 500$, $c = 1000$. By simple calculation, $f_0 = g_0 = \frac{k}{c+2} = \frac{500}{1002}$, $f_\infty = g_\infty = k = 500$. By Theorem 3.1, it follows that for every λ, μ such that $432.4529 < \lambda, \mu < 7709.65$, the boundary value problem has at least one positive solution.

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